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Non-Linear Dynamics (Vibration Analysis) of a Flexible Single Link Cartesian Manipulator

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Abstract: The present work deals with the non-linear vibration analysis of a harmonically excited single link roller-supported flexible Cartesian manipulator with a mass at the end. The governing equation of motion of this system is developed using D' Alemberts principle. Using generalized Galerkin's method the governing equation of motion is reduced to the second-order temporal differential equation of motion. In order to study the stability and bifurcations of the system the nonlinear equation of motion is solved using method of multiple scales. The influence of amplitude of the base excitation and mass ratio on the steady state response of the system is investigated for simple resonance conditions. Critical bifurcation points are determined from the fixed-point responses and periodic responses are found for different system parameters. The perturbation analysis results are compared with the previously published experimental work and are found to be in good agreement. This work will be useful for designing of a flexible manipulator.

I. INTRODUCTION

Robotic manipulators are extensively used in hazardous environments like nuclear reactors, mining, space exploration etc. The advantage of flexible manipulators is low cost, lightweight, large operational speed, low power consumption, better transportability, and safer operation due to reduced inertia. Due to many advantages study on conventional manipulators is continued to make them light weight and flexible. Manipulators can be classified into two types: 1) prismatic manipulator, 2) revolute manipulator depending on the joints used in their construction. Cartesian manipulators have prismatic joints and can reach any position in the work envelop by translation motions of the links. Flexible manipulators have less weight and stiffness due to which vibration problems occur while operating them. Research has been done to improve the vibration problems by studying dynamic models and incorporating control strategies. The following literature study is helpful in knowing the current status of the research in this field.
C.W.S.TO[1] In his study a simple model of a mast antenna structure is extended to comprehend cases in which the Centre of gravity of the tip mass does not coincide with the point of attachment. Expressions for natural frequencies and mode shapes are derived. Olkan Cuvalci[2] In the neighborhood of auto-parametric resonance the Auto-parametric interaction was investigated by varying the forcing amplitude, the internal frequency ratio, and the mass. His study was to define an absorption region numerically and experimentally with respect to forcing amplitude, internal frequency ratio and mass ratio for the passive vibration absorber. Lawrence D. Zavodney[3] The response of one- and two-degree-of-freedom (SOOF and 2DOF) systems with quadratic and cubic nonlinearities to fundamental, principal, and combination harmonic parametric excitations is investigated theoretically and experimentally. Rajesh Kumar Moolam[4] , A systematic approach for the dynamic modeling and control of spatial flexible link manipulators is presented. A general purpose code has been developed in MATLAB to derive the multi-body dynamic model of flexible manipulators for numerical simulation and control design. Jinyong Ju, Wei Li, Mengbao Fan, Yuqiao Wang, and Xuefeng Yang[5], Considering the relationship between the coefficients of the differential equations of motion and the mode shapes of the flexible manipulator, the mathematical expressions of the mode shapes with terminal load are derived. Then, based on method of multiple scales and rectangular coordinate transformation, the average equations of the FCRM are derived to analyze the influence mechanism of base disturbance and terminal load on the system parametric vibration stability. R. C. Kar and T. Sujata[6], Investigated the dynamic stability of a uniform beam elastically restrained at one end and free at the other, subjected to a directional controlled pulsating longitudinal force. The first-, second- and third-order regions of parametric instability of simple and combination resonance are determined simultaneously by an eigensolution approach. The effects of end-flexibilities and tangency coefficient of the applied force on the instability regions have been studied through the use of graphs. Atef A. Ataa, Waleed F. Faresb, Mohamed Y. Sa'adehc [7] presented work on Dynamic analysis of a two-link flexible manipulator subject to different sets of conditions. The effect of different sets of initial and boundary conditions on the joints torques is investigated in this paper. Carmelo di, Castri and Arcangelo Messina[8] presented work on Vibration analysis of Multilink Manipulators Based on Timoshenko Beam theory. Timoshenko's theory is adopted in order to accurately describe the freely vibrating dynamics of a multilink flexible manipulator. Alaa Shawky, Dawid Zydek, Yehia Z. Elhalwagy, Andrzej Ordys[8] presented work on Modeling and nonlinear

