# Odd Graceful Labeling of the Union of Paths and Cycles

M. Meenakshi[1],

[1]Asst. Professor Department of Mathematics, St. Joseph's College of Arts and Science College for women, Hosur- 635 126.

***Abstract: In this dissertation the odd graceful labeling of the union of paths and cycles and the graph $c_m \cup p_n$ is odd graceful if m is even.***

## I. INTRODUCTION

### A. *Graceful Labeling*

The study of graceful graphs and graceful labeling methods was introduced by Rosa[2]. Rosa defined a β – valuation of a graph G with q edges an injection from the vertices of G to the set {0,1,2,......,q} such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edges are distinct. β – Valuation is a function that produces graceful labeling. However, the term graceful labeling was not used until Golomb studied such labeling several years later [3]. A graph G of size q is odd graceful, if there is an injection f from V(G) to {0,1,2……2q–1} such that, when each edge uv is assigned the label or weight $|f(u) - f(v)|$, the resulting edge labels are {1,3,5,……2q–1}.

Rosa [1967] has identified essentially three reasons why a graph fails to be graceful:

*1)* G has too many vertices and not enough edges,

*2)* G has too many edges and

*3)* G has the wrong parity.

As an example of the third condition Rosa [1967] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful.

### B. *Odd Graceful Labeling Of The Union Of Paths And Cycles*

*1) Algorithm 1: Procedure Initialization:* The union graph $P_n \cup C_m$ has a vertex set $V(P_n \cup C_m) = V(P_n) \cup V(C_m)$ with cardinality n+m and an edge set $E(P_n \cup C_m) = E(P_n) \cup E(C_m)$ with cardinality q=m + n – 1. Let the cycle $C_m$ is demonstrated by listing the vertices and edges in the order $u_1,e_1,u_2,e_2,......u_{m-1}, e_{m-1},u_m,e_m,u_1$. We name the vertex $u_m$ ACTIVE vertex, the vertex $u_m$ is an endpoint of the edge $e_{m-1}$, and we name the edge $e_{m-1}$ DOUBLE-JUMP edge. The path $P_n$ is demonstrated by listing the vertices and edges in the order $v_1,e'_1,v_2,e'_2,……v_{n-1},e'_{n-1},v_n$. The algorithm has two passes; they can run in a sequential or a parallel way. In one pass, the algorithm labels the vertices and the edges in the cycle $C_m$. For the other pass, it labels the vertices and edges of the path $P_n$. At the beginning of the algorithm, we are computing the odd label function for the ACTIVE vertex and the DOUBLE-JUMP edge. The ACTIVE vertex has the odd graceful labeling function $f(u_m) = 2q – (2m – 3)$, the vertex $u_m$ has the smallest odd label value between the vertices in the cycle $C_m$. The DOUBLE-JUMP is assigned the label function $f^*(e_{m-1}) = 2q – 3m + 5$. The given label to the ACTIVE vertex and the DOUBLE-JUMP edge computed independently from other vertices or edges in the graph. Number the ACTIVE vertex with the value $f(u_m) = 2q – (2m – 3)$ Number the DOUBLE-JUMP edge with the value $f^*(e_{m-1}) = 2q – 3m + 5$

*2) Algorithm 2: Odd graceful labeling of $C_m$:* In the first pass, the algorithm starts at the vertex $u_1$, there are two main steps that can be performed. These steps (in particular order) are: performing an action on the current vertex (referred to as "numbering" the vertex), number the current vertex with the value $f(u_1)=0$, traversing to the left adjacent vertex $u_2$ and number it with the value $f(u_2)=2q – 1$, and traversing to the left adjacent vertex $u_3$ and number it with the value $f(u_3)=2$ traversing to the left adjacent vertex $u_4$ and number it with the value $f(u_4)=2q – 3$, traversing to the left adjacent vertex $u_5$ and number it with the value $f(u_5)=4$… Thus the process is most easily described through recursion. Finally, reach to the ACTIVE vertex which has the exception label and number it with the value $f(u_m) = 2q – (2m – 3)$, the edge's labeling induced by the absolute value of the difference of the vertex's labeling. To label the cycle $c_m$ odd graceful, perform the following operations, starting with $u_1$:

Number the vertex $u_1$ with the value $f(u_1)=0$

For ( i=3; $i \le m–2$; i+=2 )

$$f(u_i) = f(u_{i-2}) + 2$$

taking the absolute value of the difference of incident vertex labels.

