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Quaternion Doubly Stochastic Matrices Over Quaternion Vector Spaces and the Extreme Points on a Birkhoff's Theorem

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Abstract: Quaternion doubly stochastic matrices are described which have entries from an quaternion vector space H. The extreme points of this convex set of matrices are studied, and convex subsets of H are identified from which these extreme matrices are of a permutation matrix type. i.e., for which a Birkhoff theorem holds.
Keywords: quaternion doubly stochastic matrix, extreme points, Birkhoff's theorem, convex set.

I. INTRODUCTION

Much of the research in this area has concentrated on extending this result. Problem III, [3], recently a new directions for this work was introduced in independent papers by peter M.Gibson (4) and M.H. Clapp and R.C.Shiflett (3), which generalize the entries of the matrix from real numbers to more general algebraic systems.

Gibson considers doubly stochastic matrices with entries from a ring, while clap and shiflett work entries from convex subsets of the unit square.

Our main concern to understand the geometric nature of the set of quternion doubly stochastic matrices through the extreme points of the set and to determine when a Birkhoff's theorem holds.

II. NOTATION AND TERMINOLOGY

In this paper H will be an quaternion vector space and Z will be a convex subset of H. We will denote the set of extreme points of any convex set C by Ext C.

The convex set Z assumed to have the zero element in H in its extreme points. i.e., $0 \in \mathrm{Ext}\ Z$, and a fixed non-zero in u in $\mathrm{Ext}\ Z$ letters such as s,t,m and n are natural numbers.

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1) *Definition 3.1:* An $n \times n$ quaternion doubly stochastic matrix M with entries from Z is called Z-quaternion doubly stochastic relative to u if the row and column sums of m are all u.

The set of all $n \times n$ Z-quaternion doubly stochastic matrices relative to u is denoted by $M_n(Z)$. For $M \in M_n(Z)$, we write

$M=[x_{st}]$, where $x_{st} \in Z$ and $\sum_{t=1}^{n} x_{st} = \sum_{s=1}^{n} x_{st} = u$

2) *Definition 3.2:* Let $\{u_s : s = 1, 2, \dots, Z\}$ be a set of Z-linearly independents vectors from H. The Z-box[1] spanned by these vectors is the set. $s[0, u_1, u_2, \dots, u_Z] = \left\{ x = \sum_{s=1}^{n} \alpha_s u_s : 0 \le \alpha_s \le 1 \right\}$. Where $u = \sum_{s=1}^{n} u_s$.

III. BIRKHOFF'S THEOREM ON Z-BOXES

A matrix M is an extreme point of the set of $n \times n$ quaternion doubly stochastic matrices if and only if each entry of M is extreme in $[0,1]$

A. Proposition 4.1

$M_n(Z)$ is convex

Proof

Consider $M_1 = [x_{st}]$ and $M_2 = [y_{st}]$ in $M_n(Z)$. Then $\alpha M_1 + (1-\alpha)M_2 = [\alpha x_{st} + (1-\alpha)y_{st}]$ and $\alpha x_{st} + (1-\alpha)y_{st} \in Z$ for every $\alpha \in [0,1]$ finally $\sum_{t=1}^{n}[\alpha x_{st} + (1-\alpha)y_{st}] = \alpha\sum_{t=1}^{n} x_{st} + (1-\alpha)\sum_{t=1}^{n} y_{st} = \alpha u + (1-\alpha)u = u$ .Since the same works for s, $\alpha M_1 + (1-\alpha)M_2 \in M_n(Z)$

B. Proposition 4.2

If for $M = [x_{st}] \in M_n(Z)$ every entry x_{st} is in Ext Z, then $M \in \text{Ext } M_n(Z)$

Proof

If $M \notin \text{Ext } M_n(Z)$ then there exists $M_1 = [y_{st}]$ and $M_2 = [z_{st}]$ in $M_n(Z)$ such that $M_1 \neq M_2$ and $M = \frac{1}{2}(M_1 + M_2)$.