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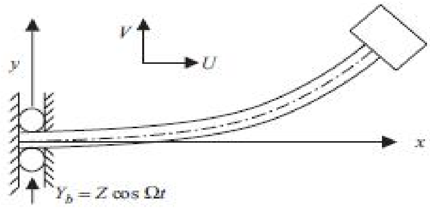
control of a flexible-link manipulator. The problem of modelling and controlling the tip position of a one-link flexible manipulator is considered. The proposed model has been used to investigate the effect of the open-loop control torque profile, and the payload. Barun Pratiher, Suman Bhowmick [9] Presented work on Nonlinear dynamic analysis of a Cartesian manipulator carrying an end effector placed at an intermediate position. Nonlinear dynamic analysis of a Cartesian manipulator carrying an end effector which is placed at different intermediate positions on the span is theoretically investigated with a single mode approach.

II. MATHEMATICAL MODELING

A. Derivation Of The Temporal Equation Of Motion

Fig. 1 shows a single link flexible Cartesian manipulator with mass M at the free end and roller supported at the other end which is subjected to harmonic excitation. The roller-supported end is assumed to have periodic motion $Y_b(t) = Z\cos\Omega_1 t$ where Z and are amplitude and Ω_1 frequency of the base excitation. The motion is considered to be in the x-y plane. Here, the manipulator is modeled as an Euler–Bernoulli beam with a tip mass.



Using D' Alembert's principle the governing differential equation of motion of the system is derived as:

$$EI\left(v_{ssss} + \frac{1}{2}v_s^2 v_{ssss} + 3v_s v_{ss} v_{sss} + v_{ss}^3\right) +$$
$$\rho A v_s\left(\int_0^s (\dot{v}_\eta^2 + v_\eta \ddot{v}_\eta)\, d\xi\right) + M(\ddot{v} + \ddot{Y}_b) v_s v_{ss} +$$
$$v_s v_{ss}\left(\rho A \ddot{Y}_b (L - s) + \int_s^L (\rho A \ddot{v} + C_d \dot{v})\, d\eta\right) - \qquad (1)$$
$$v_{ss}\left(\int_s^L \rho A\left(\int_0^\xi \left(\dot{v}_\eta^2 + v_\eta \ddot{v}_\eta\right) d\eta\right) d\xi + M\int_0^s (\dot{v}_\eta^2 + v_\eta \ddot{v}_\eta)\, d\xi\right) +$$
$$\left(1 - \frac{1}{2}v_s^2\right)\left(\rho A(\ddot{v} + \ddot{Y}_b) + C_d \dot{v}\right) + (P(t) v_s)_s = 0$$

Here V denotes the transverse displacement in y direction. $(\cdot)$ and (s) denote, respectively, the first derivative with respect to time t and displacement along the elastic line s. Here, E, I, ρ, A,L, and m are the Young modulus, moment of inertia, mass density, cross-sectional area, length of the cantilever beam, and mass of the beam (i.e. ρAL), respectively. c is the viscous damping constant. To discretize the governing equation of motion (1) one may use the following assumed mode expression:

$V(s, t) = r v_y(s) u(t)$. (2)

Here, r is the scaling factor u(t) is the time modulation and $V_y(s)$ is the eigenfunction of the cantilever beam with tip mass, which is given by

$$\psi(s) = -\frac{\sin\beta l + \sinh\beta l}{\cos\beta l + \cosh\beta l}(\cos\beta s - \cosh\beta s) + (\sin\beta s - \sinh\beta s)$$

One may determine L from the following equation:

$$1 + \cos\beta L\cosh\beta L + \left(\frac{M}{\rho AL}\right)\beta L(\cos\beta L\sinh\beta L - \sin\beta L\cosh\beta L) = 0$$

