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On the Positive Pell Equation $y^2 = 32x^2 + 41$

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Abstract: The binary quadratic equation represented by the positive Pellian $y^2 = 32x^2 + 41$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola and parabola.
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I. INTRODUCTION

A binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-2]. For an extensive review of various problems, one may refer [3-12]. In this communication, yet another interesting hyperbola given by $y^2 = 32x^2 + 41$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are obtained.

A. *Method of Analysis*

The positive Pell equation representing hyperbola under consideration is

$$y^2 = 32x^2 + 41 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 2, \; y_0 = 13$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 32x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{8\sqrt{2}} g_n , \; \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(17 + 12\sqrt{2}\right)^{n+1} + \left(17 - 12\sqrt{2}\right)^{n+1}$$

$$g_n = \left(17 + 12\sqrt{2}\right)^{n+1} - \left(17 - 12\sqrt{2}\right)^{n+1} \quad , \; n = -1, 0, 1, 2......$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = f_n + \frac{13}{8\sqrt{2}} g_n$$

$$y_{n+1} = \frac{13}{2} f_n + 8\sqrt{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+3} - 34x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table: 1 below:

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Table: 1 Numerical examples

n	x_{n+1}	y_{n+1}
-1	2	13
0	73	413
1	2480	14029
2	84247	476573
3	2861918	16189453
4	97220965	549964829

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_{n+1} is alternatively even and odd, y_{n+1} is always odd.

