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Existence of Fixed Point Theorems in Linear 2-Normed Spaces

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***Abstract*–In this paper, we discuss the concept of some existence of fixed point theorems for a pair of mappings in a setting of 2–normed spaces.**

***Keywords*–Normed space, 2–normed space, Fixed point, Common fixed point.**

I. INTRODUCTION

In 1963, S.Gahler [5] was initiated by 2–normed space & 2–Banach space. In 1975, K.Iseki was discussed the concept of the fundamental results on fixed point theorems of 2–metric space. Next Albert White and Y.J.Cho [1] investigate the important properties of linear mappings on linear 2–normed space in the year of 1984. Here after, many authors establish the fixed point theorem in 2–normed spaces and 2–Banach spaces, See [4,6,16–20]. Recently, generalise the concept of 2–normed space as well as 2–Banach space into 2–cone Banach space [3,12,22].

We now state some definitions before presenting our main results.

II. PRELIMINARIES

1) *Definition 1.* [4] Let X be a real linear space with dimension of X is greater than 1 and $\|.,.\|: X \times X \to [0,\infty)$ be a function. Then

(i) $\| x, y \| = 0$ if and only if x and y are linearly dependent,

(ii) $\| x, y \| = \| y, x \|$,

(iii) $\| \alpha x, y \| = |\alpha| \| x, y \|$,

(iv) $\| x + y, z \| \le \| x, z \| + \| y, z \|$, where for all $x, y, z \in X$ and $\alpha \in R$.

If $\| .,. \|$ is called a 2–norm and the pair $(X, \| .,. \|)$ is called a linear 2–normed space. So a 2–norm $\| x, y \|$ always satisfies [24] $\| x, y + \alpha x \| = \| x, y \|$ for all $x, y \in X$ and all scalars α.

2) *Definition 2.* [4] A sequence $\{x_n\}$ in X is convergent to an element $x \in X$, if for each $a \in X$, $\lim_{n\to\infty} \| x_n - x, a \| = 0$. If $\{x_n\}$ converges to x we write $x_n \to x$ as $n \to \infty$.

3) *Definition 3.* [4] A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $a \in X$, $\lim \| x_n - x_m, a \| = 0$ as $n, m \to \infty$.

4) *Definition 4.* [4] A complete 2–normed space is one in which every Cauchy sequence in *X* converges to an element of *X*. A complete 2–normed space *X* is called 2–Banach space.

5) *Definition 5.* [16] Let $(X, \| .,. \|)$ be a linear 2–normed space. Then the mapping $T : X \to X$ is said to be a *contraction* if there exists $k \in [0,1)$ such that $\| Tx - Ty, z \| \le k \| x - y, z \|$ for all $x, y, z \in X$.

6) *Definition 6.* [11] A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

a) ψ is non-decreasing and continuous,

b) $\psi(t) = 0$ if and only if $t = 0$.

7) *Definition 7.* [14] An ultra-altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0, t \in [0,\infty)$ and $\varphi(0) \geq 0$.

We denote this set with Φ_u.

Delbosco [2] and Skof [23] have proved a fixed point theorem for self-maps of complete normed spaces by introducing a class Φ of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

a) $\phi : [0,\infty) \to [0,\infty)$ is continuous in R^+ and strictly increasing in R^+.

b) $\phi(t) = 0$ if and only if $t = 0$.

c) $\phi(t) \geq Mt^{\mu}$ for every $t > 0, \mu > 0$ are constants.

III. MAIN RESULT

1) *Theorem 1.* Let T be a self-map of a complete 2–normed space $(X, \|.,.\|)$ and ϕ satisfying (i) and (ii).

Furthermore, let f, g, h be three decreasing functions from R^+ into $[0,1)$ such that $f(t) + 2g(t) + h(t) < 1$ for every $t > 0$. Suppose T satisfies the following condition

$$\begin{aligned}\phi(\| Tx - Ty, u \|) \leq f(\| x - y, u \|)\phi(\| x - y, u \|) + g(\| x - y, u \|)\left[\phi(\| x - T_x, u \|) + \phi(\| y - T_y, u \|)\right] \\ + h(\| x - y, u \|)\min\left\{\phi(\| x - T_y, u \|), \phi(\| y - T_x, u \|)\right\}\end{aligned} \tag{1}$$

where $x, y, u \in X$, each two of x, y and u are distinct. Then T has an unique fixed point.