The following non-dimensional parameters are used in the analysis: $\bar{x} = \frac{s}{L}, \tau = \omega_s t, \bar{\omega}_1 = \frac{\Omega_1}{\omega_s}, \bar{\omega}_2 = \frac{\Omega_2}{\omega_s}, \bar{r} = \frac{r}{L}\ \bar{m} = \frac{M}{\rho AL}, \bar{P}_0 = \frac{P_0}{P_c}, \bar{P}_1 = \frac{P_1}{P_c}, \chi = \frac{EI}{\rho AL^4}, \bar{k} = \frac{Z}{r}$

Substituting Eq. (2) into Eq. (1) and using the generalized Galerkin's method the resulting non-dimensional temporal equation of motion is obtained which is given by

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$$\ddot{q} + q + 2\varepsilon\varsigma\dot{q} + \varepsilon(\alpha_1 q^3 + \alpha_2 q^2\ddot{q} + \alpha_3\dot{q}^2 q + \alpha_4\varpi_1^2\cos(\varpi_1\tau)q^2 + \alpha_5\varpi_1^2\cos(\varpi_1\tau) + \alpha_6\cos(\varpi_2\tau)q) = 0 \quad (3)$$

The solution of the above equation is carried out using perturbation method as described in the following section.

B. Perturbation Analysis

As the equation of motion (3) contains many non-linear terms, it is very difficult to find the close form solution. Therefore, one may go for the approximate solution of Eq. (3) by using perturbation techniques. Here, method of multiple scales is used to find the approximate solution. In this method the displacement u can be represented in terms of different times scales (T0, T1) and a book keeping parameter ε as follows:

$$q(\tau,\varepsilon) = q_0(T_0,T_1) + \varepsilon q_1(T_0,T_1) + O(\varepsilon^2) \quad (4)$$

where T0= τ , T1= $\varepsilon\tau$, and the transformation of first and second time derivatives are given by

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + O(\varepsilon^2) \quad \frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2) \quad (5)$$

Here, D0 = partial differentiation w.r.tT0 and D1 = partial differentiation w.r.t T1. Substituting Eqs. (4) and (5) into Eq. (3) and equating coefficients of like powers of yields the following expressions:

$$D_0^2 q_0 + q_0 = 0 \quad (6)$$

$$D_0^2 q_1 + q_1 = -2D_0 D_1 q_0 - 2\varsigma D_0 q_0 - \alpha_1 q_0^3 - \alpha_2 q_0^2 D_0^2 q_0 - \alpha_3 (D_0 q_0)^2 q_0 - \alpha_4 \varpi^2 \cos(\varpi\tau) q_0^2 - \alpha_5 \varpi^2 \cos(\varpi\tau) \quad (7)$$

General solution of Eq. (6) can be written as $q_0 = A(T_1)\exp(iT_0) + \bar{A}(T_1)\exp(-iT_0)$

Substituting Eq. (6) into Eq. (7) leads to

$$D_0^2 q_1 + q_1 = -(2iD_1 A + 2i\varsigma A + 3\alpha_1 A^2\bar{A} - 3\alpha_2 A^2\bar{A} + \alpha_3 A^2\bar{A})\exp(iT_0) - (\alpha_1 - \alpha_2 - \alpha_3)A^3\exp(3iT_0) - \frac{1}{2}\alpha_4\varpi^2 A^2 \exp i(2+\varpi)T_0 - \frac{1}{2}\alpha_4\varpi^2\bar{A}^2 \exp i(\varpi-2)T_0 - \varpi^2\left(\alpha_4 A\bar{A} + \frac{1}{2}\alpha_5\right)\exp i(\varpi T_0) + cc \quad (8)$$

Where: cc – complex conjugate of preceding terms. Though the actual response of the system is bounded, due to the presence of secular or small divisor terms in equation 8, the solution of the system sometimes will not be bounded. For a bounded solution, these terms should be removed. Equation 8 contains secular or a small divisor terms when frequency of excitation is nearly equals to 1or three times the natural frequency. In simple resonance case both forcing and nonlinear parametric excitation terms will contribute to the resonance condition. In sub-harmonic resonance condition only nonlinear parametric term contribution is felt.

