Strongly Unique Best Co-approximation in Linear 2-Normed Spaces

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Abstract: This paper deals with some fundamental properties of the set of strongly unique best co-approximation in a linear 2-normed space.

Keywords: Linear 2-normed space, best co- approximation and strongly unique best Co-approximation.

I.INTRODUCTION

The problem of best co-approximation was first introduced by Franchetti and Furi [1] to study some characteristic properties of real Hilbert spaces and was followed up by Papini and Singer [2].

This theory is largely concerned with the questions of existence, uniqueness and characterization of best co-approximation. Newman and Shapiro [3] studied the problems of strongly unique best approximation in the space of continuous functions under supremum norm. The notion of strongly unique best co-approximation in the context of linear 2-normed spaces is introduced in this paper, it provides some important definitions and results that are required and deals with some fundamental properties of the set of strongly unique best co-approximation with respect to 2-norm.

II. PRELIMINARIES

A. Definition 1.2.1[4]

Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a best co-approximation to $x \in X$ from G if for every $g \in G$, $\| g - g_0, k \| \leq \| x - g, k \|$, for every $k \in X \setminus [G, x]$, where [G, x] represents a linear space spanned by elements of G and x .

B. Definition 1.2.2[4]

Let G be a non-empty subset of a linear 2-normed space X. An element $g_0 \in G$ is called a strongly unique best co-approximation to $x \in X$ from G, if there exists a constant $t > 0$ such that for every $g \in G$, $g - g_0, k \| \leq \| x - g, k \| - t \| x - g_0, k \|$, for every $k \in X \setminus [G, x]$.

The set of all elements of strongly unique best co-approximations to $x \in X$ from G is denoted by $T_G(x)$.

The subset G is called an existence set if $T_G(x)$ contains at least one element for every $x \in X$.

G is called a uniqueness set if $T_G(x)$ contains at most one element for every $x \in X$. G is called an existence and uniqueness set if $T_G(x)$ contains exactly one element for every $x \in X$.

C. Definition 1.2.3[4]

A set $K \subset X$ is convex if $\lambda x+(1-\lambda)y \in K$ whenever $x,y \in K$ and $\lambda \in [0,1]$.

A convex combination of $x_1,..........x_m$ is a sum of the form $\sum_{i=1}^{n} a_{ix_i}$, Where $\sum_{i=1}^{n} a_i = 1$, n is a positive integer, and all the a_i are non-negative.

If K is convex and $x_1,........,x_n \in K$, each $a_i \geq 0$, and $\sum_{i=1}^{n} a_i=1$, then $\sum_{i=1}^{n} a_{ix_i} \in K$.

This can be proved easily using induction on n.

D. Definition 1.2.4[4]

A set $K \subseteq V$ is called closed if the limit of every sequence $\{x_n\} \subset K$, which converges in V, belongs to K. i.e., $\{x_n\} \subset K$ and $x_n \to x$ in V then $x \in K$.

E. Definition 1.2.5[4]

A function f defined on some set X with real or complex values is called **bounded**, if the set of its values is bounded. In other words, there exists a real number M such that

$|f(x)| \leq M$
For all x in X. A function that is not bounded is said to be **unbounded**.

III. SOME FUNDAMENTAL PROPERTIES OF $T_G(x)$

Some basic properties of strongly unique best co-approximation are obtained in the following Theorems.

A. *Theorem 1.3.1[2]*

For any points a, b ϵ X and any α ϵ ℝ,
$\| a, b \| = \| a, b + \alpha a \|$

B. *Theorem 1.3.2[5]*

Let G be a subset of a linear 2-normed space X and x ϵ X. Then the following statements hold.

1) $T_G(x)$ is closed if G is closed.
2) $T_G(x)$ is convex if G is convex.
3) $T_G(x)$ is bounded.