Proof. Let us take x_0 be arbitrary point in X.

define $x_{n+1} = Tx_n; n = 0,1,2,...$ and $\alpha_n = \| x_n - x_{n+1}, u \|$ for $n = 0,1,2,...;$ and $\beta_n = \phi(\alpha_n)$.

Then we have

$$\begin{aligned}\beta_{n+1} &= \phi(\alpha_{n+1}) \\ &= \phi(\| x_{n+1} - x_{n+2}, u \|) \\ &= \phi(\| Tx_n - Tx_{n+1}, u \|) \\ &\leq f(\| x_n - x_{n+1}, u \|)\phi(\| x_n - x_{n+1}, u \|) + g(\| x_n - x_{n+1}, u \|)\left[\phi(\| x_n - Tx_n, u \|) + \phi(\| x_{n+1} - Tx_{n+1}, u \|)\right] \\ &\quad + h(\| x_n - x_{n+1}, u \|)\min\left\{\phi(\| x_n - Tx_{n+1}, u \|), \phi(\| x_{n+1} - Tx_n, u \|)\right\} \\ &= f(\| x_n - x_{n+1}, u \|)\phi(\| x_n - x_{n+1}, u \|) + g(\| x_n - x_{n+1}, u \|)\left[\phi(\| x_n - x_{n+1}, u \|) + \phi(\| x_{n+1} - x_{n+2}, u \|)\right] \\ &\quad + h(\| x_n - x_{n+1}, u \|)\min\left\{\phi(\| x_n - x_{n+2}, u \|), \phi(\| x_{n+1} - x_{n+1}, u \|)\right\} \\ &= f(\alpha_n)\phi(\alpha_n) + g(\alpha_n)\left[\phi(\alpha_n) + \phi(\alpha_{n+1})\right]\end{aligned} \tag{2}$$

implies $\beta_{n+1} \leq \dfrac{f(\alpha_n) + g(\alpha_n)}{1 - g(\alpha_n)}\beta_n$.

Since $f(t) + 2g(t) + h(t) < 1$, $f(\alpha_n) + 2g(\alpha_n) < 1$ which implies $\dfrac{f(\alpha_n) + g(\alpha_n)}{1 - g(\alpha_n)} < 1$.

If we set $r = \dfrac{f(\alpha_n) + g(\alpha_n)}{1 - g(\alpha_n)}$ then from (2) we get $\beta_{n+1} \leq r\beta_n$ where $r < 1$.

So $\beta_n \leq r^n\beta_0$, such that $\beta_n \to 0$ as $n \to \infty$.

Since $\beta_n < \beta_{n-1}$ and ϕ is strictly increasing, $\alpha_n < \alpha_{n-1}$, $\quad n = 1,2,...$.

Thus $\alpha_n \to \alpha$. Then $\beta_n = \phi(\alpha_n) \to \phi(\alpha)$, since ϕ is continuous. So $\phi(\alpha) = 0$ and hence by (ii), $\alpha = 0$ implies $\alpha_n \to 0$.

Now show that $\{x_n\}$ is a Cauchy sequence.

We prove it by contradiction. Then for every positive integer ε and for every positive integer k there exist two positive integers $m(k)$ and $n(k)$ such that

$k < n(k) < m(k)$ and $\| x_{m(k)} - x_{n(k)}, u \| > \varepsilon$ (3)

For each integer k let $m(k)$ be the least integer for which $m(k) > n(k) > k$,

$\| x_{n(k)} - x_{m(k)-1}, u \| \leq \varepsilon$ and $\| x_{n(k)} - x_{m(k)}, u \| > \varepsilon$

Then we have

$$\varepsilon < \| x_{n(k)} - x_{m(k)}, u \|$$
$$\leq \| x_{n(k)} - x_{m(k)}, x_{m(k)-1} \| + \| x_{n(k)} - x_{m(k)-1}, u \| + \| x_{m(k)-1} - x_{m(k)}, u \| \quad (4)$$

