A Mathematical Model on Plant- Soil Interactions Using Difference Equations

Elizabeth Sebastian[1] ,Preethi Victor [2]

[1,2] *Department of Mathematics, Auxilium College (Autonomous), Vellore, Tamil Nadu, India.*

Abstract: In this paper, we construct a mathematical model on plant soil interactions using Difference equations. We assume that the growth rate of the plant species is dependent on the density of the water concentration in the soil. The connection between the concentration of water and the rate of uptake by the plant is described by Michaelis–Menten kinetics. We list the equilibrium points and analyze the stability of the model. We analyze the growth of the prosopis juliflora (Seemakaruvelam Tree) in this paper and provide the numerical example.

Keywords: Difference equations, Local stability, Global stability, Plant species, Nutrient concentration.

I. INTRODUCTION

Plants grow in the thin upper layer of the Earth's crust known as soil. Plants need water, carbon dioxide and a range of trace minerals known as 'nutrients' to grow. They obtain these nutrients, and most of their water, from the soil. Soil is formed over very long times from igneous or sedimentary rock, volcanic ash, sand or peat. It is a highly complex and heterogeneous system with many different components that provide plants with water and nutrients. Nutrients exist in the soil in gaseous, liquid, and solid forms. Modelling Plant Uptake of dissolved soil constituents is essential to predicting plant growth under nutrient limitation. Water helps a plant by transporting important nutrients through the plant. Nutrients are drawn from the soil and used by the plant. Without enough water in the cells, the plants droop, so water helps a plant stand. Water carries the dissolved sugar and other nutrients through the plant. So without the proper balance of water, the plant not only is malnourished, but it is also physically weak and cannot support its own weight.

Nutrient uptake of plants from soil is the result of interactions between plant and soil. The rate of uptake of a nutrient depends on the concentration of this nutrient in soil solution at the root surface. In this paper, we consider that the plants growth is dependent on the water content it absorbs, therefore we focus on absorption of water content rather than nutrients. The relation between the concentration and the rate of uptake can often be described quantitatively by Michaelis–Menten kinetics as has been published by Epstein and Nielsen [9].

In the broader ecological literature on foraging and forager functional responses, Holling's disc equation (Holling 1959) provides one commonly used framework for modelling resource capture. In the plant literature on nutrient uptake kinetics, the Michaelis–Menten equation (Michaelis and Menten 1913) provides the framework for modelling nutrient capture. Here we take into account the water content capture by the plant. The resource harvest rate (units of resource uptake per time per gram of root) is given by

$$\frac{abN}{1+ahN}$$

Where b is the biomass of the roots possessed by the plant , N is the available concentration of water content in the environment (units of resources per unit volume of soil), the encounter rate between a unit of root and a water molecule is given by a and the cost in time associated with handling a given amount of water molecules is given by h [3].

II. THE MATHEMATICAL MODEL

The Mathematical model of Plant soil interaction is given below

$$\begin{aligned} P_{t+1} &= P_t + r(1-\frac{P_t}{L})P_t + \frac{abN_tP_t}{1+ahN_t} \\ N_{t+1} &= N_t + QN_t - \frac{abN_tP_t}{1+ahN_t} - \mu N_t \end{aligned} \qquad (1)$$

where

P_t - Density of plant species at time t.

N_t - Concentration of the water content in the soil at time t.

r - Intrinsic growth rate of the plant species.

Q - Precipitation rate at which water enters the soil.

μ - Leaching rate of water from the soil.

L - Carrying Capacity of the Plant species.

b - The biomass of the roots possessed by the plant.

a - The encounter rate between a unit of root and a water molecule.

h - The cost in time associated with handling a given amount of water molecules.

