International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.177
Volume 7 Issue V, May 2019- Available at www.ijraset.com

Equivalence of Two Summability Methods for Improper Integral

Dr. *Banitamani* Mallik, Mr Radhamadhab Dash, Dr. Umakanta Misra
[1]*Centurion University of Technology and Management, Paralakhemundi, Odisha, India*
[2]*Delhi Public School, Nalco Nagar, Angul, Odisha, India*
[3]*National Institute of Science and Technology, Ganjam, Odisha, India*

Abstract: In this paper, we introduce the concept of $\left|N,p;\delta,\mu\right|_k, k\geq 1$ ***summability of improper integrals and by this definition we prove a theorem that generalizes theorems of Orhan, Ozgen and Acharya.***

2010 Mathematics Subject Classification: $26D15;40F05;40D25;40A10;35A23$

Keywords: Cesaro and Riesz method, summability of improper integrals

I. INTRODUCTION

Let f be a real valued function which is continuous on $[0,\infty)$. Let $s(x)=\int_0^x f(t)dt$. We denote the Cesaro mean of $s(x)$ by $\sigma(x)$.The improper integral $\int_0^\infty f(t)dt$ is said to be integrable $|C,1|_k, k\geq 1,$ in the sense of Flett[3] if

(1.1) $$\int_0^\infty x^{k-1}\left|\sigma'(x)\right|^k dx=\int_0^\infty \frac{\left|v(x)\right|^k}{x}dx$$

is convergent , where $v(x)=\frac{1}{x}\int_0^\infty t\, f(t)dt$ usually called a generator of the integral $\int_0^\infty f(t)dt$.

Let $p(x)$ be a non-decreasing real valued function on $[0,\infty)$.We define

$$P(x)=\int_0^x p(t)dt,\ p(x)\neq 0,\ p(0)=0.$$

Then we define the Norlund mean of $s(x)$ by

$$\sigma_p(x)=\frac{1}{P(x)}\int_0^x p(t)s(t)dt.$$

We say that the improper integral $\int_0^\infty f(t)dt$ is summable $\left|N,p\right|_k, k\geq 1,$ if

(1.2) $$\int_0^\infty x^{k-1}\left|\sigma_p'(x)\right|^k dx$$

is convergent .In the special case if we take $p(x)=1$ for all values of x, then $\left|N,p\right|_k$ summability reduces to $|C,1|_k$ summability of improper integrals.

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.177
Volume 7 Issue V, May 2019- Available at www.ijraset.com

For any two functions f *and* g, it is customary to write $g(x) = O(f(x))$, if there exist K and N, for every $x > N, \left|\frac{g(x)}{f(x)}\right| \le K$. Clearly the difference between $s(x)$ and its nth weighted mean $\sigma_p(x)$, which is called the weighted Kronecker identity, is given by the identity

(1.3) $$s(x) - \sigma_p(x) = v_p(x),$$

where

$$v_p(x) = \frac{1}{P(x)}\int_0^\infty p(u)f(u)du \ .$$

We note that if we take $p(x) = 1$, for all values of x, then we have the following identity (see [2])

$$s(x) - \sigma_p(x) = v(x).$$

Since

$$\sigma'_p(x) = \frac{p(x)}{P(x)}v_p(x),$$

condition (1.3) can be written as

(1.4) $$s(x) = v_p(x) + \int_0^x \frac{p(u)}{P(u)}v_p(u)du \quad .$$

In view of the identity (1.4), the function $v_p(x)$ is called the generator function of $s(x)$.

Condition (1.1) can also be written as

(1.5) $$\int_0^\infty x^{k-1}\left(\frac{p(x)}{P(x)}\right)^k |v_p(x)|^k dx$$

is convergent. It may be noted that for infinite series, an analogous definition was introduced by Orhan [4] . The improper integral $\int_0^\infty f(t)dt$ is summable $|N, p; \delta|_k, k \ge 1, 0 \le \delta k < 1$, if

(1.6) $$\int_0^\infty x^{\delta k + k - 1}\left(\frac{p(x)}{P(x)}\right)^k |v_p(x)|^k dx$$

and summable $|N, p; \delta, \mu|_k, k \ge 1, 0 \le \delta k < 1, \mu \ge 1,$ if

(1.7) $$\int_0^\infty x^{\mu(\delta k + k - 1)}\left(\frac{p(x)}{P(x)}\right)^k |v_p(x)|^k dx \quad .$$

II. KNOWN RESULTS

Dealing with $|R, p_n|_k$ and $|R, q_n|_k$ summability methods, Orhan [4] proved the following theorem:

1) *Theorem 2.1.* The $|R, p_n|_k, (k \ge 1)$ summability implies the $|R, q_n|_k, (k \ge 1)$ summability provided that

a) $nq_n = O(Q_n)$,

b) $P_n = O(np_n)$ and

c) $Q_n = O(nq_n)$.