Now by (1), we have

$$\begin{aligned}
\phi\left(\| x_{n(k)} - x_{m(k)}, x_{m(k)-1} \|\right) &= \phi\left(\| Tx_{n(k)-1} - Tx_{m(k)-1}, x_{m(k)-1} \|\right) \\
&\leq f\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right)\phi\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right) \\
&\quad + g\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right)\left[\phi\left(\| x_{n(k)-1} - Tx_{n(k)-1}, x_{m(k)-1} \|\right) + \phi\left(\| x_{m(k)-1} - Tx_{m(k)-1}, x_{m(k)-1} \|\right)\right] \\
&\quad + h\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right)\min\left\{\phi\left(\| x_{n(k)-1} - Tx_{m(k)-1}, x_{m(k)-1} \|\right), \phi\left(\| x_{m(k)-1} - Tx_{n(k)-1}, x_{m(k)-1} \|\right)\right\} \\
&= f\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right)\phi\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right) \\
&\quad + g\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right)\left[\phi\left(\| x_{n(k)-1} - Tx_{n(k)-1}, x_{m(k)-1} \|\right) + \phi\left(\| x_{m(k)-1} - Tx_{m(k)-1}, x_{m(k)-1} \|\right)\right] \\
&\quad + h\left(\| x_{n(k)-1} - x_{m(k)-1}, x_{m(k)-1} \|\right)\min\left\{\phi\left(\| x_{n(k)-1} - Tx_{m(k)-1}, x_{m(k)-1} \|\right), \phi\left(\| x_{m(k)-1} - Tx_{n(k)-1}, x_{m(k)-1} \|\right)\right\} \\
&= 0
\end{aligned}$$

which implies by (ii) $\| x_{n(k)} - x_{m(k)}, x_{m(k)-1} \| = 0$. (5)

So by (4) and (5) we get, $\varepsilon < \| x_{n(k)} - x_{m(k)}, u \| \leq 0 + \varepsilon + \alpha_{m(k)-1}$. Since $\{\alpha_n\}$ converges to 0,

$\| x_{n(k)} - x_{m(k)}, u \| \to \varepsilon$ as $k \to \infty$.

Again

$$\| x_{n(k)+1} - x_{m(k)}, u \| \leq \| x_{n(k)+1} - x_{m(k)}, x_{n(k)} \| + \| x_{n(k)+1} - x_{n(k)}, u \| + \| x_{n(k)} - x_{m(k)}, u \|$$
$$\leq \alpha_{n(k)} + \| x_{n(k)} - x_{m(k)}, u \| ,$$

since, $\| x_{n(k)+1} - x_{m(k)}, x_{n(k)} \|$ can be made 0 as we have done in equation (5).

So $\| x_{n(k)+1} - x_{m(k)}, u \| \leq \alpha_{n(k)} + \| x_{n(k)} - x_{m(k)}, u \| \to \varepsilon$ as $k \to \infty$.

In the similar way,

$$\| x_{n(k)+2} - x_{m(k)}, u \| \leq \| x_{n(k)+2} - x_{m(k)}, x_{n(k)+1} \| + \| x_{n(k)+2} - x_{n(k)+1}, u \| + \| x_{n(k)+1} - x_{m(k)}, u \|$$
$$\leq \alpha_{n(k)+1} + \| x_{n(k)+1} - x_{m(k)}, u \| ,$$

since $\| x_{n(k)+2} - x_{m(k)}, x_{n(k)+1} \|$ can be made 0 as we have done in equation (5).

So $\| x_{n(k)+2} - x_{m(k)}, u \| \leq \alpha_{n(k)+1} + \| x_{n(k)+1} - x_{m(k)}, u \| \to \varepsilon$ as $k \to \infty$ and in similar fashion we can show $\| x_{n(k)+2} - x_{m(k)+1}, u \| \to \varepsilon$ as $k \to \infty$.

