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# Colouring of Generalized Petersen Graph of Type- k

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**Abstract:** Let  $G = \{V, E\}$  be a connected simple graph. A vertex colouring of a graph  $G$  is a function  $f: V(G) \rightarrow C$ , where  $C$  is a set of distinct colours. The vertex colouring problem is one of the fundamental problem on graphs which often appears in various scheduling problems like the file transfer problem on computer networks. In this paper we determine the vertex colouring of generalized petersen graph of type  $k$ .

**Keywords:** Cycle, Chromatic number, Vertex colouring, Generalized Petersen graph.

## I. INTRODUCTION

The theory of graph colouring has existed for more than 150 years from its modest beginning of determining whether a geographic map can be coloured with four colours. The theory has become central in discrete mathematics with many contemporary generalization and application. In this paper, we are concerned with finite, connected, simple graph. Let

$G = \{V(G), E(G)\}$  be a graph, if there is an edge  $e$  joining any two vertices  $u$  and  $v$  of  $G$ , we say that  $u$  and  $v$  are adjacent. A  $k$ -vertex colouring  $C$  of a graph  $G$  is an assignment of  $k$ -colours to the vertices of  $G$ .

### A. Definition:1.1

A graph  $G$  is an ordered pair  $(V(G), E(G))$  consisting of a non-empty set  $V(G)$  of vertices and a set  $E(G)$ , disjoint from  $V(G)$  of edges together with an incidence function  $\psi_G$  that associates with each edge of  $G$  is an unordered pair of vertices of  $G$ .

### B. Definition: 1.2

A colouring of a simple connected graph  $G$  is colouring the vertices of  $G$  in such a way that no two adjacent vertices of  $G$  get the same colour. A graph is properly coloured if it is coloured with the minimum possible number of colours.

### C. Definition: 1.3

The chromatic number of a graph  $G$  is the minimum number of colours required to colouring the vertices of  $G$  in properly and is denoted by  $\chi(G)$ .

### D. Definition:1.4

The generalized petersen graph  $GP(n, k)$  has vertices and edges of the form  $V(GP(n, k)) = \{a_i, b_i / 0 \leq i \leq n - 1\}$ ,  $E(GP(n, k)) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+k} / 0 \leq i \leq n - 1\}$ .

### E. Definition: 1.5

Walk is an alternating sequence of vertices and edges starting and ending with vertices. A walk in which all the vertices are distinct is called a path. A path containing  $n$ -vertices is denoted by  $P_n$ . A closed path is called cycle. A cycle containing  $n$ -vertices is denoted by  $C_n$ , the length of a cycle is the number of edges occurring on it.

Let  $G = GP(n, k)$  be the generalized petersen graph.

Let  $V(G)$  be the vertex set of  $G = GP(n, k)$ . It can be partitioned into two sets  $V_1$  and  $V_2$  such that

- i.  $V_1(G) = \{v_i / i = 1, 2, \dots, n\}$
- ii.  $V_2(G) = \{u_i / i = 1, 2, \dots, n\}$

Clearly, the vertices of  $V_{i;\{i=1,2\}}$  satisfies the condition  $V_1(G) \cap V_2(G) = \emptyset$ . Also, each vertex  $v_i \in V_1$  is adjacent to the corresponding  $u_i \in V_2$ .

The elements of  $V_1(G)$  form a cycle of length  $n$ . Let it be  $C_1$ .

1) *Type: I*: If  $gcd(n, k) = k$  and  $\frac{n}{k}$  is even, the vertex set of  $V_2$  contains  $k$ -disjoint cycles of length  $n/k$ . Let it be  $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$ .