a) *Proof:*

(i) Let G be closed.
Let $\{g_m\}$ be a sequence in $T_G(x)$ such that $g_m \to \bar{g}$.
To prove that $T_G(x)$ is closed, it is enough to prove that $\bar{g} \in T_G(x)$.
Since G is closed, $\{g_m\} \in G$ and $g_m \to \bar{g}$, we have $\bar{g} \in G$.
Since $\{g_m\} \in T_G(x)$, we have
$\| g - g_m, k \| \leq \| x - g, k \| - t \| x - g_m, k \|$, for every k ϵ X\[G, x] and for some t > 0
$\Rightarrow \| g - g_m + \bar{g} - \bar{g}, k \| \leq \| x - g, k \| - t \| x - g_m, k \|$
$\Rightarrow \| g - \bar{g}, k \| - \| g_m - \bar{g}, k \| \leq \| x - g, k \| - t \| x - g_m, k \|$, for every g ϵ G
Since $g_m \to \bar{g}$, $g_m - \bar{g} \to 0$. …….(1)
So $\| g_m - \bar{g}, k \| \to 0$, as 0 and k are linearly dependent.
Therefore, it follows from (1) that
$\| g - \bar{g}, k \| \leq \| x - g, k \| - t \| x - \bar{g}, k \|$, for every g ϵ G and for some t > 0 .
Thus $\bar{g} \in T_G(x)$
Hence $T_G(x)$ is closed.

(ii). Let G be convex, $g_1, g_2 \in T_G(x)$ and α ϵ(0, 1).
To prove that $\alpha g_1 + (1 - \alpha) g_2 \in T_G(x)$,
Let k ϵ X \ [G, x] .Then

$$\| g - (\alpha g_1 + (1 - \alpha) g_2, k \| = \| \alpha(g - g_1) + (1 - \alpha)(g - g_2), k \|$$
$$\leq \alpha \| g - g_1, k \| + (1 - \alpha) \| g - g_2, k \|$$
$$\leq \alpha \| x - g, k \| - \alpha t \| x - g1, k \|$$
$$+ (1 - \alpha) \| x - g, k \| - (1 - \alpha) t \| x - g_2, k \|, \text{ for every } g \in G \text{ and for some } t > 0.$$
$$= \| x - g, k \| - t (\| \alpha x - \alpha g_1, k \| + \| (1 - \alpha) x - (1 - \alpha) g_2, k \|)$$
$$\leq \| x - g, k \| - t \| \alpha x - \alpha g_1 + (1 - \alpha) x - (1 - \alpha) g_2, k \|$$
$$= \| x - g, k \| - t \| x - (\alpha g_1 + (1 - \alpha) g_2), k \| .$$

Thus $\alpha g_1 + (1 - \alpha) g_2 \in T_G(x)$.
Hence $T_G(x)$ is convex.

(iii). To prove that $T_G(x)$ is bounded, it is enough to prove for arbitrary $g_0, \bar{g}_0 \in T_G(x)$ that $\| g_O - \bar{g}_0, k \| < c$ for some c > 0 , since $\| g_0 - \bar{g}_0, k \| < c$ implies that $\sup_{g_0, \bar{g}_0 \in T_G(x)} \| g_0 , \bar{g}_0 , k \|$, is finite and hence the diameter of $T_G(x)$ is finite.
Let $g_0, \bar{g}_0 \in T_G(x)$. Then there exists a constant t > 0 such that for every
g ϵ G and k ϵ X \ [G, x],

$$\| g - g_0, k \| \leq \| x - g, k \| - t \| x - g_0, k \| \text{ and}$$
$$\| g - \bar{g}_0, k \| \leq \| x - g, k \| - t \| x - \bar{g}_0, k \| .$$

Now,

$$\| x - g_0, k \| \leq \| x - g, k \| + \| g - g_0, k \|$$
$$\leq 2 \| x - g, k \| - t \| x - g_0, k \| .$$

Hence $\| x - g_0, k \| \leq \frac{2}{1+t}$ d,
where $d = \inf_{g \epsilon G} \|x-g, k\|$
Similarly, $\| x - \bar{g}_0, k \| \leq \frac{2}{1+t} d$
Therefore, it follows that
$\| g_0 - \bar{g}_0, k \| \leq \| g0 - x, k \| + \| x - \bar{g}_0, k \|$
$\leq \frac{4}{1+t} d = c$
Hence $T_G(x)$ is bounded
Hence the proof

C. Theorem 1.3.3[2]

Let G be a subset of a linear 2-normed space X, x ϵ X and K ϵ X \ [G, x]. Then the following statements are equivalent for every y ϵ [k].

1) $g_0 \epsilon T_G(x)$.
2) $g_0 \epsilon T_G(x + y)$.
3) $g_0 \epsilon T_G(x - y)$.
4) $g_0 + y \epsilon T_G(x + y)$.
5) $g_0 + y \epsilon T_G(x - y)$.

a) *Proof:* The proof follows immediately by using Theorem 1.3.1

D. Theorem 1.3.2

Let G be a subspace of a linear 2-normed space X and x ϵ X. Then the following statements hold.