3) *Algorithm 3: Odd graceful labeling of $P_n$:* After the above process, the algorithm starts the second pass to label the vertices and edges of the path component $P_n$. Second pass starts at the edge $e'_1 = (v_1, v_2)$, its label value is $f^*(e'_1) = f(u_m) - 2$, if the label value of the edge $e'_1$ equals to the label value of the DOUBLE-JUMP edge renumber it with the value $f^*(e'_1) = f^*(e'_1) - 2$ and number the vertex $v_1$ with the label value $f(v_1) = 1$, traversing to the vertex $v_2$ and number it with the value $f(v_2) = f^*(e'_1) + 1$. Traversing to the next incident edge $e'_2$ and number it with the value $f^*(e'_2) = f^*(e'_1) - 2$, if the label value of the edge $e'_2$ equals to the label value of the DOUBLE-JUMP edge renumber it with the value $f^*(e'_2) = f^*(e'_2) - 2$, traverse to the next vertex $v_3$ which induces the label value $f(v_3) = f(v_2) - f^*(e'_2)$, otherwise traverse to the next vertex $v_3$, without double subtracting for the label value of the edge $e'_2$,and number it with the value $f(v_3) = f(v_2) - f^*(e'_2)$, traverse to the next vertex $v_4$ which induces the label value $f(v_4) = f(v_3) + f^*(e'_3)$. Thus the process is most easily described through recursion again. To label the path $P_n$ odd graceful labeling, perform the following operations, starting with the edge $e'_1 = (v_1, v_2)$:

Number the vertex $v_1$ with the value $f(v_1) = 1$

Number an auxiliary edge $e'_0$ with $f^*(e'_0) = f(u_m)$

For ($j=1$; $j \le n-1$; $j+=1$)

Number the edge $e'_j$ with $f^*(e'_j) = f^*(e'_{j-1}) - 2$

If ( $f^*(e'_j) = f^*(e_{m-1})$) , Renumber the edge $e'_j$ with the value $f^*(e'_j) = f^*(e'_j) - 2$

Number the vertex $v_{j+1}$ with the value $f(v_{j+1}) = f^*(e'_j) + (-)^{j+1} f(v_j)$

The algorithm is traversed exactly once for each vertex and edge in the graph $P_n \cup C_m$, since the size of the graph equals q then atmost O(q) time is spent in total labeling of the vertices and edges, thus the total running time of the algorithm is O(q). The parallel algorithm for the odd graceful labeling of the graph $P_n \cup C_m$, based on the above proposed sequential algorithm is building easily. Since all the above three subroutine are independent and there is no reason to sort their executing out, so they are to join up parallel in the same time point

4) *Theorem:*Let k is a given integer and m = 2k,the graph $C_m \cup P_n$ is odd graceful for every $n > m - 2$, k is even, if k is odd number the graph $C_m \cup P_n$ is odd graceful for every $n > m - 4$.

5) *Proof :* Let $V(C_m) = \{ u_1, u_2, \ldots\ldots, u_m \}$ , $V(P_n) = \{ v_1, v_2, \ldots\ldots, v_n \}$ , where $V(C_m)$ is the vertex set of the cycle $C_m$ and $V(P_n)$ is the vertex set of the path $P_n$ , and $q = n + m - 1$.

For every vertex $u_i$ and $v_i$ , we defined the odd graceful labeling functions $f(u_i)$ and $f(v_i)$ respectively as follows:

$f(u_1) = 0$ , $f(u_2) = 2q - 1$ , $f(u_3) = 2$ , $f(u_4) = 2q - 3$ , $f(u_5) = 4$ , $f(u_6) = 2q - 5$ , $f(u_7) = 6$ ,……, $f(u_m) = 2q - (2m - 3)$.

If the value m = 2k , k is odd number, the vertex $v_2$ would be labeled $f(v_2) = 2q - 2m + 2$ which decreased by two at every new value $i = 4,6,\ldots\ldots,k - 2$, this means that $f(v_i) = f(v_{i-2}) - 2 = 2q - 2m - (i - 4)$ , and

$$f(v_i) = \begin{cases} i + 2 & k \le i \text{ odd} \\ i & i = 1, 3, \ldots\ldots, k-2 \\ 2q - 2m - (i - 4) & i \text{ even} \end{cases}$$

If m = 2k, k is even number, the vertex $v_2$ would be labeled $f(v_2) = 2q - 2m + 2$ which decreased by two at every new value $i = 4,6,\ldots\ldots,k - 2$. For $i = k - 2$ the label value is $f(v_{k-2}) = 2q - 2m + 6 - k$ while the label value of the vertex $v_k$ is four out of the value $f(v_{k-2})$ , this means that $f(v_{i=k}) = f(v_{k-2}) - 4 = 2q - 2m + 2 - i$ , and

$$f(v_i) = \begin{cases} i & i \text{ odd} \\ 2q - 2m + 4 - i & i = 2, 4, \ldots\ldots, k-2 \\ 2q - 2m + 2 - i & k \le i \text{ even} \end{cases}$$

The function $f^*$ induces the edge labels of the cycle $C_m$ as the following :

$f^*(u_1u_2) = 2q - 1$

$f^*(u_2u_3) = 2q - 3$

$f^*(u_3u_4) = 2q - 5$ , …….