However, this implies for some entry $x_{st} = \frac{1}{2}(y_{st} + Z_{st})$ with $y_{st} \neq Z_{st}$ and both in Z. Therefore $x_{st} \notin \text{Ext } Z$.

1) *Theorem 4.3:* (The Birkhoff's theorem for Z-boxes)

Let Z be the Z-box spanned by the linearly independent vectors $\{u_1, u_2, ..., u_z\}$ with $\sum_{s=1}^{n} u_s = u$. Then $M = [x_{st}]$ is in $\text{Ext } M_n(Z)$ if and only if its entries x_{st} is in Ext Z.

Proof

The proof is given in a series of lemmas.

2) *Lemma 4.4:* Let $M_n(Z_2)$ and $M_n(Z_2)$ be the $n \times n$ quaternion doubly stochastic matrices over Z_1 relative to the extreme point $u_1 \neq 0$ and Z_2 relative to the extreme point $u_2 \neq 0$, respectively, If $u_1 + u_2 = u$ and $Z_1 + Z_2 = Z$, then $M_n(Z_1) + M_n(Z_2) \subset M_n(Z)$.

Proof

If $M_1 = [x_{st}] \in M_n(Z_1)$ then $\sum_s x_{st} = \sum_t x_{st} = u_1$. Similarly for $M_2 = [y_{st}] \in M_n(Z_2)$. We have $\sum_s y_{st} = \sum_t y_{st} = u_2$.

Thus $M = M_1 + M_2 = [x_{st} + y_{st}] \in M_n(Z_1 + Z_2)$ is in $M_n(Z)$. Since $\sum_s (x_{st} + y_{st}) = \sum_t (x_{st} + y_{st}) = u_1 + u_2 = u$.

3) *Lemma 4.5:* Let C, C_1 and C_2 be convex subsets of some vector space H with $C_1 + C_2 \subset C$, and let $x_1 + x_2 = x$ with $x_1 \in C_1$ and $x_2 \in C_2$, if $x \in \text{Ext } C$ then $x_1 \in \text{Ext } C_1$ and $x_2 \in \text{Ext } C_2$.

Proof

Suppose $x_1 \notin \text{Ext } C_1$; then $x_1 = \alpha y + \alpha \in (0,1)$ where y and z are in C_1 but not equal to x_1, consequently $x = \alpha(y + x_2) + (1-\alpha)(Z + x_2)$ and so x is not in Ext C.

4) *Lemma 4.6:* If Z is the Z-box $S[0,u_1,\ldots,u_z]$, where u_S are linearly independent then $I=\sum_{S=1}^{m} u_{s_s} \in \text{Ext } Z$. For any collection of subscripts $S_1,S_2,\ldots,S_m$.

Proof

Let $Z=\sum_{S=1}^{m} u_{s_s} = E\sum_{S=1}^{k}\alpha_s u_s + (1-E)\sum_{S=1}^{k}\beta_s u_s$ where α_s and β_s come from $[0,1]$ and $E\in(0,1)$. Then $\sum_{S=1}^{k}[E\alpha_s+(1-E)\beta_s]u_s - \sum_{S=1}^{Z}[E\alpha_s+(1-E)\beta_s]u_s - \sum_{S=1}^{m} u_{s_t} = 0$. By linear independence $E\alpha_{s_s}+(1-E)\beta_{s_s}=1$ for subscripts $S_1,\ldots,S_m$ and $E\alpha_s+(1-E)\beta_s=0$. Otherwise thus $\alpha_{s_t}=\beta_{s_t}=1$ for $S_1,\ldots,S_m$ and $\alpha_s=\beta_s=0$ otherwise So $I\in \text{Ext } Z$.

