Bipolar L-Fuzzy ℓ -HX group and its Level sub ℓ -HX group

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Abstract: In this paper, we discussed some properties of bipolar L - fuzzy sub ℓ - HX group of a ℓ - HX group. We establish the relation between bipolar L - fuzzy sub ℓ - HX group and bipolar anti L - fuzzy sub ℓ - HX group . The purpose of this study is to implement the fuzzy set theory and graph theory in bipolar L - fuzzy sub ℓ - HX group. Characterizations of level subsets of a bipolar L - fuzzy sub ℓ - HX group are given. We also discussed the relation between a bipolar L - fuzzy sub ℓ - HX group and its level sub ℓ - HX groups and investigate the conditions under which a given sub ℓ - HX group has a properly inclusive chain of sub ℓ - HX groups. In particular, we formulate how to structure an bipolar L - fuzzy sub ℓ - HX group by a given chain of sub ℓ - HX groups.

Keywords: Bipolar L - fuzzy ℓ - HX group, Bipolar anti L - fuzzy ℓ - HX group, level subset, level sub ℓ-HX group.

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I. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh[9] . Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld[6] gave the idea of fuzzy subgroups. In fuzzy sets the membership degree of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Li Hongxing[3] introduced the concept of HX group and the authors Luo Chengzhong , Mi Honghai, Li Hongxing[4] introduced the concept of fuzzy HX group. The author W.R.Zhang[10] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. K.M. Lee[2] introduced Bipolar-valued fuzzy sets and their operations. In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [−1,0) indicates that elements somewhat satisfy the implicit counter-property. G.S.V.Satya Saibaba[7] initiated the study of L - fuzzy lattice ordered groups and introduced the notions of L - fuzzy sub ℓ - HX group. J.A Goguen[1] replaced the valuation set [0,1] by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L - Fuzzy sets. R.Muthuraj, M.Sridharan[5] introduced Bipolar fuzzy HX group and its level sub HX groups. The authors K.Sunderrajan, A.Senthilkumar, R.Muthuraj [8] introduced L - fuzzy sub ℓ -group and its level sub ℓ -groups.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G , .) is a group, e is the identity element of G, and xy, we mean x .y

A. *Definition* 2.1

Let μ be a bipolar L - fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a ℓ - HX group on G. A bipolar L - fuzzy set λ^μ defined on ϑ is said to be a bipolar L - fuzzy sub ℓ - HX group on ϑ if for all A,B$\in\vartheta$.

1) $(\lambda^\mu)^+ (AB) \geq (\lambda^\mu)^+ (A) \wedge (\lambda^\mu)^+ (B)$
2) $(\lambda^\mu)^- (AB) \leq (\lambda^\mu)^- (A) \vee (\lambda^\mu)^- (B)$
3) $(\lambda^\mu)^+ (A) = (\lambda^\mu)^+ (A^{-1})$
4) $(\lambda^\mu)^- (A) = (\lambda^\mu)^- (A^{-1})$
5) $(\lambda^\mu)^+ (A \vee B) \geq (\lambda^\mu)^+ (A) \wedge (\lambda^\mu)^+ (B)$

6) $(\lambda^{\mu})^{-}(A \vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

7) $(\lambda^{\mu})^{+}(A \wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

8) $(\lambda^{\mu})^{-}(A \wedge B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Where $(\lambda^{\mu})^{+}(A) = \max\{ \mu^{+}(x) \,/\, \text{for all } x \in A \subseteq G \}$

and

$(\lambda^{\mu})^{-}(A) = \min\{ \mu^{-}(x) \,/\, \text{for all } x \in A \subseteq G \}$

B. Example 2.1

Let $G=\{Z_5 - \{0\}, \cdot_5\}$ be a group and define a bipolar L- fuzzy set μ on G as $\mu^{+}(1) = 0.8$, $\mu^{+}(2) = 0.5$, $\mu^{+}(3) = 0.5$, $\mu^{+}(4) = 0.5$ and $\mu^{-}(1) = -0.7$, $\mu^{-}(2) = -0.6$, $\mu^{-}(3) = -0.6$, $\mu^{-}(4) = -0.6$

By routine computations, it is easy to see that μ is a bipolar L-fuzzy sub group of G.

Let $\vartheta = \{\{1, 4\}, \{2, 3\}\}$ be a ℓ - HX group of G.

Let us consider A = {1, 4}, B = {2, 3}.

$\cdot_5$	**A**	**B**
A	A	B
B	B	A

$\wedge$	**A**	**B**
A	A	A
B	A	B

$\vee$	**A**	**B**
A	A	B
B	B	B

Define $(\lambda^{\mu})^{+}(A) = \max\{\mu^{+}(x) \,/\, \text{for all } x \in A \subseteq G \}$

and

$(\lambda^{\mu})^{-}(A) = \min\{\mu^{-}(x) \,/\, \text{for all } x \in A \subseteq G \}$

Now

$(\lambda^{\mu})^{+}(A)=(\lambda^{\mu})^{+}(\{1,4\})=\max\{\mu^{+}(1),\mu^{+}(4)\}= \max\{0.8,0.5\} = 0.8$

$(\lambda^{\mu})^{+}(B)=(\lambda^{\mu})^{+}(\{2,3\})=\max\{\mu^{+}(2),\mu^{+}(3)\}= \max\{0.5,0.5\} = 0.5$

$(\lambda^{\mu})^{+}(AB) = (\lambda^{\mu})^{+}(B) = 0.5$

$(\lambda^{\mu})^{+}(A \wedge B) = (\lambda^{\mu})^{+}(A) = 0.8$

$(\lambda^{\mu})^{+}(A \vee B) = (\lambda^{\mu})^{+}(B) = 0.5$

$(\lambda^{\mu})^{-}(A)=(\lambda^{\mu})^{-}(\{1,4\})=\min\{\mu^{-}(1),\mu^{-}(4)\}=\min\{-0.7,-0.6\}= -0.7$

$(\lambda^{\mu})^{-}(B)=(\lambda^{\mu})^{-}(\{2,3\})=\min\{\mu^{-}(2),\mu^{-}(3)\}=\min\{-0.6,-0.6\}= -0.6$

$(\lambda^{\mu})^{-}(AB) = (\lambda^{\mu})^{-}(B) = -0.6$

$(\lambda^{\mu})^{-}(A \wedge B) = (\lambda^{\mu})^{-}(A) = -0.7$

$(\lambda^{\mu})^{-}(A \vee B) = (\lambda^{\mu})^{-}(B) = -0.6$

By routine computations, it is easy to see that λ^{μ} is a bipolar L-fuzzy sub ℓ - HX group of ϑ .

