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Fractional Roman Labeling of Graphs

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Abstract: Motivated from the ancient Roman military strategy [1, 2], Suresh Kumar and Satheesh [3] introduced the notion of the Roman labeling of a connected graph G as a function f: V(G) → {0, 1, 2} such that any vertex with label 0 is adjacent to a vertex with label 2. Weight of a Roman labeling, f is defined as the sum of all vertex labels. Roman number of a graph G is defined as the minimum weight of a Roman labeling on G and is denoted by S(G). In this paper, we study the fractional variation of the Roman labelling.

I. INTRODUCTION

Roman labeling of a connected graph G as a function f: V(G) → {0, 1, 2} such that any vertex with label 0 is adjacent to a vertex with label 2. The function value f(v) of a vertex v of the graph G is called the label of v.

It can be easily seen that if G has a Roman labeling, then for any edge e = {u, v}, either both u and v are adjacent to vertices with labels at least 1 or the edge e is incident with a vertex with label 2. Clearly, f partitions the vertex set, V(G) into 3 vertex subsets, V_0, V_1 and V_2 .which are the subsets of V(G) with labels 0,1,2 respectively. So Roman labeling is also denoted as triplet, f = (V_0, V_1, V_2). *Weight of Roman labeling* is defined as the sum of all vertex labels. That is, $w(f) = \sum_{v \in V(G)} f(v)$. Roman number of a graph G is defined as the minimum weight of a Roman labeling on G and is denoted by S(G). Roman labeling with minimum weight is called a *minimal Roman labeling.*

II. MAIN RESULTS

In Graph Theory, many concepts are dealt with treating it as presence or absence of elements in a specified set. Presence of a vertex may be denoted by the number 1 and the absence by the number 0. For example, an independent set, I of a graph G is the vertex set V of the graph together with a function $f: V \to \{0,1\}$ such that f(v) = 1 if v ∈ I and f(v) = 0 otherwise, Clearly, this is an integer-valued function. Fractional graph theory deals with generalization of integer-valued graph theoretic concepts such that they can take fractional values. One of the standard methods for converting a graph theoretic concept from integer version to fractional version is to formulate the concept as an Integer programming Program and consider the linear programming methodology for the solution.

Study on fractional version of various graph parameters is an active area of research. Fractional version allows us to express the values of the parameters as fractions, which give us more insight into the characteristics of the parameter specifically and that of the underlying graph in general. In fractional versions, we choose function values for the vertices or edges from intervals, instead of integers. This allows continuity of labelings and gives us opportunity to incorporate limit process in the study.

For terms and definitions not defined explicitly here, reader can refer to Harary [4].

A. The Minimality of Roman Labeling

We discuss the minimality of Roman Labeling before we get into the details of fractional version of Roman Labeling. The Labeling, f representing S is minimal if we cannot obtain another labeling g by reducing the values of f. In this sense f is said to be irreducible.

A Roman Labeling f = (V_0, V_1, V_2) is irreducible if it is not possible to reduce the value of f(v) for some vertex v and still the resulting function be a Roman Labeling.

But the minimality of Roman Labeling is more than irreducibility. A Roman Labeling f = (V_0, V_1, V_2) is of Type II minimal if it is irreducible and V_0 is non-empty.

B. Different types of Fractional Roman Labeling

We can define different types of fractional Roman labeling taking different aspects of Roman Labeling into account. First we consider the fact that a Roman Labeling f = (V_0, V_1, V_2) has the property that every vertex in V_0 is adjacent to at least one vertex in V_2. We can fractionalize the Roman Labeling by allowing the function values to vary freely in the closed interval [0, 2]. The vertices having function values 2 can supply one item to one of the vertices having zero as function value and adjacent to it. The vertex can do this only if it has one item for own use. We can fractionalize the situation as follows: A function f : V → [0, 2] such

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that every vertex having function value zero is adjacent to at least one vertex having function value greater than one is a kind of fractional Roman Labeling.

