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An M/G/1 Queue with Server Breakdown and with Single Working Vacation

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***Abstract:** This paper deals with the steady state behaviour of an $M/G/1$ single working vacation queue with server breakdown. The server works with different service times rather than completely stopping service during a vacation. Both service times in a vacation period and in a service period are generally distributed random variables. The system may breakdown at random and repair time is arbitrary. Further, just after completion of a customer's service the server may take a single working vacation. The supplementary variable technique is employed to find the probability generating function of the number in the system and the mean number in the system. Some particular cases of interest are discussed and Numerical results are also presented.*

***Keywords:** Poisson arrivals, Random breakdown, Repair time, Working Vacation, Supplementary Variable Technique. AMS Subject Classification Number: 60K25, 60K30.*

I. INTRODUCTION

In most of the queuing literature it is assumed that the server is available in the service station on a permanent basis and service station never fails. However, these assumptions are particularly unrealistic in practical system we often meet the case where service stations may fail and can be repaired. Similarly many phenomena always occur in the area of computer communications, networks and flexible manufacturing systems etc. Vacation queueing models subject to breakdowns have been studied by many authors including Gaver (1959) Levy and Yechilai (1976) Fuhrman (1981) Doshi (1986) Keilson and Servi (1986) Shanthikumar (1988) Cramer (1989) Madhan (1999) and Madhan and Saleh (2001) are a few among several authors who studied queues with server vacations. Sengupta (1990) Takine and Sengupta (1977) Li et.al. (1997) Madhan (2003) Gautam Choudhury (2008) and Thangaraj and Vanitha (2010) studied an M/G/1 queue with breakdowns and vacations. Recently a class of semi-vacation policies has been introduced by Servi and Finn. Such a vacation is called working vacation (WV). The server works at a lower rate rather than completely stops service during a vacation. Servi and Finn (2002) studied an M/M/1 queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works were done by Wu and Takagi (2006) Tian et.al. (2008) Aftab Begum (2011) Santhi and Pazhani Bala Murugan (2013) and Santhi and Pazhani Bala Murugan (2014). In this paper we study a non Markovian queue with single working vacation and Random breakdown. The organization of this paper is as follows. In section 2, we describe the model. In section 3, we obtain the steady state probability generating function. In section 4, some particular cases are discussed. In section 5, the performance measures are obtained and in section 6, Numerical results are presented.

II. THE MODEL DESCRIPTION

A. *We assume the following to Describe the Queueing Model of Our Study*

1) Customers arrive at the system one by one in according to a Poisson stream with arrival rate $\lambda(>0)$.
2) The service discipline is FCFS.
3) The service time is general distribution. Let $S_b(x), s_b(x)$ and $S_b^*(\theta)$ be the probability distribution function, the probability density function and the Laplace Stieltjes Transform (LST) of the service time S_b respectively.
4) Whenever the system becomes empty at a service completion instant the server starts working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant, if there are customers in the system, the server will start a new busy period. Otherwise, he remains in the system until an arrival of a customer. This type of working vacation is called single working vacation. During the working vacation period, the server provides the service with

the service time S_v of a typical customer follows a general distribution with the distribution function $S_v(x)[s_v(x)$ the pdf and $S_v^*(\theta)$ the LST].

5) The system may breakdown at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha_1(>0)$ and $\alpha_2(>0)$ such that α_1 for not WV period and α_2 for WV period respectively. Further, we assume that once the system breakdown, the customer whose service is interrupted comes back to the head of the queue and the system enters a repair process immediately. The repair time follows general distribution. Let the repair time distribution functions be $S_{r_1}(x)[s_{r_1}(x)$ the pdf $S_{r_1}^*(\theta)$ the LST] and $S_{r_2}(x)[s_{r_2}(x)$ the pdf $S_{r_2}^*(\theta)$ the LST] in not WV period and WV period respectively.
6) Various stochastic processes involved in the system are assumed to be independent of each other.

III. THE SYSTEM SIZE DISTRIBUTION

The system size distribution at an arbitrary time will be treated by the supplementary variable technique. That is from the joint distribution of the queue length and the remaining service time of the customer in service if the server is busy or the remaining service time of the customers if the server is on working vacation. We define the following random variables.

$N(t)$ – the system size at time t.

$S_b^0(t)$ – the remaining service time in not WV period.

$S_v^0(t)$ – the remaining service time in WV period.

$S_{r_1}^0(t)$ – the remaining repair time in not WV period.

$S_{r_2}^0(t)$ – the remaining repair time in WV period.

$$Y(t)=\begin{cases}0 & \text{if the server is idle in WV period at time } t\\ 1 & \text{if the server is idle in not WV period at time } t\\ 2 & \text{if the server is busy on not WV period at time } t\\ 3 & \text{if the server is busy on WV period at time } t\\ 4 & \text{if the server is waiting for repair during not WV period at time } t\\ 5 & \text{if the server is waiting for repair during WV period at time } t\end{cases}$$

So that the supplementary variables $S_b^0(t), S_v^0(t), S_{r_1}^0(t)$ and $S_{r_2}^0(t)$ are introduced in order to obtain bivariate Markov process $\{N(t),\partial(t);t\geq 0\}$ where

$$\partial(t)=\begin{cases}S_b^0(t) & \text{if } Y(t)=2\\ S_v^0(t) & \text{if } Y(t)=3\\ S_{r_1}^0(t) & \text{if } Y(t)=4\\ S_{r_2}^0(t) & \text{if } Y(t)=5\end{cases}$$

We define the following limiting probabilities

$$Q_0=\lim_{t\to\infty} Pr\{N(t)=0, Y(t)=0\},$$

$P_0 = \lim_{t\to\infty} Pr\{N(t)=0, Y(t)=1\}$,

$P_n(x) = \lim_{t\to\infty} Pr\{N(t)=n, Y(t)=2, x < S_b^0(t) \le x+dx\}$; $n \ge 1$,

$Q_n(x) = \lim_{t\to\infty} Pr\{N(t)=n, Y(t)=3, x < S_v^0(t) \le x+dx\}$; $n \ge 1$,

$R_{1,n}(x) = \lim_{t\to\infty} Pr\{N(t)=n, Y(t)=4, x < S_{r_1}^0(t) \le x+dx\}$; $n \ge 1$,

$R_{2,n}(x) = \lim_{t\to\infty} Pr\{N(t)=n, Y(t)=5, x < S_{r_2}^0(t) \le x+dx\}$; $n \ge 1$.

We define the Laplace Stieltjes Transforms and the probability generating functions as follows:

For $i = 1, 2$

$$S_b^*(\theta) = \int_0^\infty e^{-\theta x} s_b(x)dx \,;\, S_v^*(\theta) = \int_0^\infty e^{-\theta x} s_v(x)dx \,;\, S_{r_i}^*(\theta) = \int_0^\infty e^{-\theta x} s_{r_i}(x)dx;$$

$$Q_n^*(\theta) = \int_0^\infty e^{-\theta x} Q_n(x)dx \;;\; Q_n^*(0) = \int_0^\infty Q_n(x)dx; P_n^*(\theta) = \int_0^\infty e^{-\theta x} P_n(x)dx \;;$$

$$P_n^*(0) = \int_0^\infty P_n(x)dx \;;\; R_{i,n}^*(\theta) = \int_0^\infty e^{-\theta x} R_{i,n}(x)dx; R_{i,n}^*(0) = \int_0^\infty R_{i,n}(x)dx \;;$$

$$Q^*(z,\theta) = \sum_{n=1}^\infty Q_n^*(\theta) z^n ; Q(z,0) = \sum_{n=1}^\infty Q_n(0) z^n \;;\; Q^*(z,0) = \sum_{n=1}^\infty Q_n^*(0) z^n ;$$

$$P^*(z,\theta) = \sum_{n=1}^\infty P_n^*(\theta) z^n \;;\; P(z,0) = \sum_{n=1}^\infty P_n(0) z^n ; \; P^*(z,0) = \sum_{n=1}^\infty P_n^*(0) z^n ;$$

$$R_i^*(z,\theta) = \sum_{n=1}^\infty R_{i,n}^*(\theta) z^n \;;\; R_i(z,0) = \sum_{n=1}^\infty R_{i,n}(0) z^n \;;\; R_i^*(z,0) = \sum_{n=1}^\infty R_{i,n}^*(0) z^n$$

By considering the steady state, we have the following system of the differential difference equations.