$f^*(u_{m-1}u_m) = 2q - 3m + 5$

$f^*(u_mu_1) = 2q - 2m + 3$.

For m = 2k , k is odd ,

$f(v_1) = 1$

$f(v_2) = 2q - 2m + 2$

$f(v_3) = 3$ …….
$f(v_{k-2}) = k – 2$
$f(v_{k-1}) = 2q – 2m – k +5$
$f(v_k) = k + 2$
$f(v_{k+1}) = 2q – 2m – k + 3$
For $m = 2k$ , k is even ,
$f(v_1) = 1$
$f(v_2) = 2q – 2m + 2$
$f(v_3) = 3$ …….
$f(v_{k-2}) = 2q – 2m – k + 6$
$f(v_{k-1}) = k – 1$
$f(v_k) = 2q – 2m – k + 2$
$f(v_{k+1}) = k + 1$
Function $f^*$ induces the edge labels of the path as follows :
$f^*(v_1v_2) = 2q – 2m + 1$
$f^*(v_2v_3) = 2q – 2m – 1$ , ……
$f^*(v_{k-2}v_{k-1}) = 2q – 3m + 7$
$f^*(v_{k-1}v_k) = 2q – 3m + 3$
$f^*(v_kv_{k+1}) = 2q – 3m + 1$ ,…… , 1

There is a guarantee that each component in the given graph has odd graceful, the path graph is odd graceful, the cycle graph with an even number of vertices is odd graceful[5]. We have to prove that the vertex labels are distinct and all the edge labels are distinct odd numbers { 1,3,5,……,2q – 1 }. The edge labels of $C_m$ are numbered according to the decreasing sequence 2q – 1 , 2q – 3 ,…… . The edge labels of $P_n$ are numbered according to the decreasing sequence $f^*(v_iv_{i+1}) = 2q – 2i – (2m – 1)$ , i = 4,5,……,q – m. It is obvious that, if i = q – m the last edge label is one; this means that the edge labels take the values in {2q – 1, 2q – 3, ……, 1}. In order to prevent any vertex in $P_n$ to share label with a vertex in $C_m$, the difference between the largest even label and the smallest even label in $P_n$ have to be more than the largest even label in $C_m$. This leads to two cases:

6) *Case I :* If m = 2k , k is even , then $( f(u_m) – 1 ) – 2 [n / 2] > m – 2$
*a)* ( 2q – (2m – 3) – 1) – 2 [n / 2] > m – 2
*b)* 2q – 2m + 3 – 1 – n > m –
*c)* 2q – 2m 2 – n > m – 2
*d)* 2( n + m – 1 ) – 2m + 2 – n > m –
*e)* 2n + 2m – 2 – 2m + 2 – n > m – 2
*f)* n > m – 2

7) *Case II :* If m = 2k , k is odd , then $( f(u_m) – 1 ) – 2( [n / 2] – 1 ) > m – 2$
*a)* ( 2q – (2m – 3) – 1) – 2( [n / 2] – 1 ) > m – 2
*b)* 2q – 2m + 3 – 1 – n + 2 > m – 2
*c)* 2q – 2m + 2 – n + 2 > m – 2
*d)* 2q – 2m – n + 4 > m – 2
*e)* 2( n + m – 1 ) – 2m – n + 4 > m – 2
*f)* 2n + 2m – 2 – 2m – n + 4 > m – 2
*g)* 2n + 2m – 2m – n + 2 > m – 2
*h)* n + 2 > m – 2
*i)* n > m – 2 – 2
*j)* n > m – 4

This completes the proof.

## II. CONCLUSION

In this dissertation, we explicitly defined the odd graceful labeling of the graph $C_m \cup P_n$ when m = 4 , 6 , 8 , 10 and by using these results we have generalized the procedure to label the vertices and edges of the graph $C_m \cup P_n$ when k is even and odd where k = m/2. We have also used the proposed sequential algorithm to label the graph $C_m \cup P_n$ when m= 12, 14.

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