III. EQUILIBRIUM POINTS

We list the possible equilibrium points of the model (1):

A. $E_1 = (\bar{P}, 0)$

Where $\bar{P} = 0$ or $\bar{P} = L$

B. $E_2 = (P^*, N^*)$

Where $P^* = \dfrac{(Q-\mu)(1+ahN^*)}{ab}$

$Q > \mu$, $ab > 0$

N^* satisfies a quadratic polynomial

$$(Q-\mu)a^2h^2N^{*2} + \left(2ah(Q-\mu) - La^2b(h+\frac{b}{r})\right)N^* + (Q-\mu-abL) = 0$$

Given that

$La^2b(h+\frac{b}{r}) > 2ah(Q-\mu)$ and $\mu + abL > Q$

The jacobian matrix of the system (1) is given by

$$J = \begin{pmatrix} 1+r-\frac{2rP}{L}+\frac{abN}{(1+ahN)} & \frac{abP}{(1+ahN)^2} \\ -\frac{abN}{(1+ahN)} & 1+(Q-\mu)-\frac{abP}{(1+ahN)^2} \end{pmatrix} \quad (2)$$

IV. STABILITY ANALYSIS OF THE MODEL

A. Theorem 1

The Equilibrium point E_1 is stable if $r < 2$ and $abL < 2 + (Q-\mu)$. Otherwise it is unstable.

1) Proof: Consider the jacobian matrix of the system (1) with respect to the Equilibrium point E_1.

$$J_1 = \begin{pmatrix} 1-r & abL \\ 0 & 1+(Q-\mu)-abL \end{pmatrix} \quad (3)$$

The eigen values of the above matrix is given by $|\lambda_1| = r-1$ and $|\lambda_2| = abL-(Q-\mu)-1$. The Equilibrium point E_1 is stable if $r < 2$ and $abL < 2+(Q-\mu)$. Otherwise it is unstable.

B. Around the Equilibrium point $E_2 = (P^*, N^*)$ **:**

Consider the jacobian matrix of the system (1) with respect to the equilibrium point $E_2 = (P^*, N^*)$

$$J_2 = \begin{pmatrix} 1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)} & \frac{abP^*}{(1+ahN^*)^2} \\ -\frac{abN^*}{(1+ahN^*)} & 1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2} \end{pmatrix} \tag{4}$$

The characteristic equation of the above given matrix is as follows:

$$\phi(\lambda) = \lambda^2 - \lambda\left\{\left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right) + \left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right)\right\} + \left\{\left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right) + \frac{a^2b^2P^*N^*}{(1+ahN^*)^3}\right\} = 0 \tag{5}$$

It follows from the Jury's condition that the equilibrium point E_2 is stable if the following conditions hold [2]:

$$\phi(1) > 0$$
$$\phi(-1) > 0$$
$$\left|\text{Det } J_2\right| < 1$$

Consider $\phi(1)$, we get

$$\phi(1) = 1 - \left\{\left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right) + \left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right)\right\} + \left\{\left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right) + \frac{a^2b^2P^*N^*}{(1+ahN^*)^3}\right\}$$

$$\Rightarrow \left(r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left((Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right) + \frac{a^2b^2P^*N^*}{(1+ahN^*)^3}$$

Consider $\phi(-1)$, we get

$$\phi(-1) = 1 + \left\{\left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right) + \left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right)\right\} + \left\{\left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right) + \frac{a^2b^2P^*N^*}{(1+ahN^*)^3}\right\}$$

$$\Rightarrow \left(2 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left(2 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right) + \frac{a^2b^2P^*N^*}{(1+ahN^*)^3}$$

Consider $\left|\text{Det } J_2\right|$, we get

$$\left|\text{Det } J_2\right| = \left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right) + \frac{a^2b^2P^*N^*}{(1+ahN^*)^3}$$

The stability condition of Equilibrium point E_2 is given in the following the theorem

C. Theorem 2

The Equilibrium point $E_2 = (P^*, N^*)$ is stable if

$$\left(\frac{2rP^*}{L} - r - \frac{abN^*}{(1+ahN^*)}\right)\left(\frac{abP^*}{(1+ahN^*)^2} - (Q-\mu)\right) < \frac{a^2b^2P^*N^*}{(1+ahN^*)^3} <$$

$$1 - \left(1 + r - \frac{2rP^*}{L} + \frac{abN^*}{(1+ahN^*)}\right)\left(1 + (Q-\mu) - \frac{abP^*}{(1+ahN^*)^2}\right)$$

Otherwise it is unstable.

V. PROSOPIS JULIFLORA

In this paper, we consider that a plants growth is dependent on the water content in the soil, because it grows by absorbing the water around it. For example, here we study on a plant which was introduced as an alternative for fuel wood, but has become an invasive weed in India and many parts of Asia and Africa.