Using (1), we deduce that

$$\begin{aligned}
\phi(\| x_{n(k)+2} - x_{m(k)+1}, u \|) &= \phi(\| Tx_{n(k)+1} - Tx_{m(k)}, u \|) \\
&\le f(\| x_{n(k)+1} - x_{m(k)}, u \|)\phi(\| x_{n(k)+1} - x_{m(k)}, u \|) \\
&+ g(\| x_{n(k)+1} - x_{m(k)}, u \|)[\phi(\| x_{n(k)+1} - Tx_{n(k)+1}, u \|) + \phi(\| x_{m(k)} - Tx_{m(k)}, u \|)] \\
&+ h(\| x_{n(k)+1} - x_{m(k)}, u \|)\min\{\phi(\| x_{n(k)+1} - Tx_{m(k)}, u \|), \phi(\| x_{m(k)} - Tx_{n(k)+1}, u \|)\} \\
&\le f(\| x_{n(k)+1} - x_{m(k)}, u \|)\phi(\| x_{n(k)+1} - x_{m(k)}, u \|) \\
&+ g(\| x_{n(k)+1} - x_{m(k)}, u \|)[\phi(\| x_{n(k)+1} - x_{n(k)+2}, u \|) + \phi(\| x_{m(k)} - x_{m(k)+1}, u \|)] \\
&+ h(\| x_{n(k)+1} - x_{m(k)}, u \|)\min\{\phi(\| x_{n(k)+1} - x_{m(k)+1}, u \|), \phi(\| x_{m(k)} - x_{n(k)+2}, u \|)\}
\end{aligned}$$

Letting $k \to \infty$, we get $\phi(\varepsilon) \le a(\varepsilon)\phi(\varepsilon) + c(\varepsilon)\phi(\varepsilon) = \{a(\varepsilon) + c(\varepsilon)\}\phi(\varepsilon) < \phi(\varepsilon)$

which is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is complete2–normed space, $\lim_{n\to\infty} x_n = z \in X$.

Claim: Show that $Tz = z$.

Again using (1) we have,

$$\begin{aligned}
\phi(\| x_{n(k)+1} - Tz, u \|) &= \phi(\| Tx_{n(k)} - Tz, u \|) \\
&\le f(\| x_{n(k)} - z, u \|)\phi(\| x_{n(k)} - z, u \|) \\
&\quad + g(\| x_{n(k)} - z, u \|)[\phi(\| x_{n(k)} - Tx_{n(k)}, u \|) + \phi(\| z - Tz, u \|)] \\
&\quad + h(\| x_{n(k)} - z, u \|)\min\{\phi(\| x_{n(k)} - Tz, u \|), \phi(\| z - Tx_{n(k)}, u \|)\}
\end{aligned}$$

implies

$$\begin{aligned}
\phi(\| x_{n(k)+1} - Tz, u \|) \le\ & f(\| x_{n(k)} - z, u \|)\phi(\| x_{n(k)+1} - z, u \|) \\
&+ g(\| x_{n(k)} - z, u \|)[\phi(\| x_{n(k)} - x_{n(k)+1}, u \|) + \phi(\| z - Tz, u \|)] \\
&+ h(\| x_{n(k)} - z, u \|)\min\{\phi(\| x_{n(k)} - Tz, u \|), \phi(\| z - x_{n(k)+1}, u \|)\}
\end{aligned}$$

passing limit as $n \to \infty$ on both sides of the inequality we get, $\phi(\| z - Tz, u \|) = 0$ which gives by (ii), $\| z - Tz, u \| = 0$ i.e., $Tz = z$.

Next let w be another fixed point of T. Then

$$\begin{aligned}
&\phi(\| z - w, u \|) = \phi(\| Tz - Tw, u \|) \\
&\le f(\| z - w, u \|)\phi(\| z - w, u \|) + g(\| z - w, u \|)[\phi(\| z - Tz, u \|) + \phi(\| w - Tw, u \|)] \\
&\quad + h(\| z - w, u \|)\min\{\phi(\| z - Tw, u \|), \phi(\| w - Tz, u \|)\} \\
&= [f(\| z - w, u \|) + h(\| z - w, u \|)]\phi(\| z - w, u \|) \\
&< \phi(\| z - w, u \|), \text{ Since } f(t) + h(t) < 1
\end{aligned}$$

which is a contradiction leads to the fact that $z = w$ and thus completes the proof.

Next we verify the Theorem (1) by a proper example.

a) *Example 1.* Let $X = R^+ \times R^+$ and d be a 2–normed which expresses $\| x - y, u \|$ as the area of the Euclidean triangle with vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $u = (u_1, u_2)$. Then $(X, \| .,. \|)$ is a complete 2–normed space[1].