$$(\lambda+\eta)Q_0 = P_1(0) + Q_1(0) \tag{1}$$

$$-\frac{d}{dx}Q_1(x) = -(\lambda+\alpha_2+\eta)Q_1(x) + Q_2(0)s_v(x) + \lambda Q_0 s_v(x) + R_{2,1}(0)s_v(x) \tag{2}$$

$$-\frac{d}{dx}Q_n(x) = -(\lambda+\alpha_2+\eta)Q_n(x) + Q_{n+1}(0)s_v(x) + \lambda Q_{n-1}(x) + R_{2,n}(0)s_v(x); \quad n>1, \tag{3}$$

$$\lambda P_0 = \eta Q_0 \tag{4}$$

$$-\frac{d}{dx}P_1(x) = -(\lambda+\alpha_1)P_1(x) + P_2(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_1(y)dy + \lambda P_0 s_b(x) + R_{1,1}(0)s_b(x) \tag{5}$$

$$\begin{aligned}-\frac{d}{dx}P_n(x) = -(\lambda+\alpha_1)P_n(x) + P_{n+1}(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_n(y)dy \\ + \lambda P_{n-1}(x) + R_{1,n}(0)s_b(x); \quad n>1\end{aligned} \tag{6}$$

$$-\frac{d}{dx}R_{2,1}(x)=-(\lambda+\eta)R_{2,1}(x)+\alpha_2 s_{r_2}(x)\int_0^\infty Q_1(x)dx \qquad (7)$$

$$-\frac{d}{dx}R_{2,n}(x)=-(\lambda+\eta)R_{2,n}(x)+\lambda R_{2,n-1}(x)+\alpha_2 s_{r_2}(x)\int_0^\infty Q_n(x)dx\ ;\quad n>1 \qquad (8)$$

$$-\frac{d}{dx}R_{1,1}(x)=-\lambda R_{1,1}(x)+\alpha_1 s_{r_1}(x)\int_0^\infty P_1(x)dx+\eta s_{r_1}(x)\int_0^\infty R_{2,1}(y)dy \qquad (9)$$

$$-\frac{d}{dx}R_{1,n}(x)=-\lambda R_{1,n}(x)+\lambda R_{1,n-1}(x)+\alpha_1 s_{r_1}(x)\int_0^\infty P_n(x)dx+\eta s_{r_1}(x)\int_0^\infty R_{2,n}(y)dy\ ;\ n>1 \qquad (10)$$

Taking the LST of (2) , (3) and from (5) to (10),we get

$$-\int_0^\infty e^{-\theta x}dQ_1(x)=-(\lambda+\alpha_2+\eta)\int_0^\infty e^{-\theta x}Q_1(x)dx+Q_2(0)\int_0^\infty e^{-\theta x}s_v(x)dx$$
$$+\lambda Q_0\int_0^\infty e^{-\theta x}s_v(x)dx+R_{2,1}(0)\int_0^\infty e^{-\theta x}s_v(x)dx$$

$$\theta Q_1^*(\theta)-Q_1(0)=(\lambda+\alpha_2+\eta)Q_1^*(\theta)-Q_2(0)S_v^*(\theta)-\lambda Q_0 S_v^*(\theta)-R_{2,1}(0)S_v^*(\theta) \qquad (11)$$

$$-\int_0^\infty e^{-\theta x}dQ_n(x)=-(\lambda+\alpha_2+\eta)\int_0^\infty e^{-\theta x}Q_n(x)dx+Q_{n+1}(0)\int_0^\infty e^{-\theta x}s_v(x)dx$$
$$+\lambda\int_0^\infty e^{-\theta x}Q_{n-1}(x)dx+R_{2,n}(0)\int_0^\infty e^{-\theta x}s_v(x)dx$$

$$\theta Q_n^*(\theta)-Q_n(0)=(\lambda+\alpha_2+\eta)Q_n^*(\theta)-Q_{n+1}(0)S_v^*(\theta)-\lambda Q_{n-1}^*(\theta)-R_{2,n}(0)S_v^*(\theta)\ ;\ n>1 \qquad (12)$$

$$-\int_0^\infty e^{-\theta x}dP_1(x)=-(\lambda+\alpha_1)\int_0^\infty e^{-\theta x}P_1(x)dx+P_2(0)\int_0^\infty e^{-\theta x}s_b(x)dx$$
$$+\eta\int_0^\infty e^{-\theta x}s_b(x)dx\int_0^\infty Q_1(y)dy+\lambda P_0\int_0^\infty e^{-\theta x}s_b(x)dx+R_{1,1}(0)\int_0^\infty e^{-\theta x}s_b(x)dx$$

$$\theta P_1^*(\theta)-P_1(0)=(\lambda+\alpha_1)P_1^*(\theta)-P_2(0)S_b^*(\theta)-\eta S_b^*(\theta)Q_1^*(0)-\lambda P_0 S_b^*(\theta)R_{1,1}(0)S_b^*(\theta) \qquad (13)$$

$$-\int_0^\infty e^{-\theta x}dP_n(x)=-(\lambda+\alpha_1)\int_0^\infty e^{-\theta x}P_n(x)dx+P_{n+1}(0)\int_0^\infty e^{-\theta x}s_b(x)dx+\eta\int_0^\infty e^{-\theta x}s_b(x)dx\int_0^\infty Q_n(y)dy$$
$$+\lambda\int_0^\infty e^{-\theta x}P_{n-1}(x)dx+R_{1,n}(0)\int_0^\infty e^{-\theta x}s_b(x)dx$$

$$\theta P_n^*(\theta)-P_n(0)=(\lambda+\alpha_1)P_n^*(\theta)-P_{n+1}(0)S_b^*(\theta)-\eta S_b^*(\theta)Q_n^*(0)-\lambda P_{n-1}^*(\theta)$$
$$-R_{1,n}(0)S_b^*(\theta)\ ;\ n>1 \qquad (14)$$

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$$-\int_0^\infty e^{-\theta x} dR_{2,1}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x} R_{2,1}(x)dx + \alpha_2 \int_0^\infty e^{-\theta x} s_{r_2}(x)dx \int_0^\infty Q_1(x)dx$$

$$\theta R^*_{2,1}(\theta) - R_{2,1}(0) = (\lambda+\eta)R^*_{2,1}(\theta) - \alpha_2 S^*_{r_2}(\theta)Q^*_1(0) \qquad (15)$$

$$-\int_0^\infty e^{-\theta x} dR_{2,n}(x) = -(\lambda+\eta)\int_0^\infty e^{-\theta x} R_{2,n}(x)dx + \lambda\int_0^\infty e^{-\theta x} R_{2,n-1}(x)dx + \alpha_2 \int_0^\infty e^{-\theta x} s_{r_2}(x)dx \int_0^\infty Q_n(x)dx$$

$$\theta R^*_{2,n}(\theta) - R_{2,n}(0) = (\lambda+\eta)R^*_{2,n}(\theta) - \lambda R^*_{2,n-1}(\theta) - \alpha_2 S^*_{r_2}(\theta)Q^*_n(0)\ ;\ n>1 \qquad (16)$$

$$-\int_0^\infty e^{-\theta x} dR_{1,1}(x) = -\lambda\int_0^\infty e^{-\theta x} R_{1,1}(x)dx + \alpha_1 \int_0^\infty e^{-\theta x} s_{r_1}(x)dx \int_0^\infty P_1(x)dx + \eta\int_0^\infty e^{-\theta x} s_{r_1}(x)dx \int_0^\infty R_{2,1}(y)dy$$

$$\theta R^*_{1,1}(\theta) - R_{1,1}(0) = \lambda R^*_{1,1}(\theta) - \alpha_1 S^*_{r_1}(\theta)P^*_1(0) - \eta S^*_{r_1}(\theta)R^*_{2,1}(0) \qquad (17)$$

$$-\int_0^\infty e^{-\theta x} dR_{1,n}(x) = -\lambda\int_0^\infty e^{-\theta x} R_{1,n}(x)dx + \lambda\int_0^\infty e^{-\theta x} R_{1,n-1}(x)dx + \alpha_1 \int_0^\infty e^{-\theta x} s_{r_1}(x)dx \int_0^\infty P_n(x)dx$$
$$+\eta\int_0^\infty e^{-\theta x} s_{r_1}(x)dx \int_0^\infty R_{2,n}(y)dy$$

$$\theta R^*_{1,n}(\theta) - R_{1,n}(0) = \lambda R^*_{1,n}(\theta) - \lambda R^*_{1,n-1}(\theta) - \alpha_1 S^*_{r_1}(\theta)P^*_n(0)\eta S^*_{r_1}(\theta)R^*_{2,n}(0);\ n>1 \qquad (18)$$

z^n times (12) summed over n from 2 to ∞ and added up with z times (11), gives

$$[\theta-(\lambda-\lambda z+\alpha_2+\eta)]Q^*(z,\theta) = [\frac{z-S^*_v(\theta)}{z}]Q(z,0) - S^*_v(\theta)\times[\lambda z Q_0 + R_2(z,0) - Q_1(0)] \qquad (19)$$