For example,  $GP(12,3)$  is represented in figure:2.1

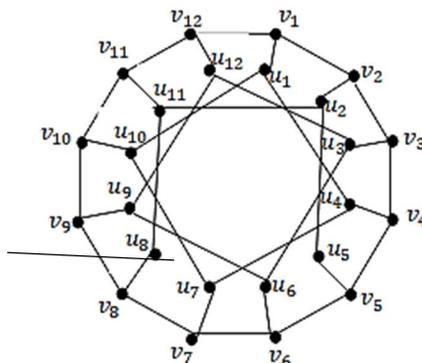


Figure:2.1

Clearly, the cycles  $C_1$  and  $C_{2_i} : (i=1,2,\dots,k)$  are of even in length. By lemma:1, We need 2-colours to colour all the vertices of  $V_1$ . Let it be  $c_1$  and  $c_2$ . Suppose the colour  $c_1$  is given to the vertex  $v_1$ . Since, each  $v_i$  is adjacent with the corresponding  $u_i$ , we can't assign the colour  $c_1$  to the vertex  $u_i$ . Hence, we can assign the colour  $c_2$  to the vertex  $u_i$ . Since the cycles  $C_{2_i} : (i=1,2,\dots,k)$  are even in length, repeat the above process for each  $u_i$  and from these above processes we can conclude that, in this type we need 2-colours to colour all the vertices of  $G$ . Therefore,  $\chi(G) = 2$ .

2) *Type: II* If  $gcd(n, k) = 1$ , the vertex set of  $V_2$  contains a cycle of length  $n$ . Let it be  $C_2$  For example,  $GP(8,3)$  is represented in figure:2.2

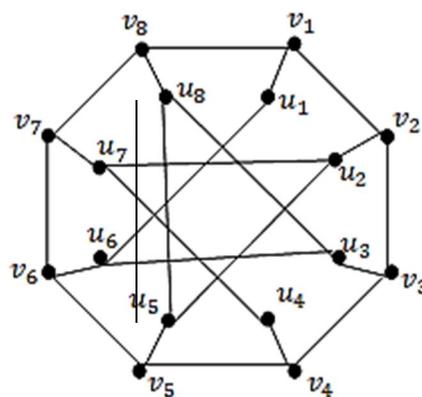


Figure:2.2

Since, the cycle  $C_1$  is of even in length. By lemma:1, We need 2-colours to colour all the vertices of  $C_1$ . Let it be  $c_1$  and  $c_2$ . Now, fix the colour  $c_1$  to the vertex  $v_1$ . Since, each  $v_i$  is adjacent with the corresponding  $u_i$ , the vertices of  $C_2$  are coloured according to their adjacency with the vertices of the cycle  $C_1$ . Hence, we need 2-colours to colour all the vertices of  $G$ . Therefore,  $\chi(G) = 2$ .

3) *Theorem:2.2* Let  $G$  be the generalized Petersen graph of type  $k$ , that is  $G = GP(n, k)$ , for all  $n$  and  $k$  are odd, then  $\chi(G) = 3$ .

4) *Proof:* Let  $G = GP(n, k)$  be the generalized Petersen graph. Let  $V(G)$  be the vertex set of  $G = GP(n, k)$ . It can be partitioned into two sets  $V_1$  and  $V_2$  such that

- i.  $V_1(G) = \{v_i / i = 1, 2, \dots, n\}$
- ii.  $V_2(G) = \{u_i / i = 1, 2, \dots, n\}$

Clearly, the vertices of  $V_{i:(i=1,2)}$  satisfies the condition  $V_1(G) \cap V_2(G) = \emptyset$ . Also, each vertex  $v_i \in V_1$  is adjacent to the corresponding  $u_i \in V_2$ . The elements of  $V_1(G)$  form a cycle of length  $n$ . Let it be  $C_1$ .

5) Type: I: If  $gcd(n, k) = k$  and  $n/k$  is odd.

The vertex set of  $V_2$  contains  $k$ -disjoint cycles of length  $n/k$ . Let it be  $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$ . For example,  $GP(9,3)$  is represented in figure:2.3