Now take $x = (1,0)$, $y = (2,0)$ and $u = (1,1)$ also let $T : X \to X$ be a mapping such that

$Tx = (2,0)$ where $x = (1,0) \in X$ and

$Ty = (3,0)$ where $y = (2,0) \in X$

Now setting $f(t) = \frac{2}{5}, g(t) = \frac{1}{5}, h(t) = \frac{1}{6}$ and $\phi(t) = t^2; t \in R^+$.

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We observe that all the conditions of Theorem (1) satisfied except the condition (1). Also it is very clear that *T* has no fixed point in *X* in this case.

Next we establish a common fixed point theorem in this line.

2) *Theorem 2.* Let *S* and *T* be self-mappings of a complete 2–normed space $(X, \|.,.\|)$ and ϕ satisfying (i) and (ii). Furthermore, let f, g, h be three decreasing functions from R^+ into $[0,1)$ such that $f(t) + 2g(t) + h(t) < 1$ for every $t > 0$. Suppose *S* and *T* satisfies the following condition

$$\begin{aligned}\phi(\| Sx - Ty, u \|) \le & f(\| x - y, u \|)\phi(\| x - y, u \|) \\ & + g(\| x - y, u \|)[\phi(\| x - Sx, u \|) + \phi(\| y - Ty, u \|)] \\ & + h(\| x - y, u \|)\min\{\phi(\| x - Ty, u \|), \phi(\| y - Sx, u \|)\}\end{aligned} \tag{6}$$

where $x, y, u \in X$, each two of x, y and u are distinct. Then *S* and *T* have a unique common fixed point in *X*.

Proof. Let $x_0 \in X$ be arbitrary.

define $x_{2n} = Sx_{2n-1}$ and $x_{2n+1} = Tx_{2n}$; $n = 0,1,2,...,$ also let $\alpha_n = \| x_n - x_{n+1}, u \|$ for $n = 0,1,2,...;$ and $\beta_n = \phi(\alpha_n)$.We also assume that $\alpha_n > 0$ for every n. Now for an even integer n, we have

$$\begin{aligned}\beta_n &= \phi(\alpha_n) \\ &= \phi(\| x_n - x_{n+1}, u \|) \\ &= \phi(\| Sx_{n-1} - Tx_n, u \|) \\ &\le f(\| x_{n-1} - x_n, u \|)\phi(\| x_{n-1} - x_n, u \|) \\ &\quad + g(\| x_{n-1} - x_n, u \|)[\phi(\| x_{n-1} - Sx_{n-1}, u \|) + \phi(\| x_n - Tx_n, u \|)] \\ &\quad + h(\| x_{n-1} - x_n, u \|)\min\{\phi(\| x_{n-1} - Tx_n, u \|), \phi(\| x_n - Sx_{n-1}, u \|)\} \\ &= f(\| x_{n-1} - x_n, u \|)\phi(\| x_{n-1} - x_n, u \|) \\ &\quad + g(\| x_{n-1} - x_n, u \|)[\phi(\| x_{n-1} - x_n, u \|) + \phi(\| x_n - x_{n+1}, u \|)] \\ &\quad + h(\| x_{n-1} - x_n, u \|)\min\{\phi(\| x_{n-1} - x_{n+1}, u \|), \phi(\| x_n - x_n, u \|)\} \\ &= f(\alpha_{n-1})\phi(\alpha_{n-1}) + g(\alpha_{n-1})[\phi(\alpha_{n-1}) + \phi(\alpha_n)]\end{aligned}$$

implies $\beta_n \le \dfrac{f(\alpha_{n-1}) + g(\alpha_{n-1})}{1 - g(\alpha_{n-1})}\beta_{n-1}$ (7)

Since $f(t) + 2g(t) + h(t) < 1$, $f(\alpha_{n-1}) + 2g(\alpha_{n-1}) < 1$ which implies $\dfrac{f(\alpha_{n-1}) + g(\alpha_{n-1})}{1 - g(\alpha_{n-1})} < 1$

If we set $r = \dfrac{f(\alpha_{n-1}) + g(\alpha_{n-1})}{1 - g(\alpha_{n-1})}$

then from (7) we get $\beta_n \le r\beta_{n-1}$ where $r < 1$.

So $\beta_n \le r^n \beta_0$, such that $\beta_n \to 0$ as $n \to \infty$.