C. Definition 2.2

Let μ be a bipolar L - fuzzy subset defined on G. Let $\vartheta \subset 2^{G} - \{\phi\}$ be a ℓ - HX group on G. A bipolar L - fuzzy set λ^{μ} defined on ϑ is said to be a bipolar anti L - fuzzy sub ℓ - HX group on ϑ if for all $A,B \in \vartheta$.

1) $(\lambda^{\mu})^{+}(AB) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

2) $(\lambda^{\mu})^{-}(AB) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

3) $(\lambda^{\mu})^{+}(A) = (\lambda^{\mu})^{+}(A^{-1})$

4) $(\lambda^{\mu})^{-}(A) = (\lambda^{\mu})^{-}(A^{-1})$

5) $(\lambda^{\mu})^{+}(A \vee B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

6) $(\lambda^{\mu})^{-}(A \vee B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

7) $(\lambda^{\mu})^{+}(A \wedge B) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

8) $(\lambda^{\mu})^{-}(A \wedge B) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

Where $(\lambda^{\mu})^{+}(A) = \min\{ \mu^{+}(x) \,/\, \text{for all } x \in A \subseteq G \}$

and

$(\lambda^{\mu})^{-}(A) = \max\{ \mu^{-}(x) \,/\, \text{for all } x \in A \subseteq G \}$

D. Example 2.2

Let G={Z_5 - {0}, $._5$} be a group and define a bipolar L- fuzzy set μ on G as $\mu^+(1) = 0.4$, $\mu^+(2) = 0.8$, $\mu^+(3) = 0.8$, $\mu^+(4) = 0.8$ and $\mu^-(1) = -0.5$, $\mu^-(2) = -0.7$, $\mu^-(3) = -0.7$, $\mu^-(4) = -0.7$

By routine computations, it is easy to see that μ is a bipolar anti L-fuzzy subgroup of G.

Let ϑ = {{1, 4}, {2, 3}} be a ℓ - HX group of G.

Let us consider A = {1, 4}, B = {2, 3}.

$._5$	**A**	**B**
A	A	B
B	B	A

∧	**A**	**B**
A	A	A
B	A	B

∨	**A**	**B**
A	A	B
B	B	B

Define $(\lambda^\mu)^+(A) = \min\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$

and

$(\lambda^\mu)^-(A) = \max\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$

Now

$(\lambda^\mu)^+(A)=(\lambda^\mu)^+(\{1,4\})=\min\{\mu^+(1),\mu^+(4)\}=\min\{0.4,0.8\}= 0.4$

$(\lambda^\mu)^+(B)=(\lambda^\mu)^+(\{2,3\})=\min\{\mu^+(2),\mu^+(3)\}=\min\{0.8,0.8\}= 0.8$

$(\lambda^\mu)^+(AB) = (\lambda^\mu)^+(B) = 0.8$

$(\lambda^\mu)^+(A \wedge B) = (\lambda^\mu)^+(A) = 0.4$

$(\lambda^\mu)^+(A \vee B) = (\lambda^\mu)^+(B) = 0.8$

$(\lambda^\mu)^-(A)=(\lambda^\mu)^-(\{1,4\})=\max\{\mu^-(1),\mu^-(4)\}=\max\{-0.5,-0.7\}=- 0.5$

$(\lambda^\mu)^-(B)=(\lambda^\mu)^-(\{2,3\})=\max\{\mu^-(2),\mu^-(3)\}=\max\{-0.7,-0.7\}= - 0.7$

$(\lambda^\mu)^-(AB) = (\lambda^\mu)^-(B) = - 0.7$

$(\lambda^\mu)^-(A \wedge B) = (\lambda^\mu)^-(A) = - 0.5$

$(\lambda^\mu)^-(A \vee B) = (\lambda^\mu)^-(B) = - 0.7$

By routine computations, it is easy to see that λ^μ is a bipolar anti L-fuzzy sub ℓ - HX group of ϑ .

III. PROPERTIES OF BIPOLAR L-FUZZY SUB ℓ - HX GROUP

In this section, We discuss some of the properties of bipolar L-fuzzy sub ℓ - HX group

A. Definition 3.1

Let $\mu=(\mu^+,\mu^-)$ and $\alpha=(\alpha^+, \alpha^-)$ are bipolar L-fuzzy subsets of G. Let $\vartheta \subset 2^G - \{\phi\}$ be a ℓ - HX group of G. Let $\lambda^\mu= ((\lambda^\mu)^+ , (\lambda^\mu)^-)$ and $\eta^\alpha= ((\eta^\alpha)^+ , (\eta^\alpha)^-)$ are bipolar L-fuzzy subsets of ϑ. The union of λ^μ and η^α is $(\lambda^\mu \cup \eta^\alpha)= ((\lambda^\mu \cup \eta^\alpha)^+ , (\lambda^\mu \cup \eta^\alpha)^-)$ defined as

1) $(\lambda^\mu \cup \eta^\alpha)^+(A) = (\lambda^\mu)^+(A) \vee (\eta^\alpha)^+(A)$

2) $(\lambda^\mu \cup \eta^\alpha)^-(A) = (\lambda^\mu)^-(A) \wedge (\eta^\alpha)^-(A)$

Where $(\lambda^\mu)^+(A)= \max\{ \mu^+(x) \,/\, \text{for all } x\in A\subseteq G\}$

$(\lambda^\mu)^-(A)= \min\{ \mu^-(x) \,/\, \text{for all } x\in A\subseteq G\}$

$(\eta^\alpha)^+(A)= \max\{\alpha^+(x) \,/\, \text{for all } x\in A\subseteq G\}$

$(\eta^\alpha)^-(A)= \min\{\alpha^-(x) \,/\, \text{for all } x\in A\subseteq G\}$

B. Theorem 3.1

Let λ^μ and η^α be any two bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ then $\lambda^\mu \cup \eta^\alpha$ is a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ.