Subsequently, dealing with summability of improper integrals, Ozgen has established the following theorem:

2) *Theorem 2.2.* [5] Let $p(x)$ and $q(x)$ be non-decreasing real valued functions on $[0,\infty)$ such that as $x\to\infty$

a) $xq(x)=O(Q(x))$,

b) $P(x)=O(xp(x))$,

and

c) $Q(x)=O(xq(x))$

If $\int_0^\infty f(t)dt$ is summable $|R,p|_k$, then it is also summable $|R,q|_k, (k\geq 1)$.

Extending the result of Ozgen, Acharya[1] has established the following result.

3) *Theorem 2.3.* Let $p(x)$ and $q(x)$ be non-decreasing real valued functions on $[0,\infty)$ such that as $x\to\infty$

a) $xq(x)=O(Q(x))$,

b) $P(x)=O(xp(x))$,

c) $Q(x)=O(xq(x))$,

d) $\int_t^m \frac{x^{\delta k}q(x)}{Q^2(x)}dx = O\left(\frac{t^{\delta k}}{Q(t)}\right)$

and

e) $\int_0^m t^{\delta k-1}|v_p(t)|^k dt = O(1)$.

If $\int_0^\infty f(t)dt$ is summable $|N,p;\delta|_k$, then it is also summable $|N,q,\delta|_k, (k\geq 1)$.

However, extending the above result , in the present paper we establish the following theorem :

III. MAIN RESULT

1) *Theorem 3.1.* Let $p(x)$ and $q(x)$ be non-decreasing real valued functions on $[0,\infty)$ such that as $x\to\infty$

a) $xq(x)=O(Q(x))$,

b) $P(x)=O(xp(x))$,

c) $Q(x)=O(xq(x))$,

d) $\int_t^m \frac{x^{\mu(\delta k+k-1)-k+1}q(x)}{Q^2(x)}dx = O\left(\frac{t^{\mu(\delta k+k-1)-k+1}}{Q(t)}\right)$

and

e) $\int_0^m t^{\mu(\delta k+k-1)-k}|v_p(t)|^k dt = O(1)$.

If $\int_0^\infty f(t)dt$ is summable $|N,p;\delta,\mu|_k$, then it is also summable $|N,q,\delta,\mu|_k, (k\geq 1)$.

IV. PROOF OF THE THEOREM

Let $\sigma_p(x)$ and $\sigma_q(x)$ be the functions of (N,p) and (N,q) means of the integral $\int_0^\infty f(t)dt$ .Since $\int_0^\infty f(t)dt$ is summable $|N,p;\delta,\mu|_k, k\geq 1, 0\leq \delta k<1, \mu\geq 1$, we can write

$$\int_0^\infty x^{\mu(\delta k+k-1)}\left(\frac{p(x)}{P(x)}\right)^k |v_p(x)|^k dx$$

is convergent. Differentiating the equation (1.4) , we have

$$f(x)=v_p'(x)+\frac{p(x)}{P(x)}v_p(x) \ .$$

By definition, we obtain

$$\sigma_q(x)=\frac{1}{Q(x)}\int_0^x q(t)s(t)dt=\frac{1}{Q(x)}\int_0^x (Q(x)-Q(t))f(t)dt$$

and

$$\sigma_q'(x)=\frac{q(x)}{Q^2(x)}\int_0^x Q(t)f(t)dt$$

$$=\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\left[v_p'(t)+\frac{p(t)}{P(t)}v_p(t)\right]dt$$

$$=\frac{q(x)}{Q^2(x)}\int_0^x Q(t)v_p'(t)dt+\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt$$

Integrating by parts of the first statement, we have

$$\sigma_q'(x)=\frac{q(x)}{Q^2(x)}\left[Q(x)v_p(x)-\int_0^x q(t)v_p(t)dt\right]+\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt$$

$$=\frac{q(x)}{Q^2(x)}\,v_p(x)+\frac{q(x)}{Q^2(x)}\int_0^x Q(t)\frac{p(t)}{P(t)}v_p(t)dt-\frac{q(x)}{Q^2(x)}\int_0^x q(t)v_p(t)dt$$

$$=\sigma_{q,1}(x)+\sigma_{q,2}(x)+\sigma_{q,3}(x),\ \text{say.}$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\int_0^m x^{\mu(\delta k+k-1)}|\sigma_{q,r}(x)|^k dx=O(1) \ \text{as}\ m\to\infty,\ \text{for}\ r=1,2,3 \ .$$