Since $\beta_n < \beta_{n-1}$ and ϕ is strictly increasing, $\alpha_n < \alpha_{n-1}$, $n = 1,2,...$.

Thus $\alpha_n \to \alpha$. Then $\beta_n = \phi(\alpha_n) \to \phi(\alpha)$, since ϕ is continuous. So $\phi(\alpha) = 0$ and hence by (ii), $\alpha = 0$ implies $\alpha_n \to 0$.

Now show that $\{x_n\}$ is a Cauchy sequence.

We prove it by contradiction. Then for every positive integer ε and for every positive integer k there exist two positive integers $2p(k)$ and $2q(k)$ such that

$k < 2q(k) < 2p(k)$ and $\| x_{2p(k)} - x_{2q(k)}, u \| > \varepsilon$. (8)

For each integer k let $2p(k)$ be the least integer for which $2p(k) > 2q(k) > k$,

$\| x_{2q(k)} - x_{2p(k)-2}, u \| \le \varepsilon$ and $\| x_{2q(k)} - x_{2p(k)}, u \| > \varepsilon$

Then we have

$$\varepsilon < \| x_{2q(k)} - x_{2p(k)}, u \| \le \| x_{2q(k)} - x_{2p(k)}, x_{2p(k)-2} \| + \| x_{2q(k)} - x_{2p(k)-2}, u \| + \| x_{2p(k)-2} - x_{2p(k)}, u \|$$

Since we can easily show that $\| x_{2q(k)} - x_{2p(k)}, x_{2p(k)-2} \| = 0$ as we have shown in (5) of Theorem (1).

$$\varepsilon < \| x_{2q(k)} - x_{2p(k)}, u \| \le \| x_{2q(k)} - x_{2p(k)-2}, u \| + \| x_{2p(k)-2} - x_{2p(k)}, u \|$$

$$\le \| x_{2q(k)} - x_{2p(k)-2}, u \| + \| x_{2p(k)-2} - x_{2p(k)}, x_{2p(k)-1} \| + \| x_{2p(k)-2} - x_{2p(k)-1}, u \| + \| x_{2p(k)-1} - x_{2p(k)}, u \|$$

again we can show like (5) of Theorem (1),

$\| x_{2p(k)-2} - x_{2p(k)}, x_{2p(k)-1} \| = 0$.

Thus

$$\varepsilon < \| x_{2q(k)} - x_{2p(k)}, u \| \le \varepsilon + 0 + \alpha_{2p(k)-2} + \alpha_{2p(k)-1} . \quad (9)$$

Since $\{\alpha_n\}$ converges to 0, $\| x_{2q(k)} - x_{2p(k)}, u \| \to \varepsilon$.

Now $\| x_{2q(k)} - x_{2p(k)+1}, u \| \le \| x_{2q(k)} - x_{2p(k)+1}, x_{2p(k)} \| + \| x_{2q(k)} - x_{2p(k)}, u \| + \| x_{2p(k)} - x_{2p(k)+1}, u \|$

$\le \| x_{2q(k)} - x_{2p(k)}, u \| + \alpha_{2p(k)}$

since we can show that $\| x_{2q(k)} - x_{2p(k)+1}, x_{2p(k)} \| = 0$ as we have done in (5) of Theorem (1).

So, $\| x_{2q(k)} - x_{2p(k)+1}, u \| \to \varepsilon$ as $k \to \infty$. (10)

Again

$$\| x_{2q(k)} - x_{2p(k)+2}, u \| \le \| x_{2q(k)} - x_{2p(k)+2}, x_{2p(k)+1} \| + \| x_{2q(k)} - x_{2p(k)+1}, u \| + \| x_{2p(k)+1} - x_{2p(k)+2}, u \|$$

$$\le \| x_{2q(k)} - x_{2p(k)+1}, u \| + \| x_{2p(k)+1} - x_{2p(k)+2}, u \| ,$$

(since $\| x_{2q(k)} - x_{2p(k)+2}, x_{2p(k)+1} \| = 0$ for similar reason as of (5) of Theorem (1).)