Inserting $\theta = (\lambda-\lambda z+\alpha_2+\eta$) in (19), we get

$$Q(z,0) = \frac{zS^*_v(\lambda-\lambda z+\alpha_2+\eta)[\lambda z Q_0 + R_2(z,0) - Q_1(0)]}{z - S^*_v(\lambda-\lambda z+\alpha_2+\eta)} \qquad (20)$$

z^n times (16) summed over n from 2 to ∞ and added up with z times (15), gives

$$[\theta-(\lambda-\lambda z+\eta)]R^*_2(z,\theta) = R_2(z,0) - \alpha_2 S^*_{r_2}(\theta)Q^*(z,0) \qquad (21)$$

Inserting $\theta = (\lambda-\lambda z+\eta$) in (21), we get

$$R_2(z,0) = \alpha_2 S^*_{r_2}(\lambda-\lambda z+\eta)Q^*(z,0) \qquad (22)$$

Substituting (22) in (21) and putting $\theta = 0$, we get

$$R^*_2(z,0) = \frac{\alpha_2 Q^*(z,0)(1-S^*_{r_2}(\lambda-\lambda z+\eta))}{\lambda-\lambda z+\eta} \qquad (23)$$

Substituting (22) in (20), we get

$$Q(z,0) = \frac{zS^*_v(\lambda-\lambda z+\alpha_2+\eta)[\lambda z Q_0 + \alpha_2 S^*_{r_2}(\lambda-\lambda z+\eta)Q^*(z,0) - Q_1(0)]}{z - S^*_v(\lambda-\lambda z+\alpha_2+\eta)} \qquad (24)$$

Substituting (22) and (24) in (19), we get

$$[\theta-(\lambda-\lambda z+\alpha_2+\eta)]Q^*(z,\theta)=\frac{\left\{\begin{array}{l} z(S_v^*(\lambda-\lambda z+\alpha_2+\eta)-S_v^*(\theta)) \\ \times[\alpha_2 S_{r_2}^*(\lambda-\lambda z+\eta)Q^*(z,0)+\lambda z Q_0-Q_1(0)]\end{array}\right\}}{z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)}$$ Inserting $\theta=0$, we get

$$Q^*(z,0)=\frac{z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))(\lambda z Q_0-Q_1(0))}{\left\{\begin{array}{l}(\lambda-\lambda z+\alpha_2+\eta)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)\end{array}\right\}} \quad (25)$$ Let

$$\begin{aligned} f(z) &= (\lambda-\lambda z+\alpha_2+\eta)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ &\quad -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta), \end{aligned}$$

we find $f(0)<0$ and $f(1)>0$. This implies that there exist a real root $z_1\in(0,1)$ for the equation $f(z)=0$. Hence at $z=z_1$ the equation (25) becomes, $Q_1(0)=\lambda z_1 Q_0$. Substituting this in (25), we get

$$Q^*(z,0)=\frac{\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))Q_0}{\left\{\begin{array}{l}(\lambda-\lambda z+\alpha_2+\eta)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)\end{array}\right\}} \quad (26)$$

Substituting (26) in (23), we get

$$R_2^*(z,0)=\frac{Q_0\alpha_2(1-S_{r_2}^*(\lambda-\lambda z+\eta))\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))}{\left\{\begin{array}{l}(\lambda-\lambda z+\eta)[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)]\end{array}\right\}} \quad (27)$$ z^n times

(14) summed over n from 2 to ∞ and added up with z times (13), yields

$$[\theta-(\lambda-\lambda z+\alpha_1)]P^*(z,\theta) = [\frac{z-S_b^*(\theta)}{z}]P(z,0)-S_b^*\times[\eta Q^*(z,0)+\lambda z P_0+R_1(z,0)-P_1(0)] \quad (28)$$ Substituting

$Q_1(0)=\lambda z_1 Q_0$ in (1), we get $P_1(0)=(\lambda(1-z_1)+\eta)Q_0$. Inserting $\theta=(\lambda-\lambda z+\alpha_1)$ and substituting $(\lambda(1-z_1)+\eta)Q_0=P_1(0)$ in (28),using (4) , we get

$$P(z,0)=\frac{zS_b^*(\lambda-\lambda z+\alpha_1)[\eta Q^*(z,0)+\eta z Q_0+R_1(z,0)-(\lambda(1-z_1)+\eta)Q_0]}{z-S_b^*(\lambda-\lambda z+\alpha_1)} \quad (29)$$ z^n times (18)

summed over n from 2 to ∞ and added up with z times (17), gives

$$(\theta-(\lambda-\lambda z))R_1^*(z,\theta)=R_1(z,0)-\alpha_1 S_{r_1}^*(\theta)P^*(z,0)-\eta S_{r_1}^*(\theta)R_2^*(z,0) \quad (30)$$ Inserting

$\theta=(\lambda-\lambda z)$ in (30), we get

$$R_1(z,0)=\alpha_1 S_{r_1}^*(\lambda-\lambda z)P^*(z,0)+\eta S_{r_1}^*(\lambda-\lambda z)R_2^*(z,0) \quad (31)$$ Substituting

(31) in (30) and putting $\theta = 0$ in (30), we get

$$R_1^*(z,0) = \frac{(1-S_{r_1}^*(\lambda-\lambda z))[\alpha_1 P^*(z,0)+\eta R_2^*(z,0)]}{\lambda-\lambda z} \quad (32)$$

Substituting (23), (26), (29), (31), $(\lambda(1-z_1)+\eta)Q_0 = P_1(0)$ and $P_0 = \frac{\eta}{\lambda}Q_0$ in (28) and also inserting $\theta = 0$ in (28), we get

$$P^*(z,0) = \frac{Q_0 z(1-S_b^*(\lambda-\lambda z+\alpha_1))\times Nr_3(z)}{D_1(z)D_2(z)} \quad (33)$$

where

$$\begin{aligned} Nr_3(z) = {} & \{\eta\lambda z(z-z_1)(1-S_v^*((\lambda-\lambda z+\alpha_2+\eta))[(\lambda-\lambda z+\eta)+\alpha_2 S_{r_1}^*(\lambda-\lambda z) \\ & \times(1-S_{r_2}^*(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta)) \\ & \times[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ & -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)]\} \end{aligned}$$

$$D_1(z) = (\lambda-\lambda z+\eta)\{(\lambda-\lambda z+\alpha_1)(z-S_b^*(\lambda-\lambda z+\alpha_1))-z\alpha_1(1-S_b^*(\lambda-\lambda z+\alpha_1))S_{r_1}^*(\lambda-\lambda z)\} \quad (34)$$

$$D_2(z) = \{(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta))-\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)\} \quad (35)$$

Substituting (27) and (33) in (32), we get

$$\begin{aligned} R_1^*(z,0) = {} & \frac{Q_0(1-S_{r_1}^*(\lambda-\lambda z))}{D_1(z)D_2(z)D_3(z)}\{\alpha_1 z(1-S_b^*(\lambda-\lambda z+\alpha_1))\{\eta\lambda z(z-z_1) \\ & \times(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))[(\lambda-\lambda z+\eta)+\alpha_2 S_{r_1}^*(\lambda-\lambda z) \\ & \times(1-S_{r_2}^*(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta)) \\ & \times[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ & -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)\}+\eta\{(\lambda-\lambda z+\alpha_1) \\ & \times(z-S_b^*(\lambda-\lambda z+\alpha_1))-\alpha_1 z(1-S_b^*(\lambda-\lambda z+\alpha_1))S_{r_1}^*(\lambda-\lambda z)\} \\ & \times\{\alpha_2(1-S_{r_2}^*(\lambda-\lambda z+\eta))\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))\}\} \end{aligned}$$

where $D_1(z)$ and $D_2(z)$ are given in (34) and (35) respectively and

$$D_3(z) = (\lambda-\lambda z) \quad (37)$$

We define $P_V(z) = Q^*(z,0) + R_2^*(z,0) + Q_0$ and using the values of $Q^*(z,0)$, $R_2^*(z,0)$, Q_0 in the above equation, we get

$$P_V(z) = \frac{Q_0 \times Nr_4(z)}{\left\{\begin{array}{l}(\lambda-\lambda z+\eta)[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\eta+\alpha_2)) \\ -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))S_{r_2}^*(\lambda-\lambda z+\eta)]\end{array}\right\}} \quad (38)$$

where

$$\begin{aligned} Nr_4(z) &= \{\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))(\lambda-\lambda z+\eta) \\ &\quad +\alpha_2\lambda z(z-z_1)(1-S_{r_2}^*(\lambda-\lambda z+\eta))(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2)) \\ &\quad +(\lambda-\lambda z+\eta)[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\eta+\alpha_2)) \\ &\quad -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))S_{r_2}^*(\lambda-\lambda z+\eta)]\} \end{aligned}$$