C. Simple Resonance case_ $(\varpi \approx 1)$:

In the case of simple resonance, one may use the detuning parameter σ to express the nearness of ϖ to 1 as

$$\varpi = 1 + \varepsilon\sigma, \qquad \sigma = O(1) \quad (9)$$

In order to eliminate the secular or small divisor terms substitute 9 into 8 we obtain the following equation.

$$2iA' + 2i\varsigma A + 3\alpha_1 A^2\bar{A} - 3\alpha_2 A^2\bar{A} + \alpha_3 A^2\bar{A} + \frac{1}{2}\varpi^2\alpha_4 A^2 \exp i(-\sigma T_1) + \varpi^2\left(\alpha_4 A\bar{A} + \frac{1}{2}\alpha_5\right)\exp i(\sigma T_1) = 0 \quad (10)$$

Substituting A in the polar form i.e. $A = \frac{1}{2}a(T_1)e^{i\beta(T_1)}$ and separating the real and imaginary parts yields the following expressions:

$$a' = -\varsigma a - \varpi^2\left(\frac{\alpha_4}{8}a^2 + \frac{1}{2}\alpha_5\right)\sin\gamma \quad (11)$$

$$a\gamma' = a\sigma - \frac{3}{8}\left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)a^3 - \varpi^2\left(\frac{3\alpha_4}{8}a^2 + \frac{1}{2}\alpha_5\right)\cos\gamma \quad (12)$$

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Here, $(\ )' = \frac{\partial}{\partial T_1}$, and $\gamma = \sigma T_1 - \beta$. For steady state response $(a_0, \gamma_0), a'$ and γ' are equal to zero. Eliminating γ from equations 11 and 12 one may find a fifth order polynomial in a^2, which can be expressed as :

$$Q_5 a^{10} + Q_4 a^8 + Q_3 a^6 + Q_2 a^4 + Q_1 a^2 + Q_0 = 0 \quad (13)$$

Where:

$$Q_0 = -\frac{1}{16}\bar{\varpi}^8 \alpha_5^4$$

$$Q_1 = \frac{1}{4}\varsigma^2 \bar{\varpi}^4 \alpha_5^2 + \frac{1}{4}\bar{\varpi}^4 \sigma^2 \alpha_5^2 - \frac{1}{8}\bar{\varpi}^8 \alpha_4 \alpha_5^3$$

$$Q_2 = \frac{3}{8}\varsigma^2 \bar{\varpi}^4 \alpha_4 \alpha_5 + \frac{1}{8}\bar{\varpi}^4 \sigma^2 \alpha_4 \alpha_5 - \frac{3}{16}\bar{\varpi}^4 \left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)\sigma \alpha_5^2 - \frac{11}{32}\bar{\varpi}^8 \alpha_4^2 \alpha_5^2$$

$$Q_3 = \frac{9}{65}\varsigma^2 \bar{\varpi}^4 \alpha_4^2 + \frac{1}{64}\bar{\varpi}^4 \sigma \alpha_4^2 + \frac{9}{256}\bar{\varpi}^4 \left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)^2 \alpha_5^2 - \frac{3}{32}\bar{\varpi}^4 \sigma \left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)\alpha_4 \alpha_5 - \frac{1}{128}\bar{\varpi}^8 \alpha_4^3 \alpha_5$$

$$Q_4 = \frac{9}{512}\bar{\varpi}^4 \left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)^2 \alpha_4 \alpha_5 - \frac{3}{256}\bar{\varpi}^4 \sigma \left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)\alpha_4^2 - \frac{9}{4096}\bar{\varpi}^8 \alpha_4^4$$, and

$$Q_5 = \frac{9}{4096}\bar{\varpi}^4 \left(\alpha_1 - \alpha_2 + \frac{\alpha_3}{3}\right)^2 \alpha_4^2$$