5) *Definition 5:* Let H be a convex subset of a quaternion vector space. A point $M\in H$ is called an extreme point if it is not an interior point of any line segment in H.That is M is extreme if and only if whenever $M=\alpha M_1+(1-\alpha)M_2$, $\alpha\in(0,1), M_1\neq M_2$ implies either $M_1\neq H$ (or) $M_2\neq H$.

a) *Example 5.1:* The set $[0,1]$ is a convex set and 0,1 are the extreme points.

b) *Lemma 5.2:* Let $Z_1+Z_2=Z$, with $u_1+u_2=u$, as in Lemma(3.5), if $M=M_1+M_2$, $M_1\in M_n(Z_1)$ and $M_2\in M_n(Z_2)$ and if $M\in \text{Ext } M_n(Z)$ then $M_1\in \text{Ext } M_n(Z_1)$ and $M_2\in \text{Ext } M_n(Z_2)$.

c) *Lemma 5.3:* Let u_1 and u_2 be linearly independent, $Z_1=S[0,u_1]$ and $Z_2=S[0,u_2]$ with Z the Z-box, $Z=Z_1+Z_2$, $u_1+u_2=u$. Then $M=[x_{st}]$ is in $\text{Ext } M_n(Z)$ if and only if each of its entries x_{st} is in $\text{Ext} Z$.

Proof

Suppose that $M=[x_{st}]\in \text{Ext } M_n(Z)$. Then $x_{st}=\alpha_{st}u_1+\beta_{st}u_2$, where α_{st} and β_{st} come from $[0,1]$ Since $u=\sum_S x_{st}=\sum_S(\alpha_s+u_1+\beta_s+u_2)=\left(\sum_S\alpha_{st}\right)u_1+\left(\sum_S\beta_{st}\right)u_2$ and $u_1+u_2=u$, the linear independent of u_1 and u_2 implies $\sum_S\alpha_{st}=\sum_S\beta_{st}=1$ .Similarly $\sum_S\alpha_{st}=\sum_S\beta_{st}=1$. This say that $M_1=\left[\alpha_{st}u_1\right]\in M_n(Z_1)$ and $M_2=\left[\beta_{st}u_2\right]\in M_n(Z_2)$ with $M=M_1+M_2$. By lemma (5.3) $M_1\in \text{Ext } M_n(Z_1)$ and $M_2\in \text{Ext } M_n(Z_2)$. Then by proposition (4.2), the entries of M_1 are either 0 or u_1 and for M_2 they are 0 or u_2. Thus the entries of M are from $\{0,u,u_1,u_2,u_1+u_2=u\}=\text{Ext } Z$.

The final step in the proof of our theorem is an induction.let Z be a Z-box $S[0,u_1,\ldots,u_Z]$ $\sum_{S=1}^{Z}u_S=u$ .Let $Z_1=S\left[0,u_Z\right]$ and $k_2\left[0,u_1,\ldots,u_{Z-1}\right]$ so that $Z=Z_1+Z_2$.

Proposition (4.2) and Lemma (5.3) get the induction started. Suppose our theorem holds for all sets which are $(Z-1)$ boxes. Let $M=\left[x_{st}\right]\in \text{Ext } M_n(Z)$ then

$$x_{st}=\sum_{l=1}^{Z}\alpha_{Stl}u_l\ ,\ \alpha_{Stl}\in[0,1]\ ,\ u=\sum_{S=1}^{n}x_{st}=\sum_{S=1}^{n}\left(\sum_{l=1}^{Z}\alpha_{stl}u_l\right)=\sum_{l=1}^{Z}\left(\sum_{S=1}^{n}\alpha_{stl}\right)u_l$$

However $u=\sum_{l=1}^{Z} u_l$, and by linear independence, $\sum_{S=1}^{n}\alpha_{stl}=1$. Similarly $\sum_{t=1}^{n}\alpha_{stl}=1$. Now, we may write