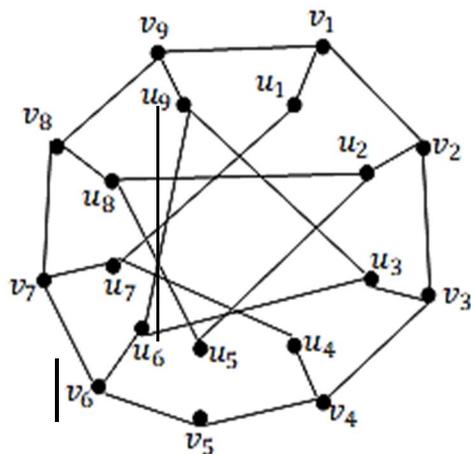


Figure:2.3

Clearly, the cycles  $C_1$  and  $C_{2_i}; (i=1,2,\dots,k)$  are of odd in length. Fix the vertex  $v_1$  and assign the colour  $c_1$  to the vertex  $v_1$ . Since the vertex  $v_2$  is adjacent to the vertex  $v_1$ , we need a new colour  $c_2$  to colour  $v_2$ . Also, the vertex  $v_3$  is independent with  $v_1$  and adjacent with  $v_2$ , so we can give the colour  $c_1$  to  $v_3 \in V_1$ . Repeat the above process for each  $v_i; (i=1,2,\dots,n-1)$ . Since, the cycle  $C_1$  is of odd length. The vertex  $v_n$  is adjacent to both the colours  $c_1$  and  $c_2$ , so we need another new colour  $c_3$  to colour the vertex  $v_n$ . Hence, we need 3-colours to colour all the vertices of the cycle  $C_1$ . Now, to colour the vertices of  $V_2$ , assign the colours  $c_1$  and  $c_2$  to the vertices  $u_j; (j=1,2,\dots,k-1) \in V_2$  according to its adjacency with the corresponding  $v_j \in V_1$ . Repeat the above process for each  $u_{j+k} \in V_2$  by assigning the colours  $c_1$  and  $c_2$ . Also, the vertices  $u_{n-k+j} \in V_2$  are independent with the vertices which are coloured by the colour  $c_3$  and adjacent with the remaining two colours  $c_1$  and  $c_2$ . So, we can assign the colour  $c_3$  to the  $(u_{n-k+j})$ -vertices. Now, we can colour the remaining  $(u_{jk})$ -vertices of the cycle  $C_{2_k}$  by fixing the colour  $c_1$  to the vertex  $u_n$  and assigning the colours  $c_1, c_2$  and  $c_3$  to all other  $(u_{jk})$ -vertices according to its adjacency with other vertices. Hence, we need 3-colours to colour all the vertices of  $G$ . Therefore,  $\chi(G) = 3$ .

6) Type: II If  $gcd(n, k) = 1$ . the vertex set  $V_2$  form a cycle  $C_2$  of odd length. Therefore, both the cycles  $C_{2_i}; (i=1,2)$  are of odd length For example,  $GP(7,3)$  is represented in figure:2.4

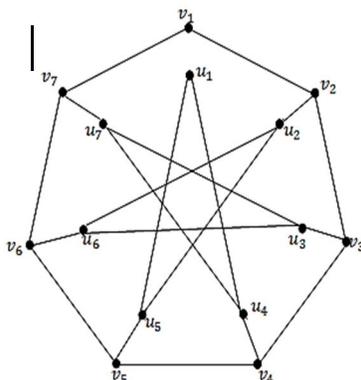


Figure:2.4

Fix the vertex  $v_1$  and assign the colour  $c_1$  to the vertex  $v_1$ .

into two sets  $V_1$  and  $V_2$  such that  $V_1(G) = \{v_i/i = 1, 2, \dots, n\}$

$V_2(G) = \{u_i/i = 1, 2, \dots, n\}$

Clearly, the vertices of  $V_{i;\{i=1,2\}}$  satisfies the condition  $V_1(G) \cap V_2(G) = \emptyset$ . Each vertex  $v_i$  is adjacent to the corresponding  $u_i$ .

The elements of  $V_1(G)$  form a cycle of length  $n$ . Let it be  $C_1$ .

7) Type: I If  $\gcd(n, k) = 1$  For example,  $GP(9,2)$  is represented in figure:2.5