As the probability generating function for the number of customers in the system when the server is on working vacation period and by defining $P_B(z) = P^*(z,0) + R_1^*(z,0) + P_0$ and using the values of $P^*(z,0)$, $R_1^*(z,0)$, P_0, we get $P_B(z)$ as follows.

$$\begin{aligned} P_B(z) &= \frac{Q_0}{\lambda D_1(z)D_2(z)D_3(z)}\{\lambda z(1-S_b^*(\lambda-\lambda z+\alpha_1))\{\eta\lambda z(z-z_1) \\ &\quad \times(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))[(\lambda-\lambda z+\eta)+\alpha_2 S_{r_1}^*(\lambda-\lambda z) \\ &\quad \times(1-S_{r_2}^*(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta)) \\ &\quad \times[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ &\quad -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)]\} \\ &\quad +\lambda(1-S_{r_1}^*(\lambda-\lambda z))\{\alpha_1 z(1-S_b^*(\lambda-\lambda z+\alpha_1))\{\eta\lambda z(z-z_1) \\ &\quad \times(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))[(\lambda-\lambda z+\eta)+\alpha_2 S_{r_1}^*(\lambda-\lambda z) \\ &\quad \times(1-S_{r_2}^*(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta)) \\ &\quad \times[(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \\ &\quad -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)\} \\ &\quad +\eta\{(\lambda-\lambda z+\alpha_1)(z-S_b^*(\lambda-\lambda z+\alpha_1))-\alpha_1 z(1-S_b^*(\lambda-\lambda z+\alpha_1)) \\ &\quad \times S_{r_1}^*(\lambda-\lambda z)\}\{\alpha_2(1-S_{r_2}^*(\lambda-\lambda z+\eta))\lambda z(z-z_1) \\ &\quad \times(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))\}\}+\eta(\lambda-\lambda z)(\lambda-\lambda z+\eta) \\ &\quad \times\{(\lambda-\lambda z+\alpha_1)(z-S_b^*(\lambda-\lambda z+\alpha_1))-z\alpha_1(1-S_b^*(\lambda-\lambda z+\alpha_1)) \\ &\quad \times S_{r_1}^*(\lambda-\lambda z)\}\{(\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\alpha_2+\eta)) \qquad (39) \\ &\quad -\alpha_2 z(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))S_{r_2}^*(\lambda-\lambda z+\eta)\}\} \end{aligned}$$

as the probability generating function for the number of customers in the system when the server is in regular service period where $D_1(z), D_2(z)$ and $D_3(z)$ are given in (34) (35) and (37) respectively.Again we define $P(z) = P_B(z) + P_V(z)$ and hence

$$P(z) = \frac{Q_0}{\lambda D_1(z)D_2(z)D_3(z)}\{\{\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))(\lambda-\lambda z+\eta)$$

$$+\alpha_2\lambda z(z-z_1)(1-S^*_{r_2}(\lambda-\lambda z+\eta))(1-S^*_v(\lambda-\lambda z+\eta+\alpha_2))$$

$$+(\lambda-\lambda z+\eta)[(\lambda-\lambda z+\eta+\alpha_2)(z-S^*_v(\lambda-\lambda z+\eta+\alpha_2))$$

$$-\alpha_2 z(1-S^*_v(\lambda-\lambda z+\eta+\alpha_2))S^*_{r_2}(\lambda-\lambda z+\eta)]\}$$

$$\times\{\lambda(\lambda-\lambda z)[(\lambda-\lambda z+\alpha_1)(z-S^*_b(\lambda-\lambda z+\alpha_1))$$

$$-\alpha_1 z(1-S^*_b(\lambda-\lambda z+\alpha_1))S^*_{r_1}(\lambda-\lambda z)]\}+\{\lambda z(1-S^*_b(\lambda-\lambda z+\alpha_1))$$

$$\times\{\eta\lambda z(z-z_1)(1-S^*_v(\lambda-\lambda z+\alpha_2+\eta))$$

$$\times[(\lambda-\lambda z+\eta)+\alpha_2 S^*_{r_1}(\lambda-\lambda z)(1-S^*_{r_2}(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)$$

$$\times(\eta z-(\lambda(1-z_1)+\eta))[(\lambda-\lambda z+\eta+\alpha_2)(z-S^*_v(\lambda-\lambda z+\alpha_2+\eta))$$

$$-\alpha_2 z(1-S^*_v(\lambda-\lambda z+\alpha_2+\eta))S^*_{r_2}(\lambda-\lambda z+\eta)]\}+\lambda(1-S^*_{r_1}(\lambda-\lambda z))$$

$$\times\{\alpha_1 z(1-S^*_b(\lambda-\lambda z+\alpha_1))\{\eta\lambda z(z-z_1)$$

$$\times(1-S^*_v(\lambda-\lambda z+\alpha_2+\eta))[(\lambda-\lambda z+\eta)+\alpha_2 S^*_{r_1}(\lambda-\lambda z)$$

$$\times(1-S^*_{r_2}(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta))$$

$$\times[(\lambda-\lambda z+\eta+\alpha_2)(z-S^*_v(\lambda-\lambda z+\alpha_2+\eta))$$

$$-\alpha_2 z(1-S^*_v(\lambda-\lambda z+\alpha_2+\eta))S^*_{r_2}(\lambda-\lambda z+\eta)\}+\eta\{(\lambda-\lambda z+\alpha_1)$$

$$\times(z-S^*_b(\lambda-\lambda z+\alpha_1))-\alpha_1 z(1-S^*_b(\lambda-\lambda z+\alpha_1))S^*_{r_1}(\lambda-\lambda z)\}$$

$$\times\{\alpha_2(1-S^*_{r_2}(\lambda-\lambda z+\eta))\lambda z(z-z_1)(1-S^*_v(\lambda-\lambda z+\eta+\alpha_2))\}\}$$

$$+\eta(\lambda-\lambda z)(\lambda-\lambda z+\eta)\{(\lambda-\lambda z+\alpha_1)(z-S^*_b(\lambda-\lambda z+\alpha_1))$$

$$-z\alpha_1(1-S^*_b(\lambda-\lambda z+\alpha_1))S^*_{r_1}(\lambda-\lambda z)\}$$

$$\times\{(\lambda-\lambda z+\eta+\alpha_2)(z-S^*_v(\lambda-\lambda z+\alpha_2+\eta))$$

$$-\alpha_2 z(1-S^*_v(\lambda-\lambda z+\alpha_2+\eta))\times S^*_{r_2}(\lambda-\lambda z+\eta)\}\}\} \quad (40)$$

as the probability generating function for the number of customers in the system where $D_1(z), D_2(z)$ and $D_3(z)$ are given in (34) (35) and (37) respectively . We shall now use the normalizing condition $P(1)=1$ to determine the only unknown Q_0 which appears in (40). Substituting $z=1$ in (40) and using L'Hospital's rule , we obtain

$$Q_0 = \frac{1-\rho_b}{\left[\dfrac{\eta^2+\eta\lambda+\lambda^2(1-z_1)}{\eta\lambda}\right]-\left[\dfrac{C_3}{C_4}\right]} \quad (41)$$

$$C_3 = \eta\lambda^2(1-z_1)\{S_v^*(\eta+\alpha_2)(1-S_b^*(\alpha_1))[\alpha_2+\eta(1+\alpha_1 E(S_{r_1}))] + \alpha_2\alpha_1 E(S_{r_1})[S_v^*(\eta+\alpha_2)-S_b^*(\alpha_1)[1-S_{r_2}^*(\eta)(1-S_v^*(\eta+\alpha_2))]]\}$$

$$C_4 = \lambda\eta\alpha_1 S_b^*(\alpha_1)(1-S_v^*(\eta+\alpha_2))(\eta+\alpha_2(1-S_{r_2}^*(\eta)))$$ and