Equation 13 is an implicit equation for amplitude of the response as a function of the external detuning parameter σ, tip mass M and the amplitude of the base excitation Z. Equation 13 does not have trivial state response, therefore the response is found by numerically solving it or numerically solving the reduced equations 11 and 12 simultaneously. Newton's method is used to solve numerically the polynomial equation to find the response of the system. The response can be found by numerically solving temporal equation 3 by using runge-kutta fourth order method.From equation 3.16, the first order nontrivial steady state approximate solution can be given by.

$$q = a\cos(\bar{\varpi}\tau - \gamma) + O(\varepsilon) \quad (14)$$

III. NUMERICAL RESULTS AND DISCUSSIONS

Metallic beam with the following parameters is considered from the Cuvalci (2000).

Length (L)	0.336 m
Cross-section area (A)	40.464 x 10^{-6} m^2
Moment of inertia (I)	8.669867 x 10^{-12} m^4
Young's modulus (E)	1.5848 x 10^{11} N/m^2
Damping constant (Cd)	0.11 N-s/m
Mass density (ρ)	7830 kg/m^3
Scaling parameter ($\bar{r}$)	0.1
Book-keeping parameter(ε)	0.1

The nonlinear response for this system is determined for different values of amplitude of harmonic support motion (Z) and payload mass M for the simple resonance condition.

A. *Simple Resonance Condition* $(\bar{\varpi} \approx 1)$

The resonance condition takes place when the frequency of the support motion of the Ω_1 nearly equals to the fundamental frequency of the system. Figure 3.2 shows the frequency response curves for simple resonance case for the non-dimensional amplitude ratio of base excitation $(\bar{Z})$ equal to 0.00372 and mass ratio $\bar{m}$ equal to 1.8787.The figure shows the stable and unstable branches of the frequency response curves.

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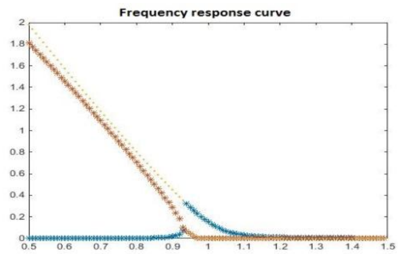
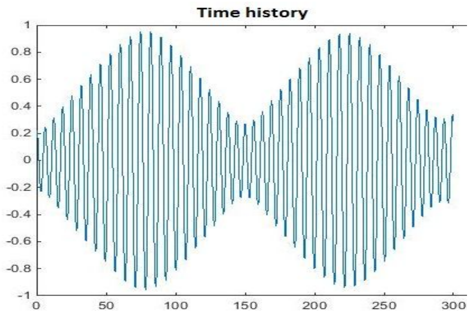


Fig.3.1. Frequency response for simple resonance case for mass ration of 1.8787 and base excitation of 0.00372.

It is observed from the frequency response curve that the system does not possess any trivial state response. So, one may note that manipulator will always oscillate about its equilibrium position with an amplitude equal to the nontrivial response as shown in fig.3.1. When the manipulator is started, with increase in frequency of external excitation, the response amplitude of the manipulator increases and it will reach a critical value, which is saddle node bifurcation point. At this point further increase in frequency, the system will experience a jump up phenomenon (observed in blue line of fig.3.3), which leads to sudden increase of amplitude. The system may fail due to this sudden jump. It is shown in figure 3.1 that the system experiences an upward jump at frequency $\bar{\omega}$ equal to 0.93 with a jump length equals to 0.3.The fundamental frequency of the system is 3.49 Hz therefore the system resonates at frequency equal to 3.24 Hz. When excitation frequency is swept down with decrease in frequency the response amplitude increases. This situation may occur when prime mover of the manipulator is stopped and in that case the system will experience a jump down phenomenon, which leads to catastrophic failure. It is observed that the system has a bi-stable region before the saddle-node bifurcation point. The initial condition will play a very important role to determine the actual system response.