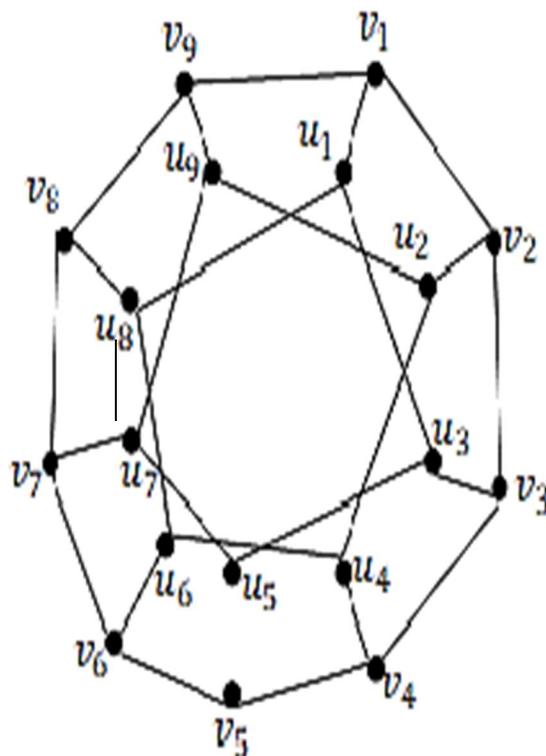


Figure:2.5

Clearly, the vertex set  $V_2$  form a cycle  $C_2$  of odd length.

Hence, both the cycles  $C_{2_i}; (i=1,2)$  are of odd length.

Fix the vertex  $v_1$  and assign the colour  $c_1$  to the vertex  $v_1$ .

Since the vertex  $v_2$  is adjacent to the vertex  $v_1$ , we need a new colour  $c_2$  to colour  $v_2$ .

Also, the vertex  $v_3$  is independent with  $v_1$  and adjacent with  $v_2$ , so we can give the colour  $c_1$  to  $v_3 \in V_1$ .

Repeat the above process for each  $v_i; (i=1,2,\dots,n-1)$ .

Since, the cycle  $C_1$  is of odd in length. The vertex  $v_n$  is adjacent to both the colours  $c_1$  and  $c_2$ , so we need another new colour  $c_3$  to colour the vertex  $v_n$ .

Hence, we need 3-colours to colour all the vertices of the cycle  $C_1$ .

Now, we can colour the vertices of the vertex set  $V_2$  by fixing the colour  $c_2$  to the vertex  $u_n$  and colour  $c_1$  to the vertex  $u_k$ .

The independent vertices  $u_{k+1}, u_{k+2}, u_{n-1}$  and  $u_{n-2}$  are adjacent to the vertices which are coloured by the colours  $c_1$  and  $c_2$  and independent to the colour  $c_3$ . So, we can give the colour  $c_3$  to these four vertices. The remaining  $u_j$ -vertices are coloured by the colours  $c_1$  and  $c_2$  according to its adjacency with the other vertices.

Hence, in this type we need 3-colours to colour all the vertices of  $G$ . Therefore,  $\chi(G) = 3$ .

If  $\gcd(n, k) = t$  and  $n/t$  is odd. The vertex set of  $V_2$  contains  $t$ -disjoint cycles of length  $n/t$ . Let it be  $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$ .

For example,  $GP(5,6)$  is represented in figure:2.6

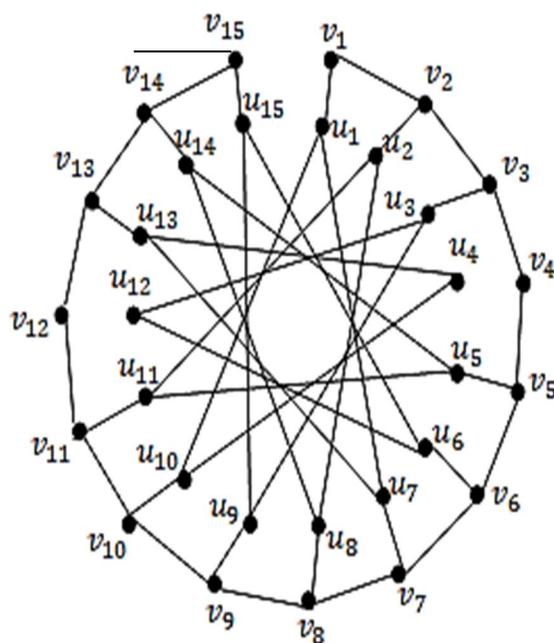


Figure:2.6

Clearly, the cycles  $C_1$  and  $C_{2i}; (i=1,2,\dots,t)$  are of odd in length.

Fix the vertex  $v_1$  and assign the colour  $c_1$  to the vertex  $v_1$ .

Since the vertex  $v_2$  is adjacent to the vertex  $v_1$ , we need a new colour  $c_2$  to colour  $v_2$ .

Also, the vertex  $v_3$  is independent with  $v_1$  and adjacent with  $v_2$ , so we can give the colour  $c_1$  to  $v_3 \in V_1$ .