$\rho_b = \dfrac{\lambda(1-S_b^*(\alpha_1))(1+\alpha_1 E(S_{r_1}))}{\alpha_1 S_b^*(\alpha_1)}$, $E(S_{r_1})$ is the mean repair time in regular service period. From (41) we obtain the system stability condition $\rho_b < 1.$

IV. PARTICULAR CASES

A. *Case(i):* If the system suffers no breakdowns then letting $\alpha_1 = 0$ and $\alpha_2 = 0$ in (40), we have

$$P(z) = P_B(z) + P_V(z) \quad (42)$$

where

$$P_B(z) = \frac{\left[\lambda z(1-S_b^*(\lambda-\lambda z))\times Nr_5(z)\right]Q_0}{\lambda(\lambda-\lambda z)(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))(z-S_b^*(\lambda-\lambda z))}$$

$$Nr_5(z) = \{[\eta\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta)) - (\lambda(1-z_1)+\eta(1-z))(z-S_v^*(\lambda-\lambda z+\eta))(\lambda-\lambda z+\eta)] + \eta(\lambda-\lambda z)(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))(z-S_b^*(\lambda-\lambda z))\}$$

$$P_V(z) = \frac{Q_0\{\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta))+(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))\}}{(\lambda-\lambda z+\eta)(z-S_v^*(\lambda-\lambda z+\eta))}$$

$$Q_0 = \frac{1-\rho_b}{\left[\dfrac{(\lambda-\lambda z_1+\eta)}{\eta}+\dfrac{\eta}{\lambda}-\dfrac{\lambda(1-z_1)S_v^*(\eta)E(S_b)}{1-S_v^*(\eta)}\right]}$$

where $\rho_b = \lambda E(S_b)$. Equation (42) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue with single working vacation (Julia Rose Mary **Error! Reference source not found.**) irrespective of the notations.

B. *Case (ii)* :If the server never do the work during vacation period then on putting $S_v^*(\lambda-\lambda z+\eta+\alpha_2)=0, \alpha_2=0$ and $S_{r_2}^*(\lambda-\lambda z+\eta)=0$ in (40) and by taking repair time to be exponentially distributed, we get

$$P(z)=P_V(z)+P_B(z) \qquad (43)$$

$$P_V(z)=\frac{Q_0(\lambda(1-z_1)+\eta)}{\lambda-\lambda z+\eta}$$

$$P_B(z)=\frac{Q_0 z[S_b^*(\lambda-\lambda z+\alpha_1)-1][\lambda(1-z_1)+\eta][(\lambda-\lambda z)(\beta+\lambda-\lambda z)+\alpha_1(\lambda-\lambda z)]}{\left\{\begin{array}{l}(\lambda-\lambda z+\eta)[(\lambda-\lambda z)(z-S_b^*(\lambda-\lambda z+\alpha_1))(\beta+\lambda-\lambda z)\\+\alpha_1 z(\lambda-\lambda z)-\alpha_1 S_b^*(\lambda-\lambda z+\alpha_1)(\beta(1-z)+\lambda-\lambda z)]\end{array}\right\}}$$

$$Q_0=\frac{1-\rho_b}{[\dfrac{\eta+\lambda(1-z_1)}{\eta}]}$$

$$\rho_b=\frac{\lambda(1-S_b^*(\alpha_1))(\alpha_1+\beta)}{\alpha_1\beta S_b^*(\alpha_1)}.$$

Equation (43) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue with Server Vacation and Random Breakdown (Thangaraj **Error! Reference source not found.** no second stage service) irrespective of the notations.

C. *Case (iii):* If the system suffers no breakdowns and the server never takes a vacation then on setting $\alpha_1=0, \alpha_2=0$ and taking limit $\eta\to\infty$ in (40), we get

$$P(z)=\frac{(1-\lambda E(S_b))(1-z)S_b^*(\lambda-\lambda z)}{S_b^*(\lambda-\lambda z)-z} \qquad (44)$$

Equation (44) is well known probability generating function of the steady state system length distribution $M/G/1$ queue (Medhi **Error! Reference source not found.**) irrespective of the notations.

V. PERFORMANCE MEASURES

A. *Mean System Size*

Let L_v and L_b denote the mean system size during the working vacation and regular service period respectively and let W_v and W_b be the mean waiting time of the customer in the system during WV period and regular service period respectively.

$$
\begin{aligned}
L_v &= \frac{d}{dz} P_V(z)|_{z=1} \\
&= \frac{d}{dz}[Q^*(z,0)+R_2^*(z,0)]|_{z=1} = \frac{d}{dz}[\frac{A(z)}{D_2(z)} + \frac{B(z)}{(\lambda-\lambda z+\eta)D_2(z)}]Q_0|_{z=1} \\
&= [\frac{D_2(z)A'(z)-A(z)D_2'(z)}{(D_2(z))^2}]Q_0|_{z=1} \\
&\quad +[\frac{(\lambda-\lambda z+\eta)(D_2(z)B'(z)-B(z)D_{2'}(z))+\lambda B(z)D_2(z)}{((\lambda-\lambda z+\eta)D_2(z))^2}]Q_0|_{z=1}
\end{aligned}
$$

where

$$A(z) = \lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2));$$

$B(z) = \alpha_2\lambda z(z-z_1)(1-S_{r_2}^*(\lambda-\lambda z+\eta))(1-S_v^*(\lambda-\lambda z+\alpha_2+\eta))$ and $D_2(z)$ is given in the equation (35). At $z=1$ the formula L_v becomes

$$L_v = [\frac{D_2(1)A'(1)-A(1)D_2'(1)}{(D_2(1))^2} + \frac{\eta(D_2(1)B'(1)-B(1)D_{2'}(1))+\lambda B(1)D_2(1)}{(\eta D_2(1))^2}]Q_0$$

Using Little's formula, we get $W_v = \frac{L_v}{\lambda}$, where $A(1) = \lambda(1-z_1)(1-S_v^*(\eta+\alpha_2));$

$$A'(1) = (1-S_v^*(\eta+\alpha_2))(\lambda+\lambda(1-z_1)) + \lambda^2(1-z_1)S_v^{*'}(\eta+\alpha_2);$$

$$D_2(1) = (1-S_v^*(\eta+\alpha_2))[\eta+\alpha_2(1-S_{r_2}^*(\eta))];$$

$$
\begin{aligned}
D_2^{'}(1) &= (1-S_v^*(\eta+\alpha_2))[-\lambda-\alpha_2 S_{r_2}^*(\eta) \\
&\quad +\lambda\alpha_2 S_{r_2}^{*'}(\eta)] + \alpha_2 + \eta(1+\lambda S_v^{*'}(\eta+\alpha_2)) + \lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)(1-S_{r_2}^*(\eta));
\end{aligned}
$$

$$B(1) = \alpha_2\lambda(1-z_1)(1-S_{r_2}^*(\eta))(1-S_v^*(\eta+\alpha_2));$$

$$
\begin{aligned}
B'(1) &= \alpha_2\{\lambda(1-z_1)[(1-S_v^*(\eta+\alpha_2))(1-S_{r_2}^*(\eta)+\lambda S_{r_2}^{*'}(\eta)) \\
&\quad +\lambda(1-S_{r_2}^*(\eta))S_v^{*'}(\eta+\alpha_2)] + \lambda(1-S_{r_2}^*(\eta))(1-S_v^*(\eta+\alpha_2))\}.
\end{aligned}
$$