Let us use the following notations: For the function $f : V \rightarrow [0, 2]$, let V_0, V_1, V_2 are the subsets of V(G) with vertex labels 0, 1, 2 respectively. In addition, we use the following notations:

$$V^f_0 = \{v \in V \mid f(v) = 0\}$$
$$V^f_1 = \{v \in V \mid f(v) = 1\}$$
$$V^f_2 = \{v \in V \mid f(v) = 2\}$$
$$V^f_{(0,1]} = \{v \in V \mid 1 \geq f(v) > 0\}$$
$$V^f_{[2,1)} = \{v \in V \mid 2 \geq f(v) > 1\}$$

If no confusion arises about the function, we can write V_0 for V^f_0, V_1 for V^f_1 and so on.

A function $f: V \rightarrow [0, 2]$ is a *Fractional Roman Labeling* if every vertex in $V^f_{[2,1)}$ is adjacent to some vertex in V^f_1. A fractional Roman labeling f is called *reducible* if it is possible to reduce the value of f(v) for some vertex v and obtain a new fractional Roman labeling of the same graph. Otherwise it is called irreducible. We can reduce the function value of a fractional Roman labeling f freely, if the function value $f(v) > 1$. This reduction will not change the vertex sets $V^f_{[2,1)}$,V^f_0 and V^f_1. But the function value can be reduced to one (then the vertex is removed from $V_{[2,1)}$ and included in V_1) only if all vertices in V_0, which are adjacent to $v \in V_{[2,1)}$ are adjacent to another vertex in $V_{[2,1)}$. Thus if the new function obtained is denoted as g, then $V^g_{[2,1)}$ is a subset of $V^f_{[2,1)}$ and V^g_1 is a superset of V^f_1 and $V^g_0 = V^f_0$. In addition to the above, if a vertex $u \in Vg_1$ is adjacent to some vertices in $V^g_{[2,1)}$, we can decrease the function values at the vertices to zero and obtain a new fractional Roman domination function, say h. Then $V^g_{[2,1)} = V^h_{[2,1)}$, V^h_0 is a superset of V^g_0 and V^h_1 is a subset of V^g_1 **.**

Now we define a special type of fractional Roman labeling. A fractional Roman labeling f is called *normal*, if it satisfies the following conditions;

1) There exists no other fractional Roman labeling g such that $V^g_{[2,1)} \supseteq V^f_{[2,1)}$, $V^g_1 \sqsubseteq V^f_1$ and $V^g_0 = V^f_0$.
2) There exist no edge connecting V^f_1 and $V^f_{[2,1)}$.

a) *Theorem 2.2.1.* If the fractional Roman Labeling f is normal, then there exists another minimal Roman Labeling g such that $V^g_{[2,1)} = V^f_{[2,1)}$, $V^g_0 = V^f_0$ and V^g_1 is a superset of V^f_1 **.** Also, there exist infinitely many normal fractional Roman labelings.

Proof. Let f be a normal fractional Roman labeling and $v \in V^f_{[2,1)}$. Since $f(v) > 1$, we can define another fractional Roman labeling g such that $g(v) = f(v) - \varepsilon$, where $\varepsilon \rightarrow 0$ and $g(v) = f(v)$ for all other vertices. So $V^f_{[2,1)} = V^g_{[2,1)}$, $V^g_0 = V^f_0$ and V^g_1 is a superset of V^f_1 . Since ε is arbitrarily chosen with $\varepsilon \rightarrow 0$, the second part of the result follows.

The above Theorem says that a normal fractional Roman labeling can still be reducible. Now we proceed to prove the condition for a normal fractional Roman Labeling to be irreducible.

b) *Theorem 2.2.2.* A normal fractional Roman Labeling f is irreducible if and only if $V^f_{[2,1)}$ is the empty set.

Proof. If $Vf_{[2,1)} = V^f_0$ for a minimal fractional Roman labeling f, then $f(v) = 0$ for all $v \in V$ and hence V^f_0 is the empty set. So f is irreducible. Conversely suppose that f is irreducible. If $V^f_{[2,1)}$ is non-empty, let $v \in Vf_{[2,1)}$. Now, we can reduce the function value at v to obtain another minimal fractional Roman labeling, a contradiction.