Repeat the above process for each  $v_i; (i=1,2,\dots,n-1)$ . Since, the cycle  $C_1$  is of odd in length. The vertex  $v_n$  is adjacent to both the colours  $c_1$  and  $c_2$ , so we need another new colour  $c_3$  to colour the vertex  $v_n$ .

Hence, we need 3-colours to colour all the vertices of the cycle  $C_1$ .

Now, t colour the vertices of  $V_2$ , assign the colours  $c_1$  and  $c_2$  to the vertices  $u_j; (j=1,2,\dots,k-1) \in V_2$  according to its adjacency with the corresponding  $v_j \in V_1$ .

Repeat the above process for each  $u_{j+k} \in V_2$  by assigning the colours  $c_1$  and  $c_2$ . Also, the vertices  $u_{n-k+j} \in V_2$  are independent with the vertices which are coloured by the colour  $c_3$  and adjacent with the remaining two colours  $c_1$  and  $c_2$ . So, we can assign the colour  $c_3$  to the  $(u_{n-k+j})$ -vertices.

Now, we can colour the remaining  $(u_{jk})$ -vertices of the cycle  $C_{2k}$  by fixing the colour  $c_1$  to the vertex  $u_n$  and assigning the colours  $c_1, c_2$  and  $c_3$  to all other  $(u_{jk})$ -vertices according to its adjacency with other vertices.

Hence, we need 3-colours to colour all the vertices of G.

Therefore,  $\chi(G) = 3$ .

8) *Theorem:2.4* Let  $G$  be the generalized Petersen graph of type  $k$ , that is  $G = GP(n, k)$ , for all  $n$  and  $k$  are even, then  $\chi(G) = 3$ .

9) *Proof:* Let  $G = GP(n, k)$  be the generalized Petersen graph.

$(G)$  be the vertex set of  $G = GP(n, k)$ . It can be partitioned into two sets  $V_1$  and  $V_2$  such that

i.  $V_1(G) = \{v_i / i = 1, 2, \dots, n\}$

ii.  $V_2(G) = \{u_i / i = 1, 2, \dots, n\}$

From (i) and (ii), the vertices of  $V_{i; (i=1,2)}$  satisfies the condition  $V_1(G) \cap V_2(G) = \emptyset$ .

The elements of  $V_1(G)$  form a cycle of length  $n$ . Let it be  $C_1$ .

10) *Type: I* If  $\gcd(n, k) = k$ . For example,  $GP(8,2)$  is represented in figure:2.7

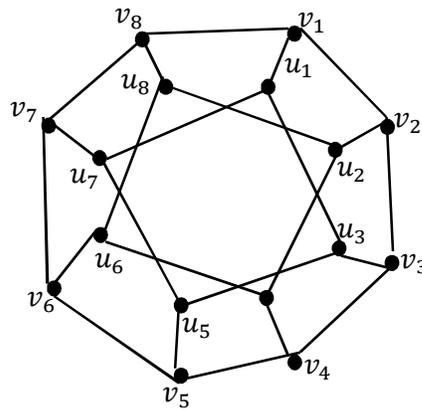


Figure:2.7

Since the cycle  $C_1$  is of even length.

By lemma:1,

We need 2-colours to colour all the vertices of  $C_1$ . Let it be  $c_1$  and  $c_2$ .

The vertices of  $V_2$  split into  $k$ -cycles. Let it be  $\{C_{2_1}, C_{2_2}, \dots, C_{2_k}\}$ . Each cycle  $C_{2_i}; (i=1,2,\dots,k)$  is of even in length.

The vertex  $\{v_i; (i=1,2,\dots,k)\}$  is adjacent to the vertex  $\{u_i; (i=1,2,\dots,k)\}$  which is coloured by the colour  $c_1$ . Therefore, we can assign the colour  $c_2$  to the vertex  $u_i$ .

But the vertices  $\{u_{i+k}; (i=1,2,\dots,k)\}$  and  $\{u_{n-k+i}; (i=1,2,\dots,k)\}$  are adjacent with the vertices which are coloured by the colours  $c_1$  and  $c_2$ .

Hence, we need a new colour  $c_3$  to colours these two disjoint vertices.

Hence, we need 3-colours to colour all the vertices of this type.