1) Proof

a)
$$\begin{aligned}((\lambda^\mu)\cup(\eta^\alpha))^+ (AB) &= (\lambda^\mu)^+(AB) \vee (\eta^\alpha)^+ (AB)\\ &\geq ((\lambda^\mu)^+(A) \wedge (\lambda^\mu)^+(B)) \vee ((\eta^\alpha)^+ (A) \wedge (\eta^\alpha)^+ (B))\\ &= ((\lambda^\mu)^+(A) \vee (\eta^\alpha)^+(A)) \wedge ((\lambda^\mu)^+(B) \vee (\eta^\alpha)^+ (B))\\ &= (\lambda^\mu\cup\eta^\alpha)^+ (A) \wedge (\lambda^\mu\cup\eta^\alpha)^+ (B)\end{aligned}$$

$\geq (\lambda^{\mu}\cup\eta^{\alpha})^{+} (A) \wedge (\lambda^{\mu}\cup\eta^{\alpha})^{+} (B)$

b) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{-} (AB) = (\lambda^{\mu})^{-}(AB) \wedge (\eta^{\alpha})^{-} (AB)$

$\leq ((\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)) \wedge ((\eta^{\alpha})^{-} (A) \vee (\eta^{\alpha})^{-} (B))$

$= ((\lambda^{\mu})^{-}(A) \wedge (\eta^{\alpha})^{-}(A)) \vee ((\lambda^{\mu})^{-}(B) \wedge (\eta^{\alpha})^{-} (B))$

$= (\lambda^{\mu}\cup\eta^{\alpha})^{-} (A) \vee (\lambda^{\mu}\cup\eta^{\alpha})^{-} (B)$

$\leq (\lambda^{\mu}\cup\eta^{\alpha})^{-} (A) \vee (\lambda^{\mu}\cup\eta^{\alpha})^{-} (B)$

c) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{+} (A^{-1}) = (\lambda^{\mu})^{+}(A^{-1}) \vee (\eta^{\alpha})^{+} (A^{-1})$

$= (\lambda^{\mu})^{+}(A) \vee (\eta^{\alpha})^{+} (A)$

$= ((\lambda^{\mu})\cup(\eta^{\alpha}))^{+} (A)$

d) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{-} (A^{-1}) = (\lambda^{\mu})^{-}(A^{-1}) \wedge (\eta^{\alpha})^{-} (A^{-1})$

$= (\lambda^{\mu})^{-}(A) \wedge (\eta^{\alpha})^{-} (A)$

$= ((\lambda^{\mu})\cup(\eta^{\alpha}))^{-} (A)$

e) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{+} (A\vee B) = (\lambda^{\mu})^{+}(A\vee B) \vee (\eta^{\alpha})^{+} (A\vee B)$

$\geq ((\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)) \vee ((\eta^{\alpha})^{+} (A) \wedge (\eta^{\alpha})^{+} (B))$

$= ((\lambda^{\mu})^{+}(A) \vee (\eta^{\alpha})^{+}(A)) \wedge ((\lambda^{\mu})^{+}(B) \vee (\eta^{\alpha})^{+} (B))$

$= (\lambda^{\mu}\cup\eta^{\alpha})^{+} (A) \wedge (\lambda^{\mu}\cup\eta^{\alpha})^{+} (B)$

$\geq (\lambda^{\mu}\cup\eta^{\alpha})^{+} (A) \wedge (\lambda^{\mu}\cup\eta^{\alpha})^{+} (B)$

f) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{-} (A\vee B) = (\lambda^{\mu})^{-}(A\vee B) \wedge (\eta^{\alpha})^{-} (A\vee B)$

$\leq ((\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)) \wedge ((\eta^{\alpha})^{-} (A) \vee (\eta^{\alpha})^{-} (B))$

$= ((\lambda^{\mu})^{-}(A) \wedge (\eta^{\alpha})^{-}(A)) \vee ((\lambda^{\mu})^{-}(B) \wedge (\eta^{\alpha})^{-} (B))$

$= (\lambda^{\mu}\cup\eta^{\alpha})^{-} (A) \vee (\lambda^{\mu}\cup\eta^{\alpha})^{-} (B)$

$\leq (\lambda^{\mu}\cup\eta^{\alpha})^{-} (A) \vee (\lambda^{\mu}\cup\eta^{\alpha})^{-} (B)$

g) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{+} (A\wedge B) = (\lambda^{\mu})^{+}(A\wedge B) \vee (\eta^{\alpha})^{+} (A\wedge B)$

$\geq ((\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)) \vee ((\eta^{\alpha})^{+} (A) \wedge (\eta^{\alpha})^{+} (B))$

$= ((\lambda^{\mu})^{+}(A) \vee (\eta^{\alpha})^{+}(A)) \wedge ((\lambda^{\mu})^{+}(B) \vee (\eta^{\alpha})^{+} (B))$

$= (\lambda^{\mu}\cup\eta^{\alpha})^{+} (A) \wedge (\lambda^{\mu}\cup\eta^{\alpha})^{+} (B)$

$\geq (\lambda^{\mu}\cup\eta^{\alpha})^{+} (A) \wedge (\lambda^{\mu}\cup\eta^{\alpha})^{+} (B)$

h) $((\lambda^{\mu})\cup(\eta^{\alpha}))^{-} (A\wedge B) = (\lambda^{\mu})^{-}(A\wedge B) \wedge (\eta^{\alpha})^{-} (A\wedge B)$

$\leq ((\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)) \wedge ((\eta^{\alpha})^{-} (A) \vee (\eta^{\alpha})^{-} (B))$

$= ((\lambda^{\mu})^{-}(A) \wedge (\eta^{\alpha})^{-}(A)) \vee ((\lambda^{\mu})^{-}(B) \wedge (\eta^{\alpha})^{-} (B))$

$= (\lambda^{\mu}\cup\eta^{\alpha})^{-} (A) \vee (\lambda^{\mu}\cup\eta^{\alpha})^{-} (B)$

$\leq (\lambda^{\mu}\cup\eta^{\alpha})^{-} (A) \vee (\lambda^{\mu}\cup\eta^{\alpha})^{-} (B)$

Hence $\lambda^{\mu} \cup \eta^{\alpha}$ is a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ.