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$$L_b = \frac{d}{dz}P_B(z)|_{z=1} = \frac{d}{dz}[P^*(z,0)+R_1^*(z,0)]|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{N_1(z)N_2(z)}{D_1(z)D_2(z)} + \frac{N_3(z)N_4(z)}{D_1(z)D_2(z)D_3(z)}\right]Q_0|_{z=1}$$

$$= \frac{\begin{bmatrix} 2D_1^{'}(z)N_2^{'}(z)(D_2(z)N_1^{'}(z)-N_1(z)D_2^{'}(z)) \\ +D_2(z)N_1(z)(D_1^{'}(z)N_2^{''}(z)-N_2^{'}(z)D_1^{''}(z)) \end{bmatrix}}{4(D_1^{'}(z)D_2(z))^2}Q_0|_{z=1}$$

$$+ \frac{\begin{bmatrix} D_1^{'}(z)D_2(z)D_3^{'}(z)(N_3^{''}(z)N_4^{'}(z)+N_3^{'}(z)N_4^{''}(z)) \\ -D_3^{'}(z)N_3^{'}(z)N_4^{'}(z)(D_1^{''}(z)D_2(z)+2D_1^{'}(z)D_2^{'}(z)) \end{bmatrix}}{2(D_1^{'}(z)D_2(z)D_3^{'}(z))^2}Q_0|_{z=1}$$

$$N_1(z) = z(1-S_b^*(\lambda-\lambda z+\alpha_1))$$

$$N_2(z) = \eta\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))[(\lambda-\lambda z+\eta)+\alpha_2 S_{r_1}^*(\lambda-\lambda z)$$

$$\times(1-S_{r_2}^*(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta))\{(\lambda-\lambda z+\eta+\alpha_2)$$

$$\times(z-S_v^*(\lambda-\lambda z+\eta+\alpha_2))-\alpha_2 z S_{r_2}^*(\lambda-\lambda z+\eta)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))\}$$

$$N_3(z) = (1-S_{r_1}^*(\lambda-\lambda z))$$

$$N_4(z) = \alpha_1 z(1-S_b^*(\lambda-\lambda z+\alpha_1))\{\eta\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))[(\lambda-\lambda z+\eta)$$

$$+\alpha_2 S_{r_1}^*(\lambda-\lambda z)(1-S_{r_2}^*(\lambda-\lambda z+\eta))]+(\lambda-\lambda z+\eta)(\eta z-(\lambda(1-z_1)+\eta))$$

$$\times((\lambda-\lambda z+\eta+\alpha_2)(z-S_v^*(\lambda-\lambda z+\eta+\alpha_2))-\alpha_2 z(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))$$

$$\times S_{r_2}^*(\lambda-\lambda z+\eta))\}+\eta\{(\lambda-\lambda z+\alpha_1)(z-S_b^*(\lambda-\lambda z+\alpha_1))$$

$$-\alpha_1 z(1-S_b^*(\lambda-\lambda z+\alpha_1))S_{r_1}^*(\lambda-\lambda z)\}$$

$$\times\{\alpha_2(1-S_{r_2}^*(\lambda-\lambda z+\eta))\lambda z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\alpha_2))\}.$$

where $D_1(z), D_2(z)$ and $D_3(z)$ are given in equations (34),(35) and (37) respectively. Differentiating $N_1(z), N_2(z), N_3(z), N_4(z), D_1(z), D_2(z)$ and $D_3(z)$ with respect to z, we get $N_1^{'}(z), N_2^{'}(z), N_3^{'}(z), N_4^{'}(z), D_2^{'}(z)$ and $D_2^{'}(z)$ Again differentiating, we get $N_1^{"}(z), N_2^{"}(z), N_3^{"}(z), N_4^{"}(z), D_2^{"}(z)$ and $D_2^{*}(z)$ At $z=1$ the formula L_b, becomes

$$L_b = \frac{\begin{bmatrix} 2D_1^{'}(1)N_2^{'}(1)(D_2(1)N_1^{'}(1) - N_1(1)D_2^{'}(1)) \\ +D_2(1)N_1(1)(D_1^{'}(1)N_2^{"}(1) - N_2^{'}(1)D_1^{"}(1)) \end{bmatrix}}{4(D_1^{'}(1)D_2(1))^2} Q_0$$

$$+ \frac{\begin{bmatrix} D_1^{'}(1)D_2(1)D_3^{'}(1)(N_3^{"}(1)N_4^{'}(1) + N_3^{'}(1)N_4^{"}(1)) \\ -D_3^{'}(1)N_3^{'}(1)N_4^{'}(1)(D_1^{"}(1)D_2(1) + 2D_1^{'}(1)D_2^{'}(1)) \end{bmatrix}}{2(D_1^{'}(1)D_2(1)D_3^{'}(1))^2} Q_0$$

Applying Little's formula, we get $W_b = \dfrac{L_b}{\lambda}$, where

$$N_1(1) = (1 - S_b^*(\alpha_1))$$

$$N_1^{'}(1) = (1 - S_b^*(\alpha_1)) + \lambda S_b^{*'}(\alpha_1)$$

$$N_2^{'}(1) = (\eta + \alpha_2(1 - S_{r_2}^{*}(\eta)))(1 - S_v^*(\eta + \alpha_2))(\eta^2 + \eta\lambda + \lambda^2(1 - z_1)) - \eta\lambda(1 - z_1)$$

$$\left[(\eta + \alpha_2)S_v^*(\eta + \alpha_2) - \lambda\alpha_2(1 - S_v^*(\eta + \alpha_2))E(S_{r_1})(1 - S_{r_2}^{*}(\eta))\right]$$

$$N_2^{"}(1) = 2(\eta + \alpha_2(1 - S_{r_2}^{*}(\eta)))\{\eta\lambda^2(1 - z_1)S_v^{*'}(\eta + \alpha_2) + S_v^{*'}(\eta + \alpha_2)\} + (1 - S_v^*(\eta + \alpha_2))$$

$$\times(-\lambda + \lambda\alpha_2 E(S_{r_1})(1 - S_{r_2}^{*}(\eta)) + \lambda\alpha_2 S_{r_2}^{*'}(\eta))(\eta\lambda^2 + \eta\lambda + 2\eta\lambda^2(1 - z_1)) + \eta\lambda(1 - z_1)$$

$$\times(1 - S_v^*(\eta + \alpha_2))[2(-\lambda + \lambda\alpha_2 E(S_{r_1})(1 - S_{r_2}^{*}(\eta))) + \lambda\alpha_2 S_{r_2}^{*'}(\eta)S_v^{*'}(\eta + \alpha_2)]$$

$$+(2\lambda^2(1 - z_1) + \eta^2)\{(1 - S_v^*(\eta + \alpha_2))[-\lambda - \alpha_2 S_{r_2}^{*}(\eta) + \alpha_2\lambda S_{r_2}^{*'}(\eta)] + \lambda\alpha_2 S_v^{*'}(\eta + \alpha_2)$$

$$\times(1 - S_{r_2}^{*}(\eta)) + \eta + \alpha_2 + \eta\lambda S_v^{*'}(\eta + \alpha_2)\} + 2\eta\lambda(1 - z_1)\{\lambda + \lambda S_v^{*'}(\eta + \alpha_2)$$

$$\times(1 + \alpha_2 S_{r_2}^{*}(\eta)) - \alpha_2\lambda^2 S_{r_2}^{*'}(\eta)S_v^{*'}(\eta + \alpha_2)\}$$

$$N_3^{'}(1) = -\lambda E(S_{r_1})$$

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$$N_3''(1) = -\lambda^2 E(S_{r_1}^2)$$

$$N_4'(1) = \alpha_1(1-S_b^*(\alpha_1))\{-\eta\lambda(1-z_1)(\eta+\alpha_2)S_v^*(\eta+\alpha_2)+(\eta+\alpha_2(1-S_{r_2}^*(\eta)))$$
$$\times(1-S_v^*(\eta+\alpha_2))(\eta^2+\eta\lambda+\lambda^2(1-z_1))\}+\eta\alpha_2(1-S_{r_2}^*(\eta))\lambda(1-z_1)$$
$$\times(1-S_v^*(\eta+\alpha_2))\left[\alpha_1 S_b^*(\alpha_1)-\lambda(1-S_b^*(\alpha_1))\right]$$