And there are also trees that can harm the environment. One such tree that is quite popular in South India is babool tree - known as "seemai karuvelam" in Tamil, wherein “seemai” means “foreign”. Babool is an invasive plant - a species that is not native to the ecosystem being considered. Therefore, it can - or likely to - cause harm to environment as well as economy and health of people and animals. Many global organizations are into creating awareness on the harmful effects of babool: Though, this tree grows in drought-hit areas and can be used as firewood, they extensively exploit ground water and drastically reduce water tables. No other trees can find water. Even if there is no ground water, babool can absorb humidity in the air, and thus prevent the chances of rain fall. Moreover, the roots of babool can destroy soil nutrients, and poison ground water. Since it does not produce enough oxygen (it produces only carbon dioxide in huge quantities), no birds find a shelter in these trees. Its leaves, seed or any other parts of the tree are of no use to human beings and animals [8].

Babool trees are common sight in the districts of Ramanathapuram, Sivagangai, Pudukottai, Virudunagar and Theni in Tamil Nadu, because people do not yet know the dangers of these invasive trees. They even plant these trees for firewood. In fact, droughts in these southern districts could be linked to the presence of babool trees. There are many versions as to how these trees entered several Indian States like Tamil Nadu, but States such as Kerala had not allowed the planting of these trees. In those days, the trees would have been imported for its ability to adapt to drought. However, having known the harmful effects of seemai karuvelam, we should take quick measures to uproot these trees from the soils of Tamil Nadu.

VI. NUMERICAL EXAMPLE

We explain our results by the following example. For this example, we use the data that was given in “Prosopis Juliflora in the Irrigation Tanks of Tamil Nadu” by R. Sakthivadivel [7]. Let us consider

$$r = 0.3685, Q = 0.9, \mu = 0.4, L = 10, b = 0.5, h = 0.1$$

We consider the encounter rate between a unit of root and a water molecule, $a = 0.2$. And analyze the stability of the model. The equilibrium points are $E_1(10,0)$ and $E_2(3.75,25)$. The eigenvalues at $E_1(10,0)$ are 0.6315 and 0.5 which satisfy the conditions given in Theorem 1 and therefore the system (1) is stable at $E_1(10,0)$. At the equilibrium point $E_2(3.75,25)$ the system (1) is unstable as for this choice of parameter, as it fails to satisfy the stability conditions of Theorem 2.

Thus from above discussion we have seen that given system (1) is stable at the $E_1 = (\bar{P}, 0)$. On the other hand we say that system is unstable at the equilibrium point $E_2 = (P^*, N^*)$.

VII. CONCLUSION

In this paper, we have constructed a discrete time model on plant soil interactions using Difference equations. We list the possible equilibrium points of the model and analyze both the local stability conditions of the model. We have considered the invasive weed Prosopis Juliflora which absorbs abundant of water from the soil in this paper, and have analyzed about it using the system (1). Here we see that the equilibrium point $E_1 = (\bar{P}, 0)$ is stable and $E_2 = (P^*, N^*)$ is unstable for the given set of parameters.

REFERENCES

[1] Tiina Roose, Mathematical Model of Plant Nutrient Uptake, Thesis, (2000).

[2] J.D. Murray, Mathematical biology. New York: Springer-Verlag, (1993).

[3] Gordon G Mcnickle, Joel S. Brown: When Michaelis and Menten met Holling:Towards a mechanistic theory of plant nutrient foraging behaviour, AOB Plants, October(2014).

[4] Ruth D. Yanai, A Steady-State Model of Nutrient Uptake Accounting for Newly Grown Roots, Soil Science Society of America Journal, 58:1562-1571 (1994)

[5] J.B. Shukla, Kusum Lata and A.K. Misra, Modeling the depletion of a renewableresource by population and Industrialization: Effect of Technology on its conservation, Natural Resource Modeling,Volume 24, Number 2, 242-267, (2011)

[6] https://en.wikipedia.org/wiki/Prosopis_juliflora

[7] R. Sakthivadivel, Prosopis Juliflora in the Irrigation Tanks of Tamil Nadu, Water Policy Research Highlight.

[8] http://www.aravindhhealthcentre.org/blog/kill-babool-save-eco-system.

[9] Wen Lin, J. M. Kelly: Nutrient uptake estimates for woody species as described by theNST 3.0, SSAND, and PCATS mechanistic nutrient uptake models, Thesis, July(2009).