C. Definition 3.2

Let $\mu=(\mu^{+},\mu^{-})$ and $\alpha=(\alpha^{+}, \alpha^{-})$ are bipolar L-fuzzy subsets of G. Let $\vartheta\subset 2^{G} - \{\phi\}$ be a ℓ - HX group of G. Let $\lambda^{\mu}= ((\lambda^{\mu})^{+} , (\lambda^{\mu})^{-})$ and $\eta^{\alpha}= ((\eta^{\alpha})^{+} , (\eta^{\alpha})^{-})$ are bipolar L-fuzzy subsets of ϑ. The intersection of λ^{μ} and η^{α} is $(\lambda^{\mu} \cap \eta^{\alpha})= ((\lambda^{\mu}\cap \eta^{\alpha})^{+} , (\lambda^{\mu} \cap \eta^{\alpha})^{-})$ defined as

$$(\lambda^{\mu} \cap \eta^{\alpha})^{+}(A) = (\lambda^{\mu})^{+}(A) \wedge (\eta^{\alpha})^{+}(A)$$

$$(\lambda^{\mu} \cap \eta^{\alpha})^{-}(A) = (\lambda^{\mu})^{-}(A) \vee (\eta^{\alpha})^{-}(A)$$

Where $(\lambda^{\mu})^{+}(A)= \max\{ \mu^{+}(x) \ / \text{ for all } x\in A\subseteq G\}$

$(\lambda^{\mu})^{-}(A)= \min\{ \mu^{-}(x) \ / \text{ for all } x\in A\subseteq G\}$

$(\eta^{\alpha})^{+}(A)= \max\{\alpha^{+}(x) \ / \text{ for all } x\in A\subseteq G\}$

$(\eta^{\alpha})^{-}(A)= \min\{\alpha^{-}(x) \ / \text{ for all } x\in A\subseteq G\}$

D. Theorem 3.2

Let λ^{μ} and η^{α} be any two bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ then $\lambda^{\mu} \cap \eta^{\alpha}$ is a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ.

1) *Proof*

a) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{+}(AB) = (\lambda^{\mu})^{+}(AB) \wedge (\eta^{\alpha})^{+}(AB)$
$\geq ((\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)) \wedge ((\eta^{\alpha})^{+}(A) \wedge (\eta^{\alpha})^{+}(B))$
$= ((\lambda^{\mu})^{+}(A) \wedge (\eta^{\alpha})^{+}(A)) \wedge ((\lambda^{\mu})^{+}(B) \wedge (\eta^{\alpha})^{+}(B))$
$= (\lambda^{\mu}\cap\eta^{\alpha})^{+}(A) \wedge (\lambda^{\mu}\cap\eta^{\alpha})^{+}(B)$
$\geq (\lambda^{\mu}\cap\eta^{\alpha})^{+}(A) \wedge (\lambda^{\mu}\cap\eta^{\alpha})^{+}(B)$

b) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{-}(AB) = (\lambda^{\mu})^{-}(AB) \vee (\eta^{\alpha})^{-}(AB)$
$\leq ((\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)) \vee ((\eta^{\alpha})^{-}(A) \vee (\eta^{\alpha})^{-}(B))$
$= ((\lambda^{\mu})^{-}(A) \vee (\eta^{\alpha})^{-}(A)) \vee ((\lambda^{\mu})^{-}(B) \vee (\eta^{\alpha})^{-}(B))$
$= (\lambda^{\mu}\cap\eta^{\alpha})^{-}(A) \vee (\lambda^{\mu}\cap\eta^{\alpha})^{-}(B)$
$\leq (\lambda^{\mu}\cap\eta^{\alpha})^{-}(A) \vee (\lambda^{\mu}\cap\eta^{\alpha})^{-}(B)$

c) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{+}(A^{-1}) = (\lambda^{\mu})^{+}(A^{-1}) \wedge (\eta^{\alpha})^{+}(A^{-1})$
$= (\lambda^{\mu})^{+}(A) \wedge (\eta^{\alpha})^{+}(A)$
$= ((\lambda^{\mu})\cap(\eta^{\alpha}))^{+}(A)$

d) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{-}(A^{-1}) = (\lambda^{\mu})^{-}(A^{-1}) \vee (\eta^{\alpha})^{-}(A^{-1})$
$= (\lambda^{\mu})^{-}(A) \vee (\eta^{\alpha})^{-}(A)$
$= ((\lambda^{\mu})\cap(\eta^{\alpha}))^{-}(A)$

e) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{+}(A\vee B) = (\lambda^{\mu})^{+}(A\vee B) \wedge (\eta^{\alpha})^{+}(A\vee B)$
$\geq ((\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)) \wedge ((\eta^{\alpha})^{+}(A) \wedge (\eta^{\alpha})^{+}(B))$
$= ((\lambda^{\mu})^{+}(A) \wedge (\eta^{\alpha})^{+}(A)) \wedge ((\lambda^{\mu})^{+}(B) \wedge (\eta^{\alpha})^{+}(B))$
$= (\lambda^{\mu}\cap\eta^{\alpha})^{+}(A) \wedge (\lambda^{\mu}\cap\eta^{\alpha})^{+}(B)$
$\geq (\lambda^{\mu}\cap\eta^{\alpha})^{+}(A) \wedge (\lambda^{\mu}\cap\eta^{\alpha})^{+}(B)$

f) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{-}(A\vee B) = (\lambda^{\mu})^{-}(A\vee B) \vee (\eta^{\alpha})^{-}(A\vee B)$
$\leq ((\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)) \vee ((\eta^{\alpha})^{-}(A) \vee (\eta^{\alpha})^{-}(B))$
$= ((\lambda^{\mu})^{-}(A) \vee (\eta^{\alpha})^{-}(A)) \vee ((\lambda^{\mu})^{-}(B) \vee (\eta^{\alpha})^{-}(B))$
$= (\lambda^{\mu}\cap\eta^{\alpha})^{-}(A) \vee (\lambda^{\mu}\cap\eta^{\alpha})^{-}(B)$
$\leq (\lambda^{\mu}\cap\eta^{\alpha})^{-}(A) \vee (\lambda^{\mu}\cap\eta^{\alpha})^{-}(B)$

g) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{+}(A\wedge B) = (\lambda^{\mu})^{+}(A\wedge B) \wedge (\eta^{\alpha})^{+}(A\wedge B)$
$\geq ((\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)) \wedge ((\eta^{\alpha})^{+}(A) \wedge (\eta^{\alpha})^{+}(B))$
$= ((\lambda^{\mu})^{+}(A) \wedge (\eta^{\alpha})^{+}(A)) \wedge ((\lambda^{\mu})^{+}(B) \wedge (\eta^{\alpha})^{+}(B))$
$= (\lambda^{\mu}\cap\eta^{\alpha})^{+}(A) \wedge (\lambda^{\mu}\cap\eta^{\alpha})^{+}(B)$
$\geq (\lambda^{\mu}\cap\eta^{\alpha})^{+}(A) \wedge (\lambda^{\mu}\cap\eta^{\alpha})^{+}(B)$

h) $((\lambda^{\mu})\cap(\eta^{\alpha}))^{-}(A\wedge B) = (\lambda^{\mu})^{-}(A\wedge B) \vee (\eta^{\alpha})^{-}(A\wedge B)$
$\leq ((\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)) \vee ((\eta^{\alpha})^{-}(A) \vee (\eta^{\alpha})^{-}(B))$
$= ((\lambda^{\mu})^{-}(A) \vee (\eta^{\alpha})^{-}(A)) \vee ((\lambda^{\mu})^{-}(B) \vee (\eta^{\alpha})^{-}(B))$
$= (\lambda^{\mu}\cap\eta^{\alpha})^{-}(A) \vee (\lambda^{\mu}\cap\eta^{\alpha})^{-}(B)$
$\leq (\lambda^{\mu}\cap\eta^{\alpha})^{-}(A) \vee (\lambda^{\mu}\cap\eta^{\alpha})^{-}(B)$

Hence $\lambda^{\mu} \cap \eta^{\alpha}$ is a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ.

E. Theorem 3.3

Let λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ if and only if $(\lambda^{\mu})^{C}$ is a bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ.

1) *Proof:* Let λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ if for A, B $\in\vartheta$, we have

a) $(\lambda^{\mu})^{+}(AB) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$
$\Leftrightarrow 1-((\lambda^{\mu})^{+})^{C}(AB) \geq (1-((\lambda^{\mu})^{+})^{C}(A)) \wedge (1-((\lambda^{\mu})^{+})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(AB) \leq 1-((1-((\lambda^{\mu})^{+})^{C}(A)) \wedge (1-((\lambda^{\mu})^{+})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(AB) \leq (\lambda^{\mu})^{+}(A) \vee (\lambda^{\mu})^{+}(B)$

b) $(\lambda^{\mu})^{-}(AB) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

$\Leftrightarrow 1-((\lambda^{\mu})^{-})^{C}(AB) \leq (1-((\lambda^{\mu})^{-})^{C}(A)) \vee (1-((\lambda^{\mu})^{-})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(AB) \geq 1-((1-((\lambda^{\mu})^{-})^{C}(A)) \vee (1-((\lambda^{\mu})^{-})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(AB) \geq (\lambda^{\mu})^{-}(A) \wedge (\lambda^{\mu})^{-}(B)$

c) $(\lambda^{\mu})^{+}(A^{-1}) = (\lambda^{\mu})^{+}(A)$
$\Leftrightarrow 1-((\lambda^{\mu})^{+})^{C}(A^{-1}) = 1-((\lambda^{\mu})^{+})^{C}(A)$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(A^{-1}) = ((\lambda^{\mu})^{+})^{C}(A)$

d) $(\lambda^{\mu})^{-}(A^{-1}) = (\lambda^{\mu})^{-}(A)$
$\Leftrightarrow 1-((\lambda^{\mu})^{-})^{C}(A^{-1}) = 1-((\lambda^{\mu})^{-})^{C}(A)$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(A^{-1}) = ((\lambda^{\mu})^{-})^{C}(A)$

e) $(\lambda^{\mu})^{+}(A\vee B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$
$\Leftrightarrow 1-((\lambda^{\mu})^{+})^{C}(A\vee B) \geq (1-((\lambda^{\mu})^{+})^{C}(A)) \wedge (1-((\lambda^{\mu})^{+})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(A\vee B) \leq 1-((1-((\lambda^{\mu})^{+})^{C}(A)) \wedge (1-((\lambda^{\mu})^{+})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(A\vee B) \leq ((\lambda^{\mu})^{+})^{C}(A) \vee ((\lambda^{\mu})^{+})^{C}(B)$

f) $(\lambda^{\mu})^{-}(A\vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$
$\Leftrightarrow 1-((\lambda^{\mu})^{-})^{C}(A\vee B) \leq (1-((\lambda^{\mu})^{-})^{C}(A)) \vee (1-((\lambda^{\mu})^{-})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(A\vee B) \geq 1-((1-((\lambda^{\mu})^{-})^{C}(A)) \vee (1-((\lambda^{\mu})^{-})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(A\vee B) \geq ((\lambda^{\mu})^{-})^{C}(A) \wedge ((\lambda^{\mu})^{-})^{C}(B)$

g) $(\lambda^{\mu})^{+}(A\wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$
$\Leftrightarrow 1-((\lambda^{\mu})^{+})^{C}(A\wedge B) \geq (1-((\lambda^{\mu})^{+})^{C}(A)) \wedge (1-((\lambda^{\mu})^{+})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(A\wedge B) \leq 1-((1-((\lambda^{\mu})^{+})^{C}(A)) \wedge (1-((\lambda^{\mu})^{+})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{+})^{C}(A\wedge B) \leq ((\lambda^{\mu})^{+})^{C}(A) \vee ((\lambda^{\mu})^{+})^{C}(B)$

h) $(\lambda^{\mu})^{-}(A\wedge B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$
$\Leftrightarrow 1-((\lambda^{\mu})^{-})^{C}(A\wedge B) \leq (1-((\lambda^{\mu})^{-})^{C}(A)) \vee (1-((\lambda^{\mu})^{-})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(A\wedge B) \geq 1-((1-((\lambda^{\mu})^{-})^{C}(A)) \vee (1-((\lambda^{\mu})^{-})^{C}(B))$
$\Leftrightarrow ((\lambda^{\mu})^{-})^{C}(A\wedge B) \geq ((\lambda^{\mu})^{-})^{C}(A) \wedge ((\lambda^{\mu})^{-})^{C}(B)$

Hence $(\lambda^{\mu})^{C}$ is a Bipolar anti L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ.