Using conditions (7.3.1.1) and (7.3.1.2), we have

$$\int_0^m x^{\mu(\delta k+k-1)}|\sigma_{q,1}(x)|^k dx=\int_0^m x^{\mu(\delta k+k-1)}\left|\frac{q(x)}{Q(x)}v_p(x)\right|^k dx$$

$$=\int_0^m x^{\mu(\delta k+k-1)}\left(\frac{q(x)}{Q(x)}\right)^k |v_p(x)|^k dx$$

$$=O(1)\int_0^m x^{\mu(\delta k+k-1)}\left(\frac{p(x)}{P(x)}\right)^k |v_p(x)|^k dx$$

$$=O(1)\int_0^m x^{\mu(\delta k+k-1)}|\sigma_p'(x)|^k dx$$

$$=O(1)\ , \text{as}\ m\to\infty \ ,$$

by virtue of the hypotheses of theorem 3.1.

Applying Holder's inequality with $k>1$, we get

$$\int_0^m x^{\mu(\delta k+k-1)}|\sigma_{q,2}(x)|^k dx=O(1)\int_0^m x^{\mu(\delta k+k-1)}\left(\frac{q(x)}{Q^2(x)}\right)^k\left(\int_0^x \frac{Q(t)p(t)}{P(t)}|v_p(t)|dt\right)^k dx$$

$$= O(1)\int_0^m \frac{x^{\mu(\delta k+k-1)-k+1}q(x)}{Q^{k+1}(x)}\left(\int_0^x \frac{Q(t)p(t)}{P(t)}|v_p(t)|dt\right)^k dx$$

$$= O(1)\int_0^m \frac{x^{\mu(\delta k+k-1)-k+1}q(x)}{Q^2(x)}\left(\int_0^x\left(\frac{Q(t)}{q(t)}\right)^k q(t)\left(\frac{p(t)}{P(t)}\right)^k |v_p(t)|^k dt\right)x\left(\frac{1}{Q(x)}\int_0^x q(t)dt\right)^{k-1} dx$$

$$= O(1)\int_0^m t^k q(t)\left(\frac{p(t)}{P(t)}\right)^k |v_p(t)|^k dt\int_t^m \frac{x^{\mu(\delta k+k-1)-k+1}q(x)}{Q^2(x)}dx$$

$$= O(1)\int_0^m t^{\mu(\delta k+k-1)+1}\frac{q(t)}{Q(t)}\left(\frac{p(t)}{P(t)}\right)^k |v_p(t)|^k dt$$

$$= O(1)\int_0^m t^{\mu(\delta k+k-1)}\left(\frac{p(t)}{P(t)}\right)^k |v_p(t)|^k dt$$

$$= O(1) \text{ as } m\to\infty,$$

by virtue of the hypotheses of theorem 7.3.1.

Finally, again by Holder's inequality with $k>1$, we have

$$\int_0^m x^{\mu(\delta k+k-1)}|\sigma_{q,3}(x)|^k dx = O(1)\int_0^m x^{\mu(\delta k+k-1)}\left(\frac{q(x)}{Q^2(x)}\right)^k\left(\int_0^x q(t)|v_p(t)|^k dt\right)^k dx$$

$$= O(1)\int_0^m \frac{x^{\mu(\delta k+k-1)-k+1}q(x)}{Q^2(x)}\left(\int_0^x q(t)|v_p(t)|^k dt\right)x\left(\frac{1}{Q(x)}\int_0^x q(t)dt\right)^{k-1} dx$$

$$= O(1)\int_0^m q(t)|v_p(t)|^k dt\int_t^m \frac{x^{\mu(\delta k+k-1)-k+1}q(x)}{Q^2(x)}dx$$

$$= O(1)\int_0^m \frac{t^{\mu(\delta k+k-1)-k+1}q(t)}{Q(t)}|v_p(t)|^k dt$$

$$O(1)\int_0^m t^{\mu(\delta k+k-1)-k}|v_p(t)|^k dt$$

$$= O(1) \text{ as } m\to\infty,$$

by virtue of the hypotheses of theorem 3.1.

This completes the proof of the theorem.

REFERENCES

[1] D. Acharya, Ph.D. thesis submitted to Fakir Mohan University, Odisha, (2019)
[2] Canak and U. Totur, "A tauberian theorem for Cesaro summability factors of integrals," Appl. Math. Lett. vol. 24, no. 3, pp. 234-245, 2011
[3] T.M. Flett, "On an extension of absolute summability and some theorems of littlewood and paley."Proc. London Math. Soc, vol.7, pp.113-141, 1957
[4] C. Orhan, "On equivalence of summability methods, Math.Slovaca" vol.40, no.2, pp.171-175, 1990.
[5] H.N. Ozgen, "On equivalence of two integrability methods", Miskole Mathematical Notes, Vol.18 (1), pp.391-396, (2017)