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# Study to Linear Topological Space

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***Abstract:** In this theis several topies from Topology linear Algebra and Real Analysis are com-bined in the study of linear topological spaces. We begin with a brief look at linear spaces before moving on to study some basic properties of this structure of linear topological basis. Then we turn our attention to linear spaces with a metric topology. In particular we consider problems involving normed liner spaces bounded linear transformation and Hilbert spaces*
***Keywords:** topology, linear algebra*

## I. INTRODUCTION

P. Thangavelu and Nithanantha Jothi introduced the concept of binary topology in (4). It is a single topological structure that carrier the subjects of a set x as well as the subsets of another set x for studying the information about the orderded pair (A.B.) of subset of x and y. A linear topological space endowed with a topology such that the vector addition and sclar multiplication are both continuos the theory of linear topological spaces provide a remarkable economy in discussion of many classical mathematical problems. We introduce the concept of binary topology to linear section 2. We define the binary linear jtopology. Section 3 Space (BLTS) We prove that the binary product of two linear topological spcae is a BLTS. Also we discuss to concept of locally convese BLTS and locally bounded BLTS and prove some of their properties. In section 4 we define binary metric and binary normal. The main result of this section is that the binary product preserve metrizablity and normbility. Section 5 deals with the construction of aBLTS using a family of binary seminorms.

## II. PRELIMINARIES

1) *Definition:* Let x and y be any two non-empty and d (x) and g (y) be their power sets respectively. A binary topology from x to y is a binary structure M Í d (x) x d (y) that satisties the following arioms (f, f) and (x, y) ÎM

If $(A_1, B_1)$ and $(A_2, B_2)$ ÎM, then $(A_1$ Ç $A_2, B_1$ Ç $B_2)$ ÎM.

If $\{(A_a, B_a)$ : a ÎD} is a family of members of M; then ($U_d$ Î$DA_d$, $U_a$ Î$DB_a$) ÎM.

If M is a binary topology from x to y then the triplet (x, y, m) is called a binary topology space and the members of M are called binary points of binary open sets. (C,D) is called binary colsed if (x | c, y | D) is binary open. The elements of x, x y are called the binary points of the binary topological space (x, y, m) yet (x, y, m) be a binary topological space and let (x, y) Î x x x The binary open set (A, B) is called a binary neighborhood of (x, y) if x Î A and ÎB. If x = y then M is called a binary topology on x and we write (x, M) as a binary space.

2) *Proposition:* Let (x, y, m) be a binary topological space. Then

(1) $_T$(M) = {A Í x : (A, B) ÎM for some B Í Y} is a topology on x.

T1(M) = {BÍ Y: (A, B) ÎM for some A Í x} is a topology on y.

## III. BINARY LINEAR TOPOLOGY

1) *Definition:* A binary topology between two vector space is said to be binary linear if the two operation are continuous i. e, if $V_1$, and $V_2$ are vector space over the some field k and for every neighbourhoods U of (x1 + x2 , y1+ y2) Î$V_1$ x $V_2$. ' two neighbourthoods $U_1$ and $U_2$ of $(x_1, y_1)$ and $(x_2, y_2)$ respectively that $U_1$ + $U_2$ Í U. Similarly for every neighbourthood W of (lx, ly) $v_1 \times v_2$ there exsits a neighbourthood w of (x, y) such that lw Í w. If M is a binary linary topology between two vector space $V_1$ and $V_2$ then triplet $(V_1, V_2, M)$ is called a binary linear topological space (BLTS).
2) *Proposition:* If $(V_1\ T_1)$ and $V_2, T_2)$ are two linear topologica space then $(V_1, V_2, T_1 \times T_{2)}$ is a called the binary linear topological space.

a) *Proof:* By proposition 2. 3, ($V_1$, $V_2$, $T_1$ x $T_2$) is a binary topological space. If remains to show that $T_2$ x $T_2$ is a binary linear topology let ($x_1$, $x_2$), ($y_1$, $y_2$) ÎV1 × V2 and ($A_1$, $A_2$) be a neighbourhood of [($x_1$, $x_2$) + ($y_1$, $y_2$)]. Then $x_1$ + $y_1$ Î $A_1$ and $x_2$ + $y_2$ Î $A_2$. Since $A_1$ Î $T_1$ Î$T_1$ and $T_2$, and $T_1$ and $T_2$ are linear topological there exist neighhbourthood $B_1$ and $C_