$M=[\alpha_{StZ}u_Z]+\sum_{l=1}^{Z-1}[\alpha_{stl}u_l]$ and $\sum_{S=1}^{n}\alpha_{stZ}u_Z=\sum_{t=1}^{n}\alpha_{stZ}u_Z=u_Z$. Thus $M_1=[\alpha_{stZ}u_Z]\in M_n(Z_1)$. Furthermore,

$$\sum_{S=1}^{n}\sum_{l=1}^{Z-1}\alpha_{stl}u_l=\sum_{l=1}^{Z-1}\left(\sum_{S=1}^{n}\alpha_{stl}\right)u_l=\sum_{l=1}^{Z-1}u_l=u-u_Z \text{ and } M_2=\left[\sum_{l=1}^{Z-1}\alpha_{stl}u_l\right]\in M_n(Z_2).$$

From the lemmas we have $M_1\in \mathrm{Ext}\ M_n(Z_1)$ and $M_2\in \mathrm{Ext}\ M_n(Z_2)$ and the entries of M_1 come from $\{0,u_Z\}$. By the induction hypothesis, the entries of M_2 come from $\mathrm{Ext}\ k_2$. Thus every entry of M is extreme by lemma (4.6).

IV. BIRKHOFF'S THEOREM

The set of $n\times n$ quaternion doubly stochastic matrices is a convex set whose extreme points are the permutation matrices.

Proof

1) The class of $n\times n$ quaternion doubly stochastic matrices is a convex set and is called under multiplication and the adjoint operators. It is however not a group.
2) Every permutation is doubly stochastic and is an extreme point of the convex set of all quaternion doubly stochastic matrices.

The harder part is showing that every extreme point is a permutation matrix. For this we need to show that each quaternion doubly stochastic matrices is a convex combination of permutation matrices.

This is proved by induction on the number of positive entries of the matrix. Note that if A is quaternion doubly stochastic, then it has at least n positive entries. If the number of positive entries is exactly n, then A is a permutation matrix.

We first show that if A is quaternion doubly stochastic , then A has at least one diagonal with no zero entry. Choose any $k\times l$ submatrix of zeros that A might have. We find permutation matrices P_1, P_2 such that P_1AP_2 has

$$P_1AP_2=\begin{bmatrix} O & B \\ C & D \end{bmatrix}$$

A. *Corollary 6.1*

1) The class of $n\times n$ quaternion doubly stochastic matrices is a convex set and is closed under multiplication and the adjoint operator. It is however, not a group.
2) Every permutation matrix is quaternion doubly stochastic and is an extreme point of the convex set.

B. *Lemma 6.2*

Let A be a quaternion doubly stochastic matrix. Show that all eigen values of A have modules less than or equal to 1, that 1 is an eigen value of A and that $\|A\|=1$.

Proof

If A is quaternion doubly stochastic then $|Ax|\le A(\ |x|\ )$ where, as usual, $|x|=(\ |x|=(\ |x_1|,\ldots,|x_n|\ )$ and we say that $x\le y$ if $x_j\le y_j$ for all j.

C. *Theorem 6.3*

A matrix A is quaternion doubly stochastic if and only if $A_x<x$ for all vectors x.

Proof

Let $A_x<x$ for all x. First choosing x to be e and then $e_i=(0,0,\ldots,1,0,\ldots,0)\ 1\le i\le n$, one can easily see that A is quaternion doubly stochastic matrix.

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Conversely, let A be quaternion doubly stochastic. Let $y = A_x$ to prove $y < x$ we may assume, with out loss of generality that the co-ordinates of both x and y are in decreasing order. Now note that for any k, $1 \le k \le n$,

$$\sum_{s=1}^{n} y_s = \sum_{s=1}^{k}\sum_{t=1}^{n} a_{st} x_s$$

If we put $U_s = \sum_{s=1}^{k} a_{st}$, then $0 \le U_i \le 1$ and $\sum_{s=1}^{n} U_s = k$. We have

$$\sum_{s=1}^{n} y_s - \sum_{s=1}^{k} x_s = \sum_{s=1}^{n} u_s x_s - \sum_{s=1}^{k} x_s = \sum_{s=1}^{n} u_s x_s - \sum_{s=1}^{k} x_s + (k - \sum_{s=1}^{n} u_i) x_k$$

$$= \sum_{s=1}^{n} (u_{s-1})(x_s - x_k) + \sum_{s=k+1}^{n} u_s (x_s - x_k)$$

Further, when $k = n$, we must have equality here simply because A is quaternion doubly stochastic matrix. Thus $y < x$.