IV. PROPERTIES OF LEVEL SUBSETS OF A BIPOLAR L-FUZZY SUB ℓ - HX GROUP

In this section, We introduce the concept of level subsets of a bipolar L-fuzzy sub ℓ - HX group and discuss some of its properties.

A. *Definition 4.1*

Let λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ. For any $<\alpha, \beta> \in [0,1] \times [-1,0]$, We define the set $\lambda^{\mu}_{<\alpha,\beta>} = \{ A \in \vartheta / (\lambda^{\mu})^{+}(A) \geq \alpha$ and $(\lambda^{\mu})^{-}(B) \leq \beta \}$ is called the $<\alpha, \beta>$ level subset of λ^{μ} or simply the level subset of λ^{μ}.

B. *Theorem 4.1*

Let λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ then for $<\alpha, \beta> \in [0,1] \times [-1,0]$ such that $(\lambda^{\mu})^{+}(E) \geq \alpha$, $(\lambda^{\mu})^{-}(E) \leq \beta$ and $\lambda^{\mu}_{<\alpha,\beta>}$ is a sub ℓ - HX group of ϑ.

1) *Proof :* For all $A,B \in \lambda^{\mu}_{<\alpha,\beta>}$ we have $(\lambda^{\mu})^{+}(A) \geq \alpha$, $(\lambda^{\mu})^{-}(A) \leq \beta$ and $(\lambda^{\mu})^{+}(B) \geq \alpha$, $(\lambda^{\mu})^{-}(B) \leq \beta$

Now $(\lambda^{\mu})^{+}(AB^{-1}) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$
$\geq \alpha \wedge \alpha$
$= \alpha$

$\Rightarrow (\lambda^{\mu})^{+}(AB^{-1}) \geq \alpha$
$(\lambda^{\mu})^{-}(AB^{-1}) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(\beta)$
$\leq \beta \vee \beta$
$= \beta$

$\Rightarrow (\lambda^{\mu})^{-}(AB^{-1}) \leq \beta$

Hence, $AB^{-1} \in \lambda^{\mu}_{<\alpha,\beta>}$

Now $(\lambda^{\mu})^{+}(A\vee B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$
$\geq \alpha \wedge \alpha$

$= \alpha$

$\Rightarrow \quad (\lambda^{\mu})^{+}(A\vee B) \geq \alpha$

$(\lambda^{\mu})^{-}(A\vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

$\leq \beta \vee \beta$

$= \beta$

$\Rightarrow \quad (\lambda^{\mu})^{-}(A\vee B) \leq \beta$

Hence, $A\vee B \in \lambda^{\mu}_{<\alpha,\beta>}$

Now $(\lambda^{\mu})^{+}(A\wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

$\geq \alpha \wedge \alpha$

$= \alpha$

$\Rightarrow \quad (\lambda^{\mu})^{+}(A\wedge B) \geq \alpha$

$(\lambda^{\mu})^{-}(A\wedge B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

$\leq \beta \vee \beta$

$= \beta$

$\Rightarrow \quad (\lambda^{\mu})^{-}(A\wedge B) \leq \beta$

Hence, $A\wedge B \in \lambda^{\mu}_{<\alpha,\beta>}$

Hence, $\lambda^{\mu}_{<\alpha,\beta>}$ is a sub ℓ - HX group of ϑ.

C. Definition 4.2

Let λ^{μ} is a bipolar L-fuzzy sub ℓ - HX group of a ℓ - HX group ϑ. The sub ℓ - HX groups $\lambda^{\mu}_{<\alpha,\beta>}$ for $<\alpha, \beta> \in [0,1] \times [-1,0]$ and $(\lambda^{\mu})^{+}(E) \geq \alpha$, $(\lambda^{\mu})^{-}(E) \leq \beta$ are called level sub ℓ - HX groups of λ^{μ}.

D. Theorem 4.2

Let ϑ be a ℓ - HX group and λ^{μ} be a bipolar L-fuzzy subset of ϑ such that $\lambda^{\mu}_{<\alpha,\beta>}$ is a sub ℓ - HX group of ϑ for $<\alpha, \beta> \in [0,1] \times [-1,0]$ such that $(\lambda^{\mu})^{+}(E) \geq \alpha$, $(\lambda^{\mu})^{-}(E) \leq \beta$ .Then λ^{μ} is a bipolar L-fuzzy sub ℓ - HX group of ϑ.

1) Proof: Let $A,B \in \vartheta$, let $A \in \lambda^{\mu}_{<\alpha 1, \beta 1>} \Rightarrow (\lambda^{\mu})^{+}(A) = \alpha_1$, $(\lambda^{\mu})^{-}(A) = \beta_1$ and $B \in \lambda^{\mu}_{<\alpha 2, \beta 2>} \Rightarrow (\lambda^{\mu})^{+}(B) = \alpha_2$, $(\lambda^{\mu})^{-}(B) = \beta_2$

Suppose $\lambda^{\mu}_{<\alpha 1, \beta 1>} < \lambda^{\mu}_{<\alpha 2, \beta 2>}$ then $A,B \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$ As $\lambda^{\mu}_{<\alpha 2, \beta 2>}$ is a sub ℓ - HX group of ϑ, $AB^{-1} \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$,

$A\wedge B \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$ and $A\vee B \in \lambda^{\mu}_{<\alpha 2, \beta 2>}$,

Now $(\lambda^{\mu})^{+}(AB^{-1}) \geq \alpha_2$

$= \alpha_1 \wedge \alpha_2$

$= (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) \leq \beta_2$

$= \beta_1 \vee \beta_2$

$= (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\vee B) \geq \alpha_2$

$= \alpha_1 \wedge \alpha_2$

$= (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\vee B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B) \leq \beta_2$

$= \beta_1 \vee \beta_2$

$= (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B) \geq \alpha_2$

$= \alpha_1 \wedge \alpha_2$

$= (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B) \leq \beta_2$

$= \beta_1 \vee \beta_2$

$=(\lambda^{\mu})^{-} (A) \vee (\lambda^{\mu})^{-} (B)$

Hence, $(\lambda^{\mu})^{-}(A \wedge B) \leq (\lambda^{\mu})^{-} (A) \vee (\lambda^{\mu})^{-} (B)$

Hence λ^{μ} is a bipolar L-fuzzy sub ℓ - HX group of ϑ.