2) Relations among the solutions

a) $x_{n+1} - 17x_{n+2} + 3y_{n+2} = 0$

b) $17x_{n+1} - 577x_{n+2} + 3y_{n+3} = 0$

c) $x_{n+1} - x_{n+3} + 6y_{n+2} = 0$

d) $x_{n+1} - 577x_{n+3} + 102y_{n+3} = 0$

e) $17x_{n+1} - x_{n+2} + 3y_{n+1} = 0$

f) $577x_{n+1} - x_{n+3} + 102y_{n+1} = 0$

g) $96x_{n+1} + 17y_{n+1} - y_{n+2} = 0$

h) $3264x_{n+1} + 577y_{n+1} - y_{n+3} = 0$

i) $96x_{n+1} + 577y_{n+2} - 17y_{n+3} = 0$

j) $1154x_{n+2} - 17x_{n+3} + 3y_{n+3} = 0$

k) $17x_{n+2} - x_{n+3} + 3y_{n+2} = 0$

l) $x_{n+2} - 17x_{n+3} + 3y_{n+3} = 0$

m) $96x_{n+2} - 17y_{n+2} + y_{n+1} = 0$

n) $192x_{n+2} + y_{n+1} - y_{n+3} = 0$

o) $96x_{n+2} - y_{n+3} + 17y_{n+2} = 0$

p) $192x_{n+2} - y_{n+3} + 697y_{n+1} = 0$

q) $96x_{n+3} - 577y_{n+2} + 17y_{n+1} = 0$

r) $3264x_{n+3} - 577y_{n+3} + y_{n+1} = 0$

s) $96x_{n+3} + y_{n+2} - 17y_{n+3} = 0$

t) $y_{n+1} - 102y_{n+2} + y_{n+3} = 0$

3) Each of the following expressions represents a nasty number

a) $\frac{4}{41}(13x_{2n+3} - 413x_{2n+2} + 123)$

b) $\frac{2}{697}(13x_{2n+4} - 14029x_{2n+2} + 4182)$

c) $\frac{4}{41}(39y_{2n+2} - 192x_{2n+2} + 123)$

d) $\frac{6}{697}(26y_{2n+3} - 4672x_{2n+2} + 1394)$

e) $\frac{4}{23657}(39y_{2n+4} - 238080x_{2n+2} + 70971)$

f) $\frac{4}{41}(413x_{2n+4} - 14029x_{2n+3} + 123)$

g) $\frac{4}{697}(1239y_{2n+2} - 192x_{2n+3} + 2091)$

h) $\frac{4}{41}(1239y_{2n+3} - 7008x_{2n+3} + 123)$

i) $\frac{4}{697}(1239y_{2n+4} - 238080x_{2n+3} + 2091)$

j) $\frac{4}{23657}(42087y_{2n+2} - 192x_{2n+4} + 70971)$

k) $\frac{4}{697}(42087y_{2n+3} - 7008x_{2n+4} + 2091)$

l) $\frac{4}{41}(42087y_{2n+4} - 238080x_{2n+4} + 123)$

m) $\frac{4}{41}(73y_{2n+2} - 2y_{2n+3} + 123)$

n) $\frac{4}{697}(1240y_{2n+2} - y_{2n+4} + 2091)$

o) $\frac{4}{41}(2480y_{2n+3} - 73y_{2n+4} + 123)$

4) Each of the following expressions represents a cubical integer

a) $\frac{1}{123}[78x_{n+2} - 2478x_{n+1} + 26x_{3n+4} - 826x_{3n+3}]$

b) $\frac{1}{2091}[39x_{n+3} - 42087x_{n+1} + 13x_{3n+5} - 14029x_{3n+3}]$

c) $\frac{1}{41}[78y_{n+1} - 384x_{n+1} + 26y_{3n+3} - 128x_{3n+3}]$

d) $\frac{1}{697}[78y_{n+2} - 14016x_{n+1} + 26y_{3n+4} - 4672x_{3n+3}]$

e) $\frac{1}{23657}\left[78y_{n+3} - 476160x_{n+1} + 26y_{3n+5} - 158720x_{3n+3}\right]$

f) $\frac{1}{123}\left[2478x_{n+3} - 84174x_{n+2} + 826x_{3n+5} - 28058x_{3n+4}\right]$

g) $\frac{1}{697}\left[2478y_{n+1} - 384x_{n+2} + 826y_{3n+3} - 128x_{3n+4}\right]$

h) $\frac{1}{41}\left[2478y_{n+2} - 14016x_{n+2} + 826y_{3n+4} - 4672x_{3n+4}\right]$

i) $\frac{1}{697}\left[2478y_{n+3} - 476160x_{n+2} + 826y_{3n+5} - 158720x_{3n+4}\right]$

j) $\frac{1}{23657}\left[84174y_{n+1} - 384x_{n+3} + 28058y_{3n+3} - 128x_{3n+5}\right]$

k) $\frac{1}{697}\left[84174y_{n+2} - 14016x_{n+3} + 28058y_{3n+4} - 4672x_{3n+5}\right]$

l) $\frac{1}{41}\left[84174y_{n+3} - 476160x_{n+3} + 28058y_{3n+5} - 158720x_{3n+5}\right]$

m) $\frac{1}{123}\left[438y_{n+1} - 12y_{n+2} + 146y_{3n+3} - 4y_{3n+4}\right]$

n) $\frac{1}{2091}\left[7440y_{n+1} - 6y_{n+3} + 2480y_{3n+3} - 2y_{3n+5}\right]$

o) $\frac{1}{123}\left[14880y_{n+2} - 438y_{n+3} + 4960y_{3n+4} - 146y_{3n+5}\right]$

5) Each of the following expressions represents a bi-quadratic integer

a) $\frac{1}{123}\left[104x_{2n+3} - 3304x_{2n+2} + 26x_{4n+5} - 826x_{4n+4} + 738\right]$

b) $\frac{1}{2091}\left[52x_{2n+4} - 56116x_{2n+2} + 13x_{4n+6} - 14029x_{4n+4} + 12546\right]$

c) $\frac{1}{41}\left[104y_{2n+2} - 512x_{2n+2} + 26y_{4n+4} - 128x_{4n+4} + 246\right]$

d) $\frac{1}{697}\left[104y_{2n+3} - 18688x_{2n+2} + 26y_{4n+5} - 4672x_{4n+4} + 4182\right]$

e) $\frac{1}{23657}\left[104y_{2n+4} - 634880x_{2n+2} + 26y_{4n+6} - 158720x_{4n+4} + 141942\right]$