Therefore,  $\chi(G) = 3$ .

11) Type: II If  $\gcd(n, k) = t$  and  $n/t$  is odd. For example,  $GP(10,4)$  is represented in figure:2.8

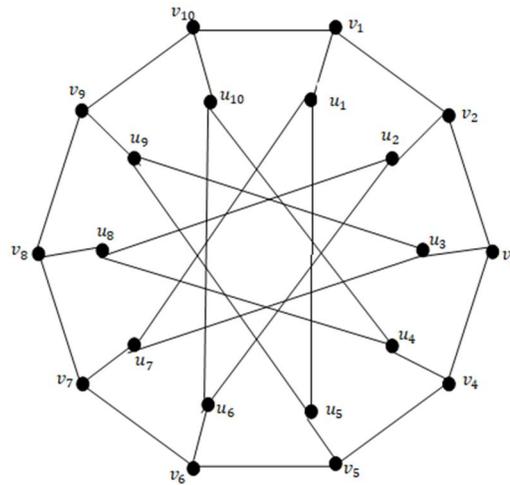


Figure:2.8

Since, the cycle  $C_1$  is of even in length.

By lemma:1,

We need 2-colours to colour all the vertices of  $C_1$ . Let it be  $c_1$  and  $c_2$ .

The vertices of  $V_2$  split into  $t$ -cycles. Let it be  $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$ . Each cycle  $C_{2_i}; (i=1,2,\dots,t)$  are of odd in length.

By lemma:1,

We need 3-colours to colour all the vertices  $C_{2_i}; (i=1,2,\dots,t)$  according to there adjacency with the vertices of  $C_1$ .

Therefore, we need 3-colours to colour all the vertices of this type.

Hence,  $\chi(G) = 3$ .

12) Type: III If  $\gcd(n, k) = t$  and  $n/t$  is even. For example,  $GP(16,6)$  is represented in figure:2.9

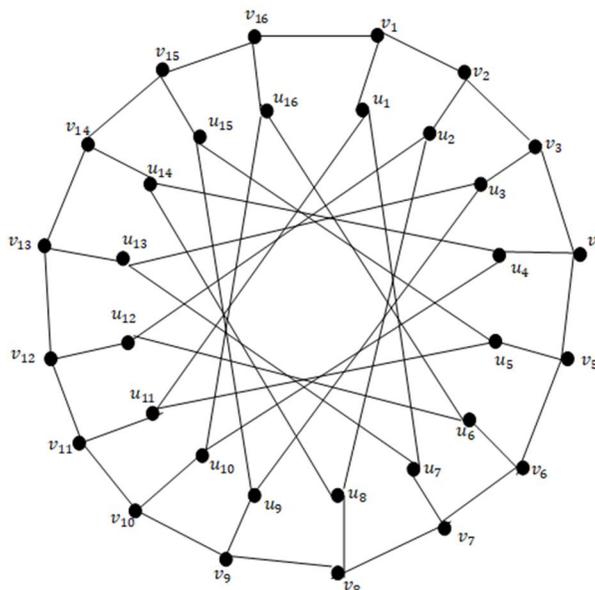


Figure:2.9

Since, the cycle  $C_1$  is of even in length.

By lemma:1,

We need 2-colours to colour all the vertices of  $C_1$ . Let it be  $c_1$  and  $c_2$ .

The vertices of  $V_2$  split into  $k$ -cycles. Let it be  $\{C_{2_1}, C_{2_2}, \dots, C_{2_t}\}$ . Each cycle  $C_{2_i}; (i=1,2,\dots,t)$  is of even in length.

The vertex  $v_i; (i=1,2,\dots,t)$  is adjacent to the vertex  $u_i; (i=1,2,\dots,t)$  which is coloured by the colour  $c_1$ . Therefore, we can assign the colour  $c_2$  to the vertex  $u_i$ .

But the vertices  $u_{i+k}; (i=1,2,\dots,t)$  and  $u_{n-k+i}; (i=1,2,\dots,t)$  are adjacent with the vertices which are coloured by the colours  $c_1$  and  $c_2$ .

Hence, we need a new colour  $c_3$  to colours these two disjoint vertices.

Hence, we need 3-colours to colour all the vertices of this type.

Therefore,  $\chi(G) = 3$ . Hence proved.

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