It is possible to define a more generalized version of fractional Roman Labeling by letting the vertices in V^f_1 to take a value from the interval (0, 1]. Then the vertex set equivalent to V^f_1 becomes $V^f_{(0,1]}$. Now we apply the condition that there is no edge from a vertex in $V^f_{(0,1]}$ to a vertex in $V^f_{[2,1)}$ to obtain a subnormal fractional Roman Labeling.

A fractional Roman Labeling f is minimal (irreducible) if there does not exist a minimal Roman Labeling $g \neq f$ for which $g(v) \leq f(v)$ for all $v \in V$ and $g(v) < f(v)$ for some $v \in V$. Now we proceed to

define a subnormal fractional Roman Labeling of type. A fractional Roman labeling f is called subnormal, if it satisfies the following conditions.

i) For all $v \in V^f0$, $f(N[v]) \geq 2$.
ii) For all $v \in V^f(0,1]$, $f(N[v]) \geq 1$.
iii) For all $v \in V^f(1,2]$, $f(N[v]) \geq 2$.

We can define minimal fractional Roman labelings in the same way the different types of minimal Roman labelings are defined.

C. Convexity of Fractional Roman Labelings

In this section, we study convex combination of the fractional Roman labelings, similar to the way we can define the various types of fractional Roman labelings, it is necessary to consider the convexity properties of each of them separately.

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Convex combination of the two fractional Roman labelings f and g of a graph G, is the function $(f * g)_\lambda = \lambda f + (7 - \lambda)g, 0 < \lambda < 1$. Since any convex combination of labelings is again a labeling, the set of all labelings forms a convex set. However it is evident that the convex combination of two minimal labelings need not be a minimal labeling. So the set of all minimal labelings is not a convex set. Now, a question arise: Is a convex combination of two fractional Roman labelings again a fractional Roman labeling? We are going to address this question in the following results:

1) *Theorem 2.3.1.* Let f and g be two fractional Roman labelings of G. Then, $V^{f*g}{}_{[2,1)} \sqsubseteq V^f{}_{[2,1)} \cup V^g{}_{[2,1)}$

Proof. Let $v \in V^{f*g}{}_{[2,1)}$. Then $[(f * g)_\lambda](v) > 1$ so that $[\lambda f + (1 - \lambda)g](v) > 1$. Hence, either $f(v) > 1$ or $g(v) > 1$. So, $v \in V^f{}_{[2,1)}$ or $v \in V^g{}_{[2,1)}$.

But the reverse inclusion need not be true. For example, let f(v) = 1.30, g(v) = 0.30, λ = 0.5. Then v is not a member of $V^{f*g}{}_{[2,1)}$.

2) *Theorem 2.3.2.* Let f and g be two fractional Roman labelings of G. Then, $V^f{}_{[2,1)} \cap V^g{}_{[2,1)} \sqsubseteq V^{f*g}{}_{[2,1)}$.

Proof. Let $v \in V^f{}_{[2,1)} \cap V^g{}_{[2,1)}$. So f(v) >1, g(v) > 1. Then, $[\lambda f + (1 - \lambda)g](v) > 1$. So, $v \in V^{f*g}{}_{[2,1)}$.

3) *Theorem 2.3.3***.** Convex combination of two fractional Roman labelings, f and g, is a fractional Roman labeling of G, if $V^{f*g}{}_{[2,1)} = V^f{}_{[2,1)} \cup V^g{}_{[2,1)}$

Proof. We recall that a function f: V→ [0, 2] is a Fractional Roman Labeling iff every vertex in $V^f{}_{[2,1)}$ is adjacent to some vertex in $V^f{}_1$. Let $v \in V^{f*g}{}_{[2,1)}$. Then, $v \in V^f{}_{[2,1)}$. and $v \in V^g{}_{[2,1)}$. Since f and g are fractional Roman labelings, $v \in V^f{}_1$ and $v \in V^g{}_1$. But, $f(v) = g(v) = 1$ implies that $(f * g)(v) = 1$. So, $f * g$ is a fractional Roman labeling of G.