$$N_4''(1) = (\alpha_1(1-S_b^*(\alpha_1))+\lambda\alpha_1 S_b^{*'}(\alpha_1))\{(1-S_v^*(\eta+\alpha_2))(\eta\lambda+\eta\lambda(1-z_1))(\eta+\alpha_2(1-$$
$$S_{r_2}^*(\eta)))+\eta\lambda^2(1-z_1)\alpha_2(1-S_v^*(\eta+\alpha_2))E(S_{r_1})(1-S_{r_2}^*(\eta))+(\lambda^2(1-z_1)+$$
$$\eta^2)(1-S_v^*(\eta+\alpha_2))(\eta+\alpha_2(1-S_{r_2}^*(\eta)))-\eta\lambda(1-z_1)(\eta+\alpha_2-\alpha_2(1-S_v^*(\eta+$$
$$\alpha_2))S_{r_2}^*(\eta))\}+\alpha_1(1-S_b^*(\alpha_1))\{(\eta+\alpha_2(1-S_{r_2}^*(\eta)))S_v^{*'}(\eta+\alpha_2)(\eta\lambda^2(1-z_1)+\eta\lambda^2)$$
$$+(\eta\lambda(1-z_1)+\eta\lambda)(1-S_v^*(\eta+\alpha_2))(\lambda\alpha_2 S_{r_2}^{*'}(\eta)-\lambda+\lambda\alpha_2 E(S_{r_1})(1-S_{r_2}^*(\eta)))$$
$$+(\eta+\alpha_2(1-S_{r_2}^*(\eta)))[\eta\lambda^2(1-z_1)S_v^{*'}(\eta+\alpha_2)+\eta\lambda^2 S_v^{*'}(\eta+\alpha_2)-\eta\lambda^3(1-z_1)$$
$$\times S_v^{*''}(\eta+\alpha_2)]+\eta\lambda(1-z_1)(\lambda S_v^{*'}(\eta+\alpha_2)+2(1-S_v^*(\eta+\alpha_2)))(\lambda\alpha_2 S_{r_2}^{*'}(\eta)$$
$$+\lambda\alpha_2 E(S_{r_1})(1-S_{r_2}^*(\eta)))-2\eta\lambda^2(1-z_1)(1-S_v^*(\eta+\alpha_2))+\eta\lambda^3\alpha_2(1-z_1)$$
$$\times S_v^{*'}(\eta+\alpha_2)(E(S_{r_1})(1-S_{r_2}^*(\eta))+S_{r_2}^{*'}(\eta))+\eta\lambda(1-z_1)(1-S_v^*(\eta+\alpha_2))$$
$$\times[\alpha_2\lambda^2 E(S_{r_1}^2)(1-S_{r_2}^*(\eta))+2\lambda^2\alpha_2 E(S_{r_1})S_{r_2}^{*'}(\eta)]+2\lambda^2(1-z_1)\{(1-S_v^*(\eta+\alpha_2))$$
$$\times[-\lambda-\alpha_2 S_{r_2}^*(\eta)+\lambda\alpha_2 S_{r_2}^{*'}(\eta)]+(\eta+\alpha_2)+\alpha_2\lambda S_v^{*'}(\eta+\alpha_2)\}+2\eta^2\{(1-S_v^*(\eta+\alpha_2))$$
$$\times[-\lambda-\alpha_2 S_{r_2}^*(\eta)+\lambda\alpha_2 S_{r_2}^{*'}(\eta)]+(\eta+\alpha_2)(1+\lambda S_v^{*'}(\eta+\alpha_2))-\alpha_2\lambda S_v^{*'}(\eta+$$
$$\alpha_2)S_{r_2}^*(\eta)\}-\eta\lambda(1-z_1)\{-2\lambda(1+\lambda S_v^{*'}(\eta+\alpha_2))-\lambda^2(\eta+\alpha_2)S_v^{*''}(\eta+\alpha_2)-$$
$$\lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)S_{r_2}^*(\eta)+\alpha_2(1-S_v^*(\eta+\alpha_2))S_{r_2}^{*'}(\eta)-\lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)S_{r_2}^*(\eta)+$$
$$\alpha_2\lambda^2 S_v^{*''}(\eta+\alpha_2)S_{r_2}^*(\eta)+\alpha_2\lambda^2 S_v^{*'}(\eta+\alpha_2)S_{r_2}^{*'}(\eta)+\lambda\alpha_2(1-S_v^*(\eta+\alpha_2))S_{r_2}^{*'}(\eta)+$$
$$\alpha_2\lambda^2 S_v^{*'}(\eta+\alpha_2)S_{r_2}^{*'}(\eta)\}+\eta\alpha_2\lambda(1-z_1)(1-S_{r_2}^*(\eta))(1-S_v^*(\eta+\alpha_2))\{-2\lambda(1+$$
$$\lambda S_b^{*'}(\alpha_1))-2\alpha_1\lambda S_b^{*'}(\alpha_1)-2\alpha_1\lambda(1-S_b^*(\alpha_1))E(S_{r_1})-\alpha_1\lambda^2(S_b^{*'}(\alpha_1)E(S_{r_1})$$
$$+(1-S_b^*(\alpha_1))E(S_{r_1}^2))\}-\eta\lambda(1-S_b^*(\alpha_1))(1+\lambda E(S_{r_1}))[(1-S_v^*(\eta+\alpha_2))$$
$$\times(\lambda^2(1-z_1)\alpha_2 S_{r_2}^{*'}(\eta)+\lambda(1-z_1)\alpha_2(1-S_{r_2}^*(\eta))+\lambda\alpha_2(1-S_{r_2}^*(\eta)))+$$

$$\alpha_2\lambda^2(1-z_1)(1-S_{r_2}^*(\eta))S_v^{*'}(\eta+\alpha_2)]+\eta\alpha_2\{\alpha_1 S_b^*(\alpha_1)-\lambda(1-S_b^*(\alpha_1))$$
$$\times(1+\alpha_1 E(S_{r_1}))\{(1-S_v^*(\eta+\alpha_2))(\lambda^2(1-z_1)S_{r_2}^{*'}(\eta)+\lambda(1-z_1)(1-S_{r_2}^*(\eta))$$
$$+\lambda(1-S_{r_2}^*(\eta))+\lambda^2(1-z_1)S_v^{*'}(\eta+\alpha_2)(1-S_{r_2}^*(\eta)))\}\}$$

$$D_1^{'}(1)=\eta\{\alpha_1 S_b^*(\alpha_1)-(1-S_b^*(\alpha_1))(\lambda+\lambda\alpha_1 E(S_{r_1}))\}$$
$$D_1^{''}(1)=-2\lambda\{\alpha_1 S_b^*(\alpha_1)-(1-S_b^*(\alpha_1))(\lambda+\lambda\alpha_1 E(S_{r_1}))\}+\eta\{-2\lambda(1+\lambda S_b^{*'}(\alpha_1))$$
$$-2\lambda\alpha_1 S_b^{*'}(\alpha_1)-2\lambda\alpha_1(1-S_b^*(\alpha_1))E(S_{r_1})-2\lambda^2\alpha_1 S_b^{*'}(\alpha_1)E(S_{r_1})$$
$$-\lambda^2\alpha_1(1-S_b^*(\alpha_1))E(S_{r_1})^2\}$$

$$D_2(1)=(1-S_v^*(\eta+\alpha_2))[\eta+\alpha_2(1-S_{r_2}^*(\eta))]$$

$$D_2^{'}(1)=(1-S_v^*(\eta+\alpha_2))[-\lambda-\alpha_2 S_{r_2}^*(\eta)+\lambda\alpha_2 S_{r_2}^{*'}(\eta)]+\alpha_2+\eta(1+\lambda S_v^{*'}(\eta+\alpha_2))$$
$$+\lambda\alpha_2 S_v^{*'}(\eta+\alpha_2)(1-S_{r_2}^*(\eta))$$

$$D_3^{'}(1)=-\lambda$$

VI. NUMERICAL RESULT

Assuming that the service time distribution for both regular service period and working vacation period as exponentially distributed and using the fact that

$$S_b^*(\alpha_1)=\frac{\mu_b}{(\alpha_1+\mu_b)},\quad S_v^*(\eta+\alpha_2)=\frac{\mu_v}{(\eta+\alpha_2+\mu_v)},\quad E(S_{r_1})=\frac{1}{\mu_{r_1}}$$
$$S_b^{*}{}'(\alpha_1)=-\frac{\mu_b}{(\alpha_1+\mu_b)^2},\ S_v^{*}{}'(\eta+\alpha_2)=-\frac{\mu_v}{(\eta+\alpha_2+\mu_v)^2},\ E(S_{r_1}^2)=\frac{2}{\mu_{r_1}^2},$$
$$S_{r_2}^*(\eta)=\frac{\mu_{r_2}}{(\eta+\mu_{r_2})},\quad S_v^{*}{}''(\eta+\alpha_2)=\frac{2\mu_v}{(\eta+\alpha_2+\mu_v)^3},\quad S_{r_2}^{*}{}'(\eta)=-\frac{\mu_{r_2}}{(\eta+\mu_{r_2})^2}$$

and by fixing the values of $z_1=0.5, \mu_v=8, \mu_b=15, \mu_{r_1}=3, \mu_{r_2}=5, \alpha_1=1, \alpha_2=2$ and ranging the values of λ from 3.2 to 3.6 insteps of 0.1 and varying the values of η from 1.5 to 1.9 insteps of 0.1, we calculated the corresponding values of L_b and W_b for single working vacation and tabulated in Table 1 and in Table 2 respectively.