1) $T_G(x + g) = T_G(x) + g$, for every g ϵ G.
2) $T_G(\alpha x) = \alpha T_G(x)$, for every α ϵ R .

a) *Proof*

(i). Let $\bar{g}$ be an arbitrary but fixed element of G .
Let $g_0 \epsilon T_G(x)$. It is clear that $g_0 + \bar{g} \epsilon T_G(x) + \bar{g}$.
To prove that $T_G(x)+ \bar{g} \subseteq T_G(x+ \bar{g})$,
it is enough to prove that $g_0 + \bar{g} \epsilon T_G(x+ \bar{g})$.
Now,
$\| g + \bar{g} - g_0 - \bar{g}, k \| \leq \| x - g, k \| - t \| x - g_0, k \|$, for all g ϵ G and for some t > 0.
$\Rightarrow \| g + \bar{g} - (g_0 + \bar{g}), k \| \leq \| x + \bar{g} - (g + \bar{g}), k \| - t \|x + \bar{g} - (g_0 + \bar{g}), k\|$
for all g ϵ G
$\Rightarrow g_0 + \bar{g} \epsilon T_G(x + \bar{g})$, since $g - \bar{g} \epsilon G$.
Conversely, let $g_0 + \bar{g} \epsilon T_G(x + \bar{g})$.
To prove that $T_G(x + \bar{g}) \subseteq T_G(x) + \bar{g}$, it is enough to prove that $g_0 \epsilon T_G(x)$.
Now,
$\| g - g_0, k \| = \| g + \bar{g} - (g_0 + \bar{g}), k \|$
$\leq \| x + \bar{g} - (g + \bar{g}), k \leq -t \| x + \bar{g} - (g_0 + \bar{g}), k \|$, for all g ϵ G and for some t > 0.
$\Rightarrow g_0 \epsilon T_G(x)$, thus the result follows.
(ii). The proof is similar to that of (i).
Hence the proof.

E. Proposition 1.3.4[2]

Let G be a subset of a linear 2-normed space X, x ϵ X and
k ϵ X \ [G, x] . If $g_0 \epsilon T_G(x)$, then there exists a constant t > 0 such that for all
g ϵ G,
$\|x - g_0, k\| \leq 2\|x - g, k\| - t\|x - g_0, k\|$
Proof:
The proof is trivial.

F. Theorem: 1.3.5 [2]

Let G be a subspace of a linear 2-normed space X, $x \in X$ and

$K \in X \setminus [G, x]$. Then $g_0 \in T_G(x) \Leftrightarrow g_0 \in T_G(\alpha^m x + (1 - \alpha^m)g_0)$, for all $\alpha \in \mathbb{R}$ and

$m = 0, 1, 2, \cdots$.

Proof:

Claim: $g_0 \in T_G(x) \Leftrightarrow g_0 \in T_G(\alpha x+(1-\alpha)g_0)$, for every $\alpha \in \mathbb{R}$.

Let $g_0 \in T_G(x)$. Then

$\| g - g_0, k \| \leq \| x - g, k \| -t \| x - g_0, k \|$, for all $g \in G$ and for some $t > 0$.

$\Rightarrow \| \alpha g - \alpha g_0, k \| \leq \| \alpha x - \alpha g, k \| -t \| \alpha x - \alpha g_0, k \|$, for all $g \in G$

$\Rightarrow \| \alpha(\frac{(\alpha-1)g_0+g}{\alpha})-\alpha g_0, k \| \leq \| \alpha x-\alpha(\frac{(\alpha-1)g_0+g}{\alpha}), k \| -t \| \alpha x - \alpha g_0, \|$, for all $g \in G$ and $\alpha \neq 0$,

Since $(\frac{(\alpha-1)g_0+g}{\alpha}) \in G$

$\Rightarrow \| g - g0, k \| \leq \| \alpha x + (1 - \alpha)g_0 - g, k \| -t \| \alpha x + (1 - \alpha)g_0 - g_0, k \|$

$\Rightarrow g_0 \in T_G(\alpha x + (1 - \alpha)g_0$, when $\alpha \neq 0$.

If $\alpha = 0$, then it is clear that $g_0 \in T_G(\alpha x + (1 - \alpha)g_0)$.

The converse is obvious by taking $\alpha = 1$.

Hence the claim is true.

By repeated application of the claim the result follows.

IV. CONCLUSION

Hence we proved that linear 2-normed space are closed, bounded and convex.

Therefore , strongly unique best co-approximation in linear 2-normed space is also closed.

Here also we proved some fundamental properties.

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