These results show that convex combination of two fractional Roman labelings need not be a fractional Roman labeling in general. Convexity of the set of all minimal fractional Roman labelings is yet to be explored. The explorations to find the various types of fractional Roman labelings possessing convex combinations and the properties of fractional Roman labelings can guide further in the area.

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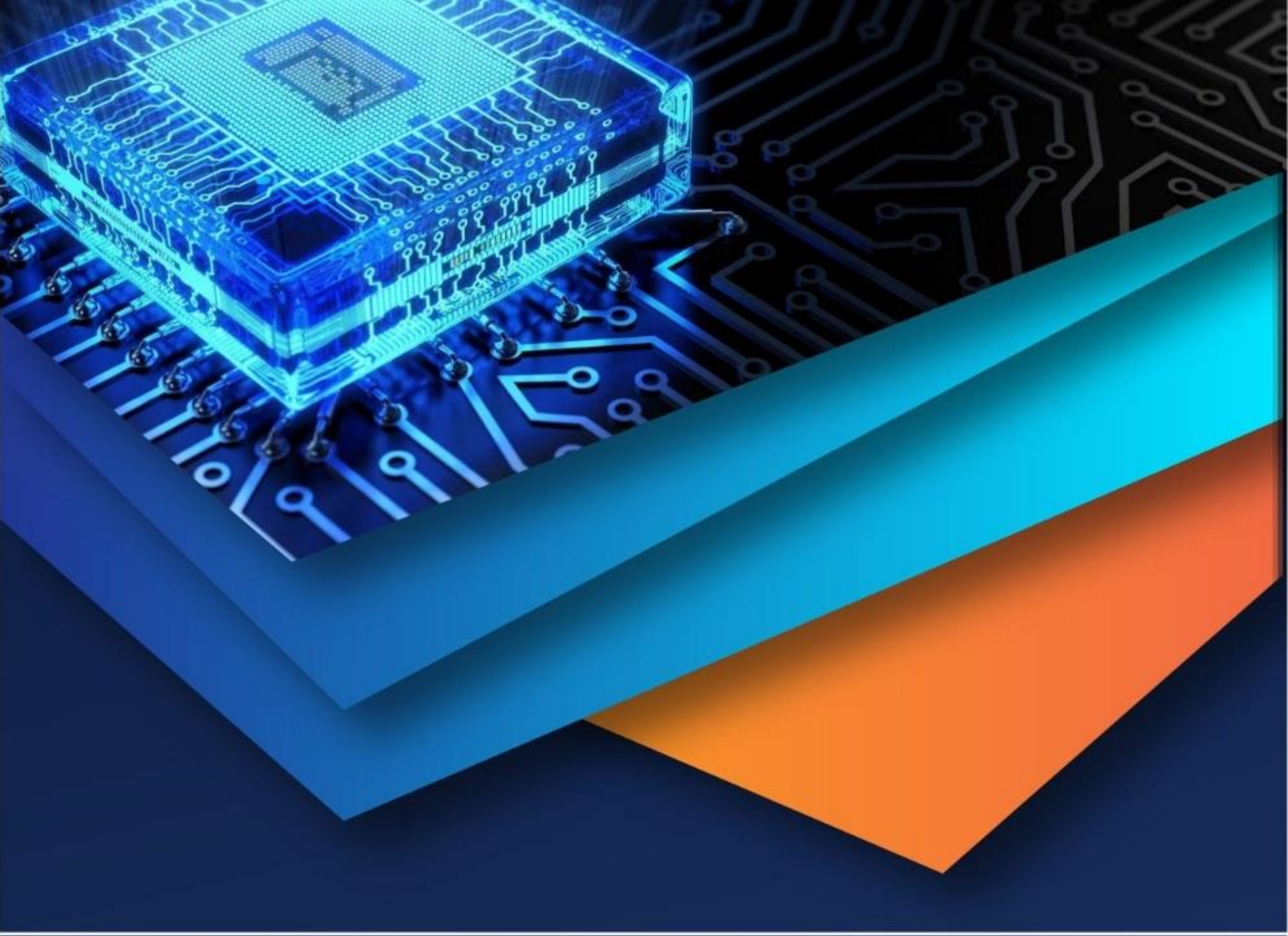

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