D. *Theorem 6.4*

Is the convex combination of quaternion doubly stochastic matrices is also a quaternion doubly stochastic matrices.

Proof

Let's say A and B be two quaternion doubly stochastic matrices.

$M = (1-m)A + mB$, $M \in R$, $0 < m < 1$

Sum of the i^{th} row as $\sum_{j=1}^{n} m_{ij}$

$$\sum_{U=1}^{n} m_{ij} = \sum_{j=1}^{n} ((1-m)a_{ij} + mb_{ij}) = (1-m)\sum_{j=1}^{n} a_{ij} + m\sum_{j=1}^{n} b_{ij} = (1-m) + m = 1$$

E. *Theorem 6.5*

Show that if a square matrix is quaternion doubly stochastic skew-symmetric then it diagonal entries must all be 0.

Proof

Let A be an $n \times n$ quaternion doubly stochastic skew-symmetric doubly stochastic matrix.

Then by definition $A^T = -A$

Let $A = (a_{ij})$ $1 \le i, j \le n$, $A^T = (a_{ij})^T = a_{ji}$, $1 \le i, j \le n$, $A^T = -A = -(a_{ij})$, $1 \le i, j \le n$, When $i = j$,

$a_{jj} = -a_{jj} = 2a_{jj} = 0$, $a_{jj} = 0$ $1 \le i, j \le n$

F. *Theorem 6.6*

Let A be $n \times n$ quaternion doubly stochastic matrix over H. The quaternion doubly stochastic matrix A will be called a symmetric if $a_{ij} = a_{ji}$ for all i and j, $A = A^T$ i.e., if a matrix is equal to its transpose matrix.

Proof

Let Z be a set of all $n \times n$ quaternion symmetric doubly stochastic matrix. Let W be a quaternion space of all $n \times n$ quaternion doubly stochastic matrix over a field H. We have to prove that W is a quaternion subspace of Z.

Let $(A)_{n \times n}$ and $(B)_{n \times n}$

$$A \bullet B = (a_{ij})_{n \times n} \bullet (b_{ij})_{n \times n} = (\left|a_{ij} \bullet b_{ij}\right|_{n \times n}) = \left|a_{ij}\right|_{n \times n} \bullet \left|b_{ij}\right|_{n \times n} = \left|a_{ij} \bullet b_{ij}\right|_{n \times n}$$

Hence the product of quaternion symmetric doubly stochastic matrices is also a quaternion symmetric doubly stochastic matrices.

V. CONCLUSION

In this paper we discuss about he extreme points of this convex set of matrices and convex subsets of H are identified for which these extreme matrices are of a permutation matrix type. i.e., for which a Birkhoff's theorem holds.

REFERENCES

[1] R.Benson, Eucliclean geometry and convexity, McGraw-Hill, 1996
[2] R.Brualdi, and P.Gibson, Convex polyhebra of doubly stochastic matrices IV, Linear Algebra Appl.15: 153-172 (1976).
[3] G.Birkhoff's , Lattice theory (revised Edition), A.M.S; New York , 1948.
[4] M.H.Clapp and R.CShiflett, A Birkhoff theorem for doubly stochastic matrices with vector entries. Stud. Appl.math 62: 273 – 279 (1980).
[5] P.M.Gibson, Generalized doubly stochastic and permutation matrices over a ring, linear algebra Appl.30: 101 – 107 (1980)
[6] J.V.Ryff, On the representation of doubly stochastic operators, Pacific.J.math.13: 1379 – 1386 (1963).
[7] R.C.Shiflett, Extreme stochastic measures and Felelman's conjecture, J.math.Anal.Appl.68: 111 – 117 (1979).