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Theory and Properties of Binary Relations

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***Abstract*: Apart from being the most common and easy topic, Relations and function still have got their own complexities to which most of the people are unfamiliar with. This paper is going to be a future reference for all those who want to research deeper in this section. Starting from the basic this paper will include all the necessary, relevant and weighty topics which need to be studied before further research. We will study Functions and relations, their properties, types, orders and range etc . An attempt has been made for novice to understand it thoroughly, starting from the scratch. All relevant notations and examples have also been included.**

I. INTRODUCTION

In mathematics, a binary relation on a set A is a collection of ordered pairs of elements of A. In other words, it is a subset of the Cartesian product A2 = A × A. More generally, one can say that a binary relation between two sets A and B is a subset of A × B. The terms correspondence, dyadic relation and 2-place relation are synonyms for binary relation.

Let X and Y are nonempty sets. A binary relation (or just relation) from X to Y is a subset R ⊆ X × Y.

If (x, y) ∈ R, we say that x is related to y by R, denoted xRy. If (x, y) 6∈ R, we say that x is not related to y, denoted xRy¯. For each element x ∈ X, we denote by R(x) the subset of elements of Y that are related to x, that is, R(x):= {y ∈ Y | xRy} = {y ∈ Y | (x, y) ∈ R}.

For each subset A ⊆ X, we define

R (A) = {y ∈ Y | ∃ x ∈ A such that xRy} = [x∈AR(x).

When X = Y, we say that R is a binary relation on X.
For a set A having n elements and B having m elements, the total number of relations from A to B is $2^{nxm.}$

For example

If a set A = {1, 2}, relations from A to A will be $2^{2x2} = 2^4 = 16$

Elements = {(1, 2), (2, 1), (1, 1), (2, 2)}

Relations = {(1,2),(2,1),(1,1),(2,2)}, {(1,2),(2,1)}, {(1,2), (1,1)}, {(1,2),(2,2)},{(2,1),(1,1)} , {(2,1),(2,2)},{(1,1),(2,2)} , {(1,2),(2,1),(1,1)} , (1,2),(1,1),(2,2)} , {(2,1),(1,1),(2,2)} , {(1,2),(2,1),(2,2)} , {(1,2),(2,1),(1,1),(2,2)} and fi .

II. BASIC CONCEPTS OF RELATIONS

A. *Domain of a relation:* The domain of a relation R is the set of elements in P which are related to some elements in Q or it is the set of all first entries of ordered pairs in R. It is denoted by DOM (R).

B. *Range of relation:* The range of a relation R is the set of elements in Q which are related to some elements in P or it is the set of all second entries of the ordered pairs in R. It is denoted by RAN (R).

For example : Let A={1,2,3,4}, B={a,b,c,d} and R={(1,a),(1,b),(1,c),(2,b),(2,c),(2,d).

Then,

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DOM (R) = {1, 2} AND RAN (R) ={a,b,c,d}

C. *Compliment of relation:* For a relation R from set A to B, the compliment of relation as its name says is denoted by R^c is a relation from A to B such that

R^c = {(a,b): (a,b) ∉ R}

D. *Inverse of Relation:* Consider a relation R from a set A to B. The inverse of a relation denoted by $R^{-1,}$ is a relation from A to B such that $bR^{-1a \text{ if}}$ aRb i.e.

$R^{-1 =}$ {(b,a) : (a,b) € R}

For example: Find the inverse of the relation {(0, 7), (4, –3), (–3, 4), (–2, –2)}.

To find the inverse of this relation just reverse the coordinates of the ordered pairs.

The inverse of the relation is {(7, 0), (–3, 4), (4, –3), (–2, –2)}.

E. *Composition of Relations:* The composition of binary relations is a concept of forming a new relation $S \circ R$ from two given relations *R* and *S*, having as its most well-known special case the composition of functions.

Consider a relation R1 from A to B and A2 from B to C. The composition of relation R1 and R2 denoted by R1 o R2, is a relation from A to C and is defined by R1 o R2 = {(a,c) : (a,b) € R1 and (b,c)€ R2 for some b€ B.

For example: Let P and Q be the relation on the set A= {1, 2, 3, 4} defined by

P={(1,2),(2,2),(2,3),(2,4),(3,2),(4,2),(4,3)}

Q={(2,2),(2,3),(3,2),(3,3),(3,4),(4,1),(4,2)}

P o P = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,2),(4,2),(4,3)}

P o Q = {(1,2),(1,3),(2,2),(2,3),(2,4),(2,1),(3,2),(3,3),(4,2),(4,3),(4,4)}

P o P o Q= {(1,2),(1,3),(1,4),(1,1),(2,2),(2,3),(2,4),(2,1),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3)}

III. PROPERTIES OF RELATIONS

Properties of the relation tell us about the nature of relation or type of relation.

A. *Reflexive Relation:* Consider a binary relation R on a set A. Relation R is called a reflexive relation if, for every a in set A, (a,a) € R i.e. (a,a) € R, ∀ a € A.

B. *Ir-reflexive Relation:* Consider a binary relation R on a set A. Relation R is called a irreflexive relation if, for every a in set A, (a,a) ∉ R i.e. (a,a) € R, ∀ a € A.

For example : The relation R on {1,2,3} given by R = {(1,1), (2,2), (2,3), (3,3)} is reflexive. (All loops are present.). But if any loop like (1, 1) or (2, 2) wouldn't have been there then the relation would have been irreflexive.

C. *Symmetric Relation:* Consider a binary relation R on a set A. Relation R is called a symmetric relation if for every (a,b) € R implies that (b,a) also belongs to R.