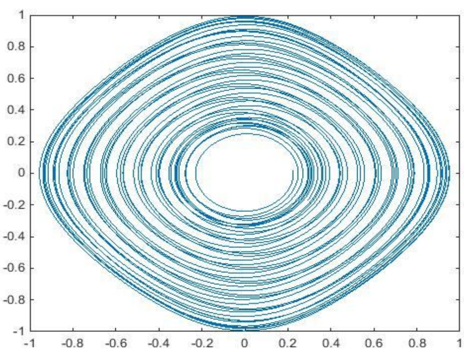
Figure 3.2. shows the time response, phase portrait, and Poincare's section at the critical point. The time response is obtained by solving the temporal equation of motion using fourth order Ringe-Kutta method. While transient response of the system with initial point (a=0.326, gamma=0.1)give a beating type phenomenon, the steady state response of the system is periodic. The phase portrait and Poincare's section for transient and steady state response are shown below.

$\dot{q}$



Time

$\dot{q}$



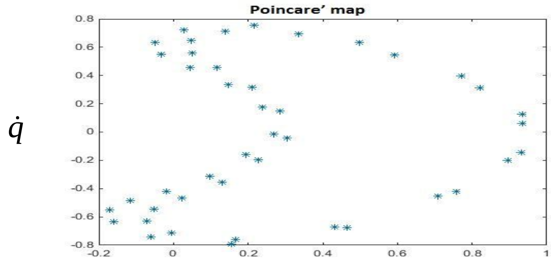


Fig. 3.2. Time response, phase portrait, and Poincare's section at the critical point respectively.

Figures below show the variation of the response of the system with varying mass ration or excitation amplitude of base.

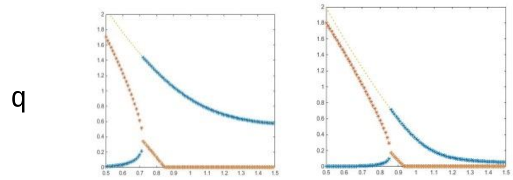


Fig 3.3 Frequency response curve for simple resonance case for mass ration of 1.8787 and excitation amplitude is z bar= 0.05, 0.0125 respectively.

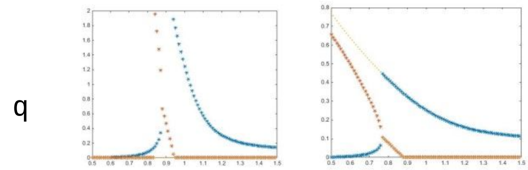


Fig 3.4 Frequency response curve for simple resonance case for excitation amplitude of 0.02 and mass ratio = 0.5656, 4.6968 respectively.

From figure 3.3 we can see that the jump length and maximum response will decrease with decrease in the excitation amplitude. The variation of response is marginal with the variation of excitation amplitude. With decrease of excitation amplitude the saddle point shifts towards the fundamental frequency.
From figure 3.4 the bifurcation point shifts towards left side with increase of mass ratio i.e. the system fails at lower frequency for a system with higher mass ratio. The maximum amplitude and jump length decreases with increase in the mass ratio value.

IV. CONCLUSION

Non-linear response of a flexible single link roller-supported Cartesian manipulator with payload subjected to a harmonic base excitation is investigated using the method of multiple scales. Frequency responses are plotted and their stability and bifurcations are studied for different values of mass ratio and amplitude of base excitation for simple resonance condition. In simple resonance case, the system does not possess any trivial state response. In this case, with decrease in amplitude of external excitation, the maximum value of nontrivial response amplitude remained almost same and system undergoes a catastrophic failure due to jump up phenomena at saddle-node bifurcation point. The designer of the manipulator may use the simplified polynomial equation 13 to find the response amplitude to successfully design a new similar manipulator without having vibration problem when the system is operating near the simple resonance frequency. The present work can be extended to sub-harmonic resonance condition.