E. Theorem 4.3

Let ϑ be a ℓ - HX group and λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of ϑ. If two bipolar level sub ℓ - HX groups $\lambda^{\mu}_{<\alpha, \gamma>}$, $\lambda^{\mu}_{<\beta, \delta>}$ with $\alpha<\beta$ and $\delta<\gamma$ of λ^{μ} are equal if and only if there is no $A \in \vartheta$ such that $\alpha \leq (\lambda^{\mu})^{+} (A)<\beta$ and $\delta<(\lambda^{\mu})^{-} (A) \leq \gamma$

1) Proof: Let $\lambda^{\mu}_{<\alpha, \gamma>} = \lambda^{\mu}_{<\beta, \delta>}$. Suppose that there exists $A \in \vartheta$ such that $\alpha \leq (\lambda^{\mu})^{+} (A)<\beta$ and $\delta<(\lambda^{\mu})^{-} (A) \leq \gamma$

Then $\lambda^{\mu}_{<\beta, \delta>} \subset \lambda^{\mu}_{<\alpha, \gamma>}$ since $A \in \lambda^{\mu}_{<\alpha, \gamma>}$ but not in $\lambda^{\mu}_{<\beta, \delta>}$ which contradicts the hypothesis. Hence there exists no $A \in \vartheta$ such that $\alpha \leq (\lambda^{\mu})^{+} (A)<\beta$ and $\delta<(\lambda^{\mu})^{-} (A) \leq \gamma$

Conversely, let there be no $A \in \vartheta$ such that $\alpha \leq (\lambda^{\mu})^{+} (A)<\beta$ and $\delta<(\lambda^{\mu})^{-} (A) \leq \gamma$

Since $\alpha<\beta$ and $\delta<\gamma$ of λ^{μ} we have $\lambda^{\mu}_{<\beta, \delta>} \subset \lambda^{\mu}_{<\alpha, \gamma>}$

Let $A \in \lambda^{\mu}_{<\alpha, \gamma>}$ then $(\lambda^{\mu})^{+} (A) \geq \alpha$ and $(\lambda^{\mu})^{-} (A) \leq \gamma$ since there exists no $A \in \vartheta$ such that $\alpha \leq (\lambda^{\mu})^{+} (A)<\beta$ and $\delta<(\lambda^{\mu})^{-} (A) \leq \gamma$, we have $(\lambda^{\mu})^{+} (A) \geq \beta$ and $(\lambda^{\mu})^{-} (A) \leq \delta$ which implies $A \in \lambda^{\mu}_{<\beta, \delta>}$ that is $\lambda^{\mu}_{<\alpha, \gamma>} \subset \lambda^{\mu}_{<\beta, \delta>}$ Hence $\lambda^{\mu}_{<\alpha, \gamma>} = \lambda^{\mu}_{<\beta, \delta>}$.

F. Theorem 4.4

A L-fuzzy subset λ^{μ} of ϑ is a bipolar L-fuzzy sub ℓ - HX group of ϑ if and only if the level subsets $\lambda^{\mu}_{<\alpha ,\beta>}$, $<\alpha, \beta> \in$ Image λ^{μ} are sub ℓ - HX group of ϑ.

1) Proof: It is clear.

G. Theorem 4.5

Any sub ℓ - HX group H of a ℓ - HX group ϑ can be realized as a level sub ℓ - HX group of some bipolar L-fuzzy sub ℓ - HX group of ϑ.

1) Proof.: Let $\lambda^{\mu} = ((\lambda^{\mu})^{+}, (\lambda^{\mu})^{-})$ be a bipolar L-fuzzy subset and $A \in \vartheta$,

2) Define $(\lambda^{\mu})^{+} (A)= \begin{cases} \alpha , \text{ if } A \in H \\ 0 , \text{ if } A \notin H \end{cases}$ and

$$(\lambda^{\mu})^{-} (A)= \begin{cases} 0 , \text{ if } A \in H \\ \beta , \text{ if } A \notin H \end{cases}$$

we shall prove λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of ϑ.

Let $A, B \in \vartheta$

i) Suppose $A, B \in H$ then $AB \in H$, $AB^{-1} \in H$, $A \wedge B \in H$ and $A \vee B \in H$

$(\lambda^{\mu})^{+} (A)= (\lambda^{\mu})^{+} (B)= \alpha$ and $(\lambda^{\mu})^{-} (A)= (\lambda^{\mu})^{-} (B)=0$

Now $(\lambda^{\mu})^{+} (AB^{-1}) = \alpha$

$\geq \alpha \wedge \alpha$

$=(\lambda^{\mu})^{+} (A) \wedge (\lambda^{\mu})^{+} (B)$

Hence, $(\lambda^{\mu})^{+} (AB^{-1}) \geq (\lambda^{\mu})^{+} (A) \wedge (\lambda^{\mu})^{+} (B)$

Now $(\lambda^{\mu})^{-} (AB^{-1}) = 0$

$\leq 0 \vee 0$

$=(\lambda^{\mu})^{-} (A) \vee (\lambda^{\mu})^{-} (B)$

Hence, $(\lambda^{\mu})^{-} (AB^{-1}) \leq (\lambda^{\mu})^{-} (A) \vee (\lambda^{\mu})^{-} (B)$

Now $(\lambda^{\mu})^{+} (A \vee B) = \alpha$

$\geq \alpha \wedge \alpha$

$=(\lambda^{\mu})^{+} (A) \wedge (\lambda^{\mu})^{+} (B)$

Hence, $(\lambda^{\mu})^{+} (A \vee B) \geq (\lambda^{\mu})^{+} (A) \wedge (\lambda^{\mu})^{+} (B)$

Now $(\lambda^{\mu})^{-} (A \vee B) = 0$

$\leq 0 \vee 0$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B) = \alpha$

$\geq \alpha \wedge \alpha$

$=(\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B) = 0$

$\leq 0 \vee 0$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\wedge B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Suppose $A\in H$, $B\notin H$ then $AB\notin H$, $AB^{-1}\notin H$, $A\wedge B\in H$ or $A\wedge B\notin H$ and $A\vee B\in H$ or $A\vee B\notin H$