f) $\frac{1}{123}\left[3304x_{2n+4} - 112232x_{2n+3} + 826x_{4n+6} - 28058x_{4n+5} + 738\right]$

g) $\frac{1}{697}\left[3304y_{2n+2} - 512x_{2n+3} + 826y_{4n+4} - 128x_{4n+5} + 4182\right]$

h) $\frac{1}{41}\left[3304y_{2n+3} - 18688x_{2n+3} + 826y_{4n+5} - 4672x_{4n+5} + 246\right]$

i) $\frac{1}{697}\left[3304y_{2n+4}-634880x_{2n+3}+826y_{4n+6}-158720x_{4n+5}+4182\right]$

j) $\frac{1}{23657}\left[112232y_{2n+2}-512x_{2n+4}+28058y_{4n+4}-128x_{4n+6}+141942\right]$

k) $\frac{1}{697}\left[112232y_{2n+3}-18688x_{2n+4}+28058y_{4n+5}-4672x_{4n+6}+4182\right]$

l) $\frac{1}{41}\left[112232y_{2n+4}-634880x_{2n+4}+28058y_{4n+6}-158720x_{4n+6}+246\right]$

m) $\frac{1}{123}\left[584y_{2n+2}-16y_{2n+3}+146y_{4n+4}-4y_{4n+5}+738\right]$

n) $\frac{1}{2091}\left[9920y_{2n+2}-8y_{2n+4}+2480y_{4n+4}-2y_{4n+6}+12546\right]$

o) $\frac{1}{123}\left[19840y_{2n+3}-584y_{2n+4}+4960y_{4n+5}-146y_{4n+6}+738\right]$

6) Each of the following expressions represents a quintic integer

a) $\frac{1}{123}\left[260x_{n+2}-8260x_{n+1}+130x_{3n+4}-4130x_{3n+3}+26x_{5n+6}-826x_{5n+5}\right]$

b) $\frac{1}{2091}\left[130x_{n+3}-140290x_{n+1}+65x_{3n+5}-70145x_{3n+3}+13x_{5n+7}-14029x_{5n+5}\right]$

c) $\frac{1}{41}\left[260y_{n+1}-1280x_{n+1}+130y_{3n+3}-640x_{3n+3}+26y_{5n+5}-128x_{5n+5}\right]$

d) $\frac{1}{697}\left[260y_{n+2}-46720x_{n+1}+130y_{3n+4}-23360x_{3n+3}+26y_{5n+6}-4672x_{5n+5}\right]$

e) $\frac{1}{23657}\left[260y_{n+3}-1587200x_{n+1}+130y_{3n+5}-793600x_{3n+3}+26y_{5n+7}-158720x_{5n+5}\right]$

f) $\frac{1}{123}\left[8260x_{n+3}-280580x_{n+2}+4130x_{3n+5}-140290x_{3n+4}+826x_{5n+7}-28058x_{5n+6}\right]$

g) $\frac{1}{697}\left[12390y_{n+1}-1920x_{n+2}-128x_{5n+6}+826y_{5n+5}\right]$

h) $\frac{1}{41}\left[8260y_{n+2}-46720x_{n+2}+4130y_{3n+4}-23360x_{3n+4}+826y_{5n+6}-4672x_{5n+6}\right]$

i) $\frac{1}{697}\left[8260y_{n+3}-1587200x_{n+2}+4130y_{3n+5}-793600x_{3n+4}+826y_{5n+7}-158720x_{5n+6}\right]$

j) $\frac{1}{23657}\left[280580y_{n+1}-1280x_{n+3}+140290y_{3n+3}-640x_{3n+5}+28058y_{5n+5}-128x_{5n+7}\right]$

k) $\frac{1}{697}\left[280580y_{n+2}-46720x_{n+3}-23360x_{3n+5}+140290y_{3n+4}+28058y_{5n+6}-4672x_{5n+7}\right]$

l) $\frac{1}{41}\left[280580y_{n+3}-1587200x_{n+3}+140290y_{3n+5}-793600x_{3n+5}+28058y_{5n+7}-158720x_{5n+7}\right]$

m) $\frac{1}{123}\left[1460y_{n+1} - 40y_{n+2} + 730y_{3n+3} - 20y_{3n+4} + 146y_{5n+5} - 4y_{5n+6}\right]$

n) $\frac{1}{2091}\left[24800y_{n+1} - 20y_{n+3} + 12400y_{3n+3} - 10y_{3n+5} + 2480y_{5n+5} - 2y_{5n+7}\right]$

o) $\frac{1}{123}\left[49600y_{n+2} - 1460y_{n+3} + 24800y_{3n+4} - 730y_{3n+5} + 4960y_{5n+6} - 146y_{5n+7}\right]$

B. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table: 2 below:

Table: 2 Hyperbolas

S. No	Hyperbolas	(X, Y)
1	$X^2 - 32Y^2 = 60516$	$(26x_{n+2} - 826x_{n+1}, 146x_{n+1} - 4x_{n+2})$
2	$X^2 - 32Y^2 = 69956496$	$(26x_{n+3} - 28058x_{n+1}, 4960x_{n+1} - 4x_{n+3})$
3	$X^2 - 32Y^2 = 6724$	$(26y_{n+1} - 128x_{n+1}, 26x_{n+1} - 4y_{n+1})$
4	$X^2 - 32Y^2 = 1943236$	$(26y_{n+2} - 4672x_{n+1}, 826x_{n+1} - 4y_{n+1})$
5	$X^2 - 32Y^2 = 2238614596$	$(26y_{n+3} - 158720x_{n+1}, 28058x_{n+1} - 4y_{n+3})$
6	$X^2 - 32Y^2 = 60516$	$(826x_{n+3} - 28058x_{n+2}, 4960x_{n+2} - 146x_{n+3})$
7	$X^2 - 32Y^2 = 1943236$	$(826y_{n+1} - 128x_{n+2}, 26x_{n+1} - 146y_{n+1})$
8	$X^2 - 32Y^2 = 6724$	$(826y_{n+2} - 4672x_{n+2}, 826x_{n+2} - 146y_{n+2})$
9	$X^2 - 32Y^2 = 1943236$	$(826y_{n+3} - 158720x_{n+2}, 28058x_{n+2} - 146y_{n+3})$
10	$X^2 - 32Y^2 = 2238614596$	$(28058y_{n+1} - 128x_{n+3}, 26x_{n+3} - 4960y_{n+1})$
11	$X^2 - 32Y^2 = 1943236$	$(28058y_{n+2} - 4672x_{n+3}, 826x_{n+3} - 4960y_{n+2})$
12	$X^2 - 32Y^2 = 6724$	$(28058y_{n+3} - 158720x_{n+3}, 28058x_{n+3} - 4960y_{n+3})$
13	$32X^2 - Y^2 = 1936512$	$(146y_{n+1} - 4y_{n+2}, 26y_{n+2} - 826y_{n+1})$
14	$32X^2 - Y^2 = 2238607872$	$(4960y_{n+1} - 4y_{n+3}, 26y_{n+3} - 28058y_{n+1})$
15	$32X^2 - Y^2 = 1936512$	$(4960y_{n+2} - 146y_{n+3}, 826y_{n+3} - 28058y_{n+2})$

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2) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table: 3 below:

Table: 3 Parabolas

S. No	Parabolas	(X, Y)
1	$123X - 32Y^2 = 30258$	$(26x_{2n+3} - 826x_{2n+2}, 146x_{n+1} - 4x_{n+2})$
2	$4182X - 32Y^2 = 34978248$	$(26x_{2n+4} - 28058x_{2n+2}, 4960x_{n+1} - 4x_{n+3})$
3	$41X - 32Y^2 = 3362$	$(26y_{2n+2} - 128x_{2n+2}, 26x_{n+1} - 4y_{n+1})$
4	$697X - 32Y^2 = 971618$	$(26y_{2n+3} - 4672x_{2n+2}, 826x_{n+1} - 4y_{n+2})$
5	$23657X - 32Y^2 = 1119307298$	$(26y_{2n+4} - 158720x_{2n+2}, 28058x_{n+1} - 4y_{n+3})$
6	$123X - 32Y^2 = 30258$	$(826x_{2n+4} - 28058x_{2n+3}, 4960x_{n+2} - 146x_{n+3})$
7	$697X - 32Y^2 = 971618$	$(826y_{2n+2} - 128x_{2n+3}, 26x_{n+2} - 146y_{n+1})$
8	$41X - 32Y^2 = 3362$	$(826y_{2n+3} - 4672x_{2n+3}, 826x_{n+2} - 146y_{n+2})$
9	$697X - 32Y^2 = 971618$	$(826y_{2n+4} - 158720x_{2n+3}, 28058x_{n+2} - 146y_{n+3})$
10	$23657X - 32Y^2 = 1119307298$	$(28058y_{2n+2} - 128x_{2n+4}, 26x_{n+3} - 4960y_{n+1})$
11	$697X - 32Y^2 = 971618$	$(28058y_{2n+3} - 4672x_{2n+4}, 826x_{n+3} - 4960y_{n+2})$
12	$41X - 32Y^2 = 3362$	$(28058y_{2n+4} - 158720x_{2n+4}, 28058x_{n+3} - 4960y_{n+3})$
13	$3936X - Y^2 = 968256$	$(146y_{2n+2} - 4y_{2n+3}, 26y_{n+2} - 826y_{n+1})$
14	$133824X - Y^2 = 1119303936$	$(4960y_{2n+2} - 4y_{2n+4}, 26y_{n+3} - 28058y_{n+1})$
15	$3936X - Y^2 = 968256$	$(4960y_{2n+3} - 146y_{2n+4}, 826y_{n+3} - 28058y_{n+2})$

REFERENCES

[1] Telang S.J., Number Theory, Tata Mc Graw Hill Publishing Company Limited, New Delhi-2000.
[2] David M.Burton, Elementary Number Theory, Tata Mc Graw Hill Publishing Company Limited, New Delhi-2002.
[3] Geetha T., Gopalan M.A., Vidhyalakshmi S., Observation on the hyperbola, $y^2 = 72x^2 + 1$, Scholars Journal of Physics, Mathematics and Statistics, 1(1), 2014, 1-13.
[4] Gopalan M.A, Vidhyalakshmi S., Umarani J., Remarkable Observations on the hyperbola $y^2 = 24x^2 + 1$, Bulletin of Mathematics and Statistics Research, 1, 2014, 9-12.
[5] Gopalan M.A , Vidhyalakshmi S, Premalatha E, Vanitha M, Observation on the hyperbola, $y^2 = 220x^2 + 9$, Star Research Journal, Vol.4, Issue3(2), 2016, 12-16.
[6] Ramya S, Kavitha A. On the positive pell equation, $y^2 = 90x^2 + 31$, Journal of Mathematics and informatics, 11(1), 2017, 11-17.
[7] Sumathi G., Observations on the hyperbola, $y^2 = 150x^2 + 16$, International Journal of Recent Trends in Engineering and Research, 3(9), 2017, 198-206.
[8] Meena K, Gopalan M.A., Sivaranjani V, On the positive pell equation $y^2 = 102x^2 + 33$. International Journal of Advanced Education and Research 2(1), 2017, 91-96
[9] Gopalan M.A, Devibala S., Swetha T., On the positive pell equation $y^2 = 5x^2 + 4$, IOSR Journal of Mathematics, Vol 13, Issue 5, Ver.1, 2017, 65-69.
[10] Usha Rani T.R, Dhivya K, On the positive pell equation $y^2 = 14x^2 + 18$, International Journal of Academic Research and Development, Volume 3, Issue 3, 2018, 43-50.
[11] Shreemathi Adiga, Anusheela N. and Gopalan M.A., On the Hyperbola $y^2 = 3x^2 + 4$, IJLEMR, 03(06), 2018, 45-48
[12] Shreemati Adiga, Anusheela N. and Gopalan M.A., On the Positive Pellian Equation $y^2 = 48x^2 + 16$, IJMTT, 59(4), 2018, 256-264