λ η	1.5	1.6	1.7	1.8	1.9
3.2	0.103263	0.078540	0.061836	0.050624	0.043194
3.3	0.482666	0.429466	0.389194	0.358284	0.334233
3.4	0.987989	0.896551	0.824544	0.767054	0.720540
3.5	1.686728	1.542186	1.425982	1.331392	1.253469
3.6	2.695585	2.474262	2.293973	2.145468	2.021821

Table 1: Arrival rate (λ) versus mean system size (L_b) in regular service period

$\lambda \mid \eta$	1.5	1.6	1.7	1.8	1.9
3.2	0.032270	0.024544	0.019324	0.015820	0.013498
3.3	0.146262	0.130141	0.117938	0.108571	0.101283
3.4	0.290585	0.263691	0.242513	0.225604	0.211923
3.5	0.481922	0.440625	0.407423	0.380398	0.358134
3.6	0.748774	0.687295	0.637215	0.595963	0.561617

Table 2: Arrival rate (λ) versus mean waiting time (W_b) in regular service period

The corresponding graphs have been drawn for λ versus L_b and λ versus W_b and are shown in Figure 1 and in Figure 2 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for various values of η.

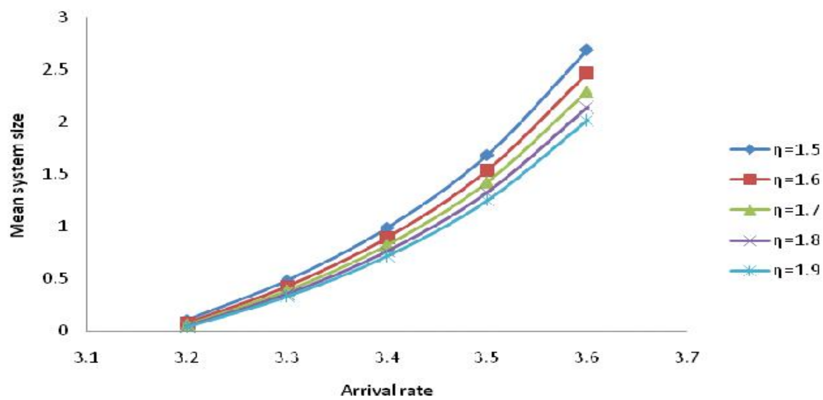

Figure 1: Arrival rate (λ) versus mean system size (L_b) in regular service period

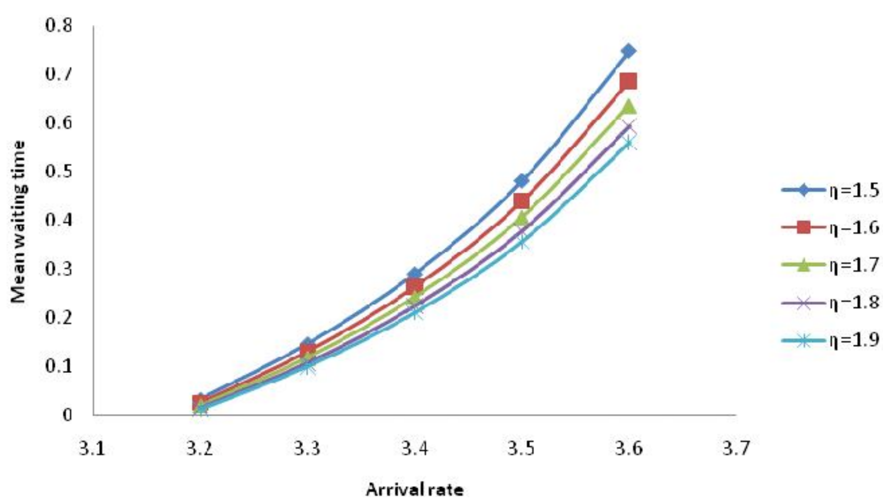

Figure 2: Arrival rate (λ) versus mean waiting time (W_b) in regular service period

Again fixing the values of $z_1 = 0.4, \mu_v = 7, \mu_b = 12, \mu_{r_1} = 2, \mu_{r_2} = 4, \alpha_1 = 2, \alpha_2 = 1$ and ranging the values of λ from 2.5 to 2.9 insteps of 0.1 and varying the values of η from 2.2 to 3.0 insteps of 0.2, we calculated the corresponding values of L_v and W_v for single working vacation and tabulated in Table 3 and in Table 4 respectively.

$\lambda \ \eta$	2.2	2.4	2.6	2.8	3.0

2.5	0.067190	0.060233	0.052123	0.046225	0.041756
2.6	0.077287	0.064986	0.056247	0.049856	0.044989
2.7	0.082729	0.069682	0.060342	0.053475	0.048218
2.8	0.088056	0.074317	0.064405	0.057077	0.051440
2.9	0.093267	0.078891	0.068435	0.060661	0.054653

Table 3: Arrival rate (λ) versus mean system size (L_v) in WV period

λ η	2.2	2.4	2.6	2.8	3.0
2.5	0.026876	0.024093	0.020849	0.018490	0.016702
2.6	0.029726	0.024994	0.021633	0.019175	0.017303
2.7	0.030640	0.025808	0.022349	0.019805	0.017859
2.8	0.031449	0.026542	0.023002	0.020385	0.018371
2.9	0.032161	0.027204	0.023598	0.020918	0.018846

Table 4: Arrival rate (λ) versus mean waiting time (W_v) in WV period

The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 3 and in Figure 4 respectively. From the graphs it is seen that as λ increases both L_v and W_v increases for various values of η.

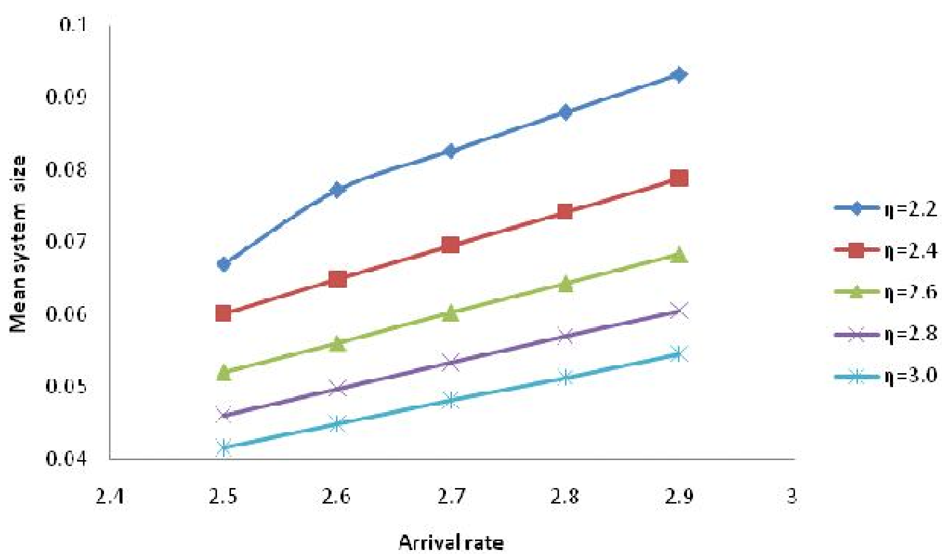

Figure 3: Arrival rate (λ) versus mean system size (L_v) in WV period

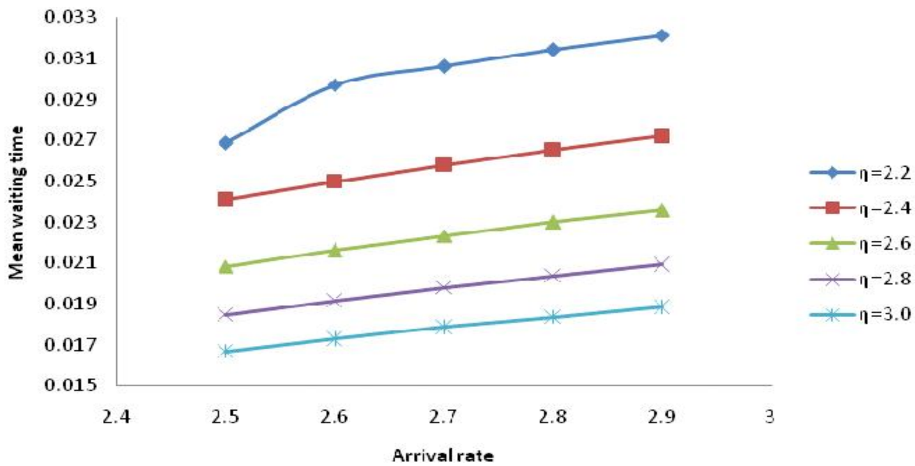

Figure 4: Arrival rate (λ) versus mean waiting time (W_v) in WV period

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