$(\lambda^{\mu})^{+}(A)=\alpha$, $(\lambda^{\mu})^{+}(B)=0$ and $(\lambda^{\mu})^{-}(A)=0$, $(\lambda^{\mu})^{-}(B)=\beta$

3) Define

a) $(\lambda^{\mu})^{+}(A\wedge B)= \begin{cases} \alpha, & \text{if } A\wedge B \in H \\ 0, & \text{if } A\wedge B \notin H \end{cases}$ and

$(\lambda^{\mu})^{-}(A\wedge B)= \begin{cases} 0, & \text{if } A\wedge B \in H \\ \beta, & \text{if } A\wedge B \notin H \end{cases}$

b) $(\lambda^{\mu})^{+}(A\vee B)= \begin{cases} \alpha, & \text{if } A\vee B \in H \\ 0, & \text{if } A\vee B \notin H \end{cases}$ and

$(\lambda^{\mu})^{-}(A\vee B)= \begin{cases} 0, & \text{if } A\vee B \in H \\ \beta, & \text{if } A\vee B \notin H \end{cases}$

Let $A\wedge B\in H$ and $A\vee B\in H$

Now $(\lambda^{\mu})^{+}(AB^{-1}) = \alpha$

$\geq \alpha \wedge 0$

$=(\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1}) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1}) = \beta$

$\leq 0 \vee \beta$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1}) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\vee B) = \alpha$

$\geq \alpha \wedge 0$

$=(\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\vee B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B) = 0$

$\leq 0 \vee \beta$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B) = \alpha$

$\geq \alpha \wedge 0$

$=(\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B) = 0$

$\leq 0 \vee \beta$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\wedge B)\leq(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Let $A\wedge B\notin H$ and $A\vee B\notin H$

Now $(\lambda^{\mu})^{+}(AB^{-1})=\alpha$

$\geq \alpha\wedge 0$

$=(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1})\geq(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1})=\beta$

$\leq 0\vee\beta$

$=(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1})\leq(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\vee B)=0$

$\geq \alpha\wedge 0$

$=(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\vee B)\geq(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B)=\beta$

$\leq 0\vee\beta$

$=(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\vee B)\leq(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\wedge B)=0$

$\geq \alpha\wedge 0$

$=(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\wedge B)\geq(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\wedge B)=\beta$

$\leq 0\vee\beta$

$=(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A\wedge B)\leq(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

iii)suppose $A,B\notin H$ then $AB^{-1}\in H$ or $AB^{-1}\notin H$, $A\wedge B\notin H$ and $A\vee B\notin H$

$(\lambda^{\mu})^{+}(A)=(\lambda^{\mu})^{+}(B)=0$, $(\lambda^{\mu})^{-}(A)=(\lambda^{\mu})^{-}(B)=\beta$, $(\lambda^{\mu})^{+}(A\wedge B)=(\lambda^{\mu})^{+}(A\vee B)=0$ and $(\lambda^{\mu})^{-}(A\wedge B)=(\lambda^{\mu})^{-}(A\vee B)=\beta$

$$(\lambda^{\mu})^{+}(AB^{-1})=\begin{cases}\alpha, & \text{if } AB^{-1}\in H\\ 0, & \text{if } AB^{-1}\notin H\end{cases}$$ and

$$(\lambda^{\mu})^{-}(AB^{-1})=\begin{cases}0, & \text{if } AB^{-1}\in H\\ \beta, & \text{if } AB^{-1}\notin H\end{cases}$$

Let $AB^{-1}\notin H$

Now $(\lambda^{\mu})^{+}(AB^{-1})=0$

$\geq 0\wedge 0$

$=(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(AB^{-1})\geq(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(AB^{-1})=\beta$

$\leq\beta\vee\beta$

$=(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(AB^{-1})\leq(\lambda^{\mu})^{-}(A)\vee(\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A\vee B)=0$

$\geq 0\wedge 0$

$=(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A\vee B)\geq(\lambda^{\mu})^{+}(A)\wedge(\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A\vee B)=\beta$

$\leq\beta\vee\beta$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A \vee B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Now $(\lambda^{\mu})^{+}(A \wedge B) = 0$

$\geq 0 \wedge 0$

$=(\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Hence, $(\lambda^{\mu})^{+}(A \wedge B) \geq (\lambda^{\mu})^{+}(A) \wedge (\lambda^{\mu})^{+}(B)$

Now $(\lambda^{\mu})^{-}(A \wedge B) = \beta$

$\leq \beta \vee \beta$

$=(\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Hence, $(\lambda^{\mu})^{-}(A \wedge B) \leq (\lambda^{\mu})^{-}(A) \vee (\lambda^{\mu})^{-}(B)$

Thus in all cases, λ^{μ} be a bipolar L-fuzzy sub ℓ - HX group of ϑ. For this bipolar L-fuzzy sub ℓ - HX group, $\lambda^{\mu}_{<\alpha,\beta>} = H$

4) *Remark:* As a Consequence of the Theorem 4.3, Theorem 4.4 the level sub ℓ - HX groups of a bipolar L-fuzzy sub ℓ - HX group λ^{μ} of a ℓ - HX group ϑ form a chain. Since $(\lambda^{\mu})^{+}(E) \geq (\lambda^{\mu})^{+}(A)$ and $(\lambda^{\mu})^{-}(E) \leq (\lambda^{\mu})^{-}(A)$ for all A in ϑ

Therefore, $\lambda^{\mu}_{<\alpha 0,\beta 0>}$, $\alpha \in [0,1]$ and $\beta \in [-1,0]$

Where $(\lambda^{\mu})^{+}(E) = \alpha_0$, $(\lambda^{\mu})^{-}(E) = \beta_0$ is the smallest sub ℓ - HX group and

we have the chain $\{E\} \subseteq \lambda^{\mu}_{<\alpha 0,\beta 0>} \subseteq \lambda^{\mu}_{<\alpha 1,\beta 1>} \subseteq \lambda^{\mu}_{<\alpha 2,\beta 2>} \subseteq \ldots\ldots \subseteq \lambda^{\mu}_{<\alpha n,\beta n>} = \vartheta$,

where $\alpha_0 > \alpha_1 > \alpha_2 > \ldots\ldots > \alpha_n$ and $\beta_0 < \beta_1 < \beta_2 < \ldots\ldots < \beta_n$

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