D. *Asymmetric Relation:* Consider a binary relation R on a set A. Relation R is called asymmetric relation if for every (a,b) € R implies that (b,a) does not belongs to R.

For Example : Consider the relation A={4,5,6} and R= {(1,2),(2,1),(3,2),(2,3)}

The relation is symmetric since for every (a,b) € R , we have (b,a) € R . But if any element like (2, 1) would have been absent then for every (a,b) € R , (b,a) is absent and the relation would then be called asymmetric.

E. *Transitive Relations:* Consider a binary relation R on a set A. Relation R is called transitive relation if whenever both (a,b) and (b,c) belong to R, implies that (a,c) also belongs to R i.e. (a,b),(b,c) € R ➔ (a,c) € R

For example: A={1,2,3} and R={(1,2),(2,1),(1,1),(2,2)

The relation R is a transitive as for every (a,b),(b,c) belong to R we have (a,c)€ R i.e.

(1, 2), (2, 1) € R ➔ (1, 1) € R

F. *Anti-symmetric Relations:* Consider a binary relation R on a set A. Relation R is called antisymmetric relation if (a,b) € R implies that (b,a) € R unless a=b. In other words, we can say if (a,b) and (b,a) belong to R, then a=b.

For example: A={4,5,6} and R={(4,4),(4,5),(5,4),(5,6),(4,6)}

The relation R is not anti-symmetric as $4 \neq 5$ but (4, 5) and (5, 4) both belong to R.

IV. CLOSURE PROPERTIES

Consider a relation R on some set A. Suppose the relation R does not possess the desired property. We will add as few new pairs as possible to relation R to get a relation that does have the desired property.

The smallest relation S on A that contains R and the desired property is called the closure of relation R with the desired property.

A. *Reflexive Closure:* A binary relation on a set is reflexive, if for every object a in the set, the pair (a,a) is in the relation.

The relation is irreflexive, if for no object in the set the pair (a,a) is in the relation.

The relation is neither reflexive nor irreflexive if there is an object a such that (a,a) is in the relation, and there is an object b such that (b,b) is not in the relation.

For example: Set A = {a,b,c} R: {(a,b),(a,c),(c,c)}

The relation is not reflexive. To make it reflexive, we add links (a,a) and (b,b)

The reflexive closure of the relation R is obtained by the union of R and the identity relation $\mathbf{I_A}$ on the set A. (Reminder: $I_A = \{(a,a) \mid a \in A\}$

Symmetric closure of R: A relation is symmetric if for each pair (a,b) in the relation, a ≠ b, the pair (b,a) is also in the relation. A relation is anti-symmetric if for each pair (a,b), a ≠ b, the pair (b,a) is not in the relation.

A relation is neither symmetric not anti-symmetric, if there is a pair (a,b), a ≠ b, such that the pair (b,a) is in the relation, and there is a pair (c,d), c ≠ d, such that the pair (d,c) is not in the relation.

Example: Set A = {a,b,c} R: {(a,b),(a,c), (c,c)}

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The relation is not symmetric.To make it symmetric, we add the elements **(c,a), (b,a)**

The symmetric closure of the relation R is obtained by the union of R and the inverse relation R^{-1} on the set A. Reminder: The inverse relation R^{-1} is defined as: {(b,a)| (a,b) **є** R}

Transitive closure of R: A relation is transitive if whenever R contains elements (a,b) and (b,c), it also contains the element (a,c).

Let's say R contains (a,b) and (b,c). We would need also (a,c) if we want to make R transitive. Certainly (a,c) is an element in the composition R ∘ R. Let us denote Rn = Rn-1 ∘ R, n > 1, R1 = R. The transitive closure R* of a relation R over a set of m elements is computed as

$R^* \subseteq R \cup R2 \cup R3 \cup \ldots \cup Rm$

If R = R*, then the set A is closed under R.

There is a theorem stating that in order find the transitive closure of a binary relation on a set with m elements we need to find at most the Rm composition

The algorithm is as follows:

1. Initialize S = R, i = 2. While S ∪ Ri

   S = S ∪ Ri

   i = i + 1

If the transitive closure is same as the original relation, then the relation is transitive.

Composition of two binary relations is computed as boolean matrix multiplication of the binary matrices that represent the relations. Let R and S be two binary relations on a set A with m elements.

The composition Q = R ∘ S is represented by a matrix with elements

qi,j = (ri □ s1,j) V (ri, □ s2,j) V ... V (ri,m □ sm,j),

Where ri,j are the elements of the matrix representing R, and si,j are the elements of the matrix representing S.

R = {(a,b),(b,c),(c,d), (a,c),(a,d),(b,d)} on the set A = {a,b,c,d}

The transitive closure is equal to the original relation, hence the original relation R is transitive.

Equivalence Relations: A relation R on set A is called an equivalence relation if it satisfies the following properties

1. Relation R is reflexive
2. Relation R is symmetric
3. Relation R is transitive

Partial Order Relations: A relation R on set A is called partial order relation if it satisfies the following properties:

1. Relation R is reflexive
2. Relation R is antisymmetric

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3. Relation R is transitive

Partial order set: The set A together with a partially order relation R on the set A and is denoted by (A, R) is called a partially ordered set or POSET.

Total order relation: Consider the relation R on the set A. If it is also the case that for all a,b € R, we have either (a,b) € or (b,a) € R or a=b, then the relation R is called total order relation on set A.

V. CONCLUSION

So, in this paper we threw light on not-so-uncommon concept associated with discrete structures, i.e. Relations. Domain, Range, Compliment of relations have been discussed in detail followed by examples. Apart from that, properties of relations, such as reflexive, irreflexiv, assymetric, etc. have been talked about. In the end, closure properties of relations such as transition closure, symmetric closure and many more have been discussed.

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