A. Appendix

The system fundamental frequency

$$\omega_s = \sqrt{\chi\left(\frac{h_{14}}{h_2}\right)}$$

$$\alpha_1 = \frac{\chi \bar{r}^2}{\varepsilon \omega_s^2}\left(\frac{h_{19}}{h_2} + \frac{h_{18}}{h_2} + 3\frac{h_{20}}{h_2}\right)$$

$$\alpha_2=\frac{\bar{r}^2}{\varepsilon}\left(\frac{h_1}{h_2}+\frac{h_4}{h_2}+\bar{m}\frac{h_5}{h_2}-\frac{h_6}{h_2}-\frac{h_8}{2h_2}-\bar{m}\frac{h_7}{h_2}\right)$$

$$\alpha_3=\frac{\bar{r}^2}{\varepsilon}\left(\frac{h_{11}}{h_2}-\frac{h_{12}}{h_2}-\bar{m}\frac{h_{13}}{h_2}\right)$$

$$\alpha_4=\frac{\overline{Zr}}{\varepsilon}\left(\frac{h_{15}}{h_2}+\bar{m}\frac{h_{16}}{h_2}-\frac{h_{17}}{2h_2}\right),\alpha 5=\frac{k}{\varepsilon}\left(\frac{h_1}{h_2}\right)$$

$$\varsigma=\frac{C_d}{2\varepsilon\rho A\omega_s}$$

Here the expressions for h1, h2, . . . , h20 are given below:

$$h_1=\int_0^1\psi(\bar{x})d\bar{x},\qquad h_2=\int_0^1\left(\psi(\bar{x})\right)^2d\bar{x},\quad h_3=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\left(\int_0^{\bar{x}}\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2d\bar{\xi}\right)\psi(\bar{x})\right)d\bar{x},\quad h_4=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\left(\int_{\bar{x}}^1 d\psi(\bar{\xi})d\bar{\xi}\right)\psi(\bar{x})\right)d\bar{x},$$

$$h_5=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\left(\psi(\bar{x})\right)^2\right)d\bar{x},\quad h_6=\int_0^1\left(\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\left(\int_{\bar{x}}^1\left(\int_0^{\eta}\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2d\bar{\xi}\right)d\bar{\eta}\right)\psi(\bar{x})\right)d\bar{x},\quad h_7=\int_0^1\left(\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\left(\int_0^{\bar{x}}\left(\frac{d\psi(\bar{\xi})}{d\bar{\xi}}\right)^2d\bar{\xi}\right)\psi(\bar{x})\right)d\bar{x},$$

$$h_8=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\right)^2\left(\psi(\bar{x})\right)^2d\bar{x},\ h_9=h_4,h_{10}=h_8,h_{11}=h_3,h_{12}=h_6,h_{13}=h_7,$$

$$h_{14}=\int_0^1\frac{d^4\psi(\bar{x})}{d\bar{x}^4}\psi(\bar{x})d\bar{x},\qquad h_{15}=\int_0^1(1-\bar{x})\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\psi(\bar{x})d\bar{x},$$

$$h_{16}=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\psi(\bar{x})d\bar{x},\quad h_{17}=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\right)^2\psi(\bar{x})d\bar{x},$$

$$h_{18}=\int_0^1\left(\frac{d\psi(\bar{x})}{d\bar{x}}\right)^2\frac{d^4\psi(\bar{x})}{d\bar{x}^4}\psi(\bar{x})d\bar{x},$$

$$h_{19}=\int_0^1\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\psi(\bar{x})d\bar{x},$$

$$h_{20}=\int_0^1\frac{d\psi(\bar{x})}{d\bar{x}}\frac{d^2\psi(\bar{x})}{d\bar{x}^2}\frac{d^3\psi(\bar{x})}{d\bar{x}^3}\psi(\bar{x})d\bar{x},$$

$$h_{21}=h_{19}.$$

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