$$\le \| x_{2q(k)} - x_{2p(k)+1}, x_{2p(k)} \| + \| x_{2q(k)} - x_{2p(k)}, u \| + \| x_{2p(k)} - x_{2p(k)+1}, u \| + \| x_{2p(k)+1} - x_{2p(k)+2}, u \|$$

$$\le 0 + \| x_{2q(k)} - x_{2p(k)}, u \| + \alpha_{2p(k)} + \alpha_{2p(k)+1}$$

which gives $\| x_{2q(k)} - x_{2p(k)+2}, u \| \to \varepsilon$ as $k \to \infty$. (11) Similarly,

$\| x_{2q(k)+1} - x_{2p(k)+2}, u \| \to \varepsilon$ as $k \to \infty$. (12)

Now from (6) we have

$$\phi\left(\| x_{2p(k)+2} - x_{2q(k)+1}, u \|\right) = \phi\left(\| Sx_{2p(k)+1} - Tx_{2q(k)}, u \|\right)$$
$$\le f\left(\| x_{2p(k)+1} - x_{2q(k)}, u \|\right)\phi\left(\| x_{2p(k)+1} - x_{2q(k)}, u \|\right)$$
$$+ g\left(\| x_{2p(k)+1} - x_{2q(k)}, u \|\right)\left[\phi\left(\| x_{2p(k)+1} - Sx_{2p(k)+1}, u \|\right) + \phi\left(\| x_{2q(k)} - Tx_{2q(k)}, u \|\right)\right]$$
$$+ h\left(\| x_{2p(k)+1} - x_{2q(k)}, u \|\right)\min\left\{\phi\left(\| x_{2p(k)+1} - Tx_{2q(k)}, u \|\right), \phi\left(\| x_{2q(k)} - Sx_{2p(k)+1}, u \|\right)\right\}$$

passing limit as $k \to \infty$ we get by (10), (11) and (12),

$$\phi(\varepsilon) \le f(\varepsilon)\phi(\varepsilon) + h(\varepsilon)\phi(\varepsilon) = \{f(\varepsilon) + h(\varepsilon)\}\phi(\varepsilon) < \phi(\varepsilon)$$

which is a contradiction. So $\{x_n\}$ is a Cauchy sequence. Since X is complete 2–normed space, $\lim_n x_n = z \in X$.

Again using (6) we have

$$\phi(\| x_{2p(k)+2} - Tz, u \|) = \phi(\| Sx_{2p(k)+1} - Tz, u \|)$$
$$\leq f(\| x_{2p(k)+1} - z, u \|)\phi(\| x_{2p(k)+1} - z, u \|)$$
$$+ g(\| x_{2p(k)+1} - z, u \|)[\phi(\| x_{2p(k)+1} - Sx_{2p(k)+1}, u \|) + \phi(\| z - Tz, u \|)]$$
$$+ h(\| x_{2p(k)+1} - z, u \|)\min\{\phi(\| x_{2p(k)+1} - Tz, u \|), \phi(\| z - Sx_{2p(k)+1}, u \|)\}$$

Taking limit as $k \to \infty$ we get $\phi(\| z - Tz, u \|) = 0$ implies $\| z - Tz, u \| = 0$ by property (ii).

Hence $Tz = z$.

Similarly it can be shown that $Sz = z$. So S and T have a common fixed point $z \in X$.

We now show that z is the unique common fixed point of S and T.

If not, then let w be another fixed point of S and T.

Then $\phi(\| z - w, u \|) = \phi(\| Sz - Tw, u \|)$

$$\leq f(\| z - w, u \|)\phi(\| z - w, u \|)$$
$$+ g(\| z - w, u \|)[\phi(\| z - Sz, u \|) + \phi(\| w - Tw, u \|)]$$
$$+ h(\| z - w, u \|)\min\{\phi(\| z - Tw, u \|), \phi(\| w - Sz, u \|)\}$$
$$= [f(\| z - w, u \|) + h(\| z - w, u \|)]\phi(\| z - w, u \|)$$
$$< \phi(\| z - w, u \|), \text{ since } f(t) + h(t) < 1$$

which is a contradiction. Hence $z = w$ and thus completes the proof.

Remark 1. In the same way we can verify the Theorem (2) by setting $S(1,0) = (2,0)$ and $T(2,0) = (3,0)$ taking all the values same on the complete 2–normed space $(X, \| .,. \|)$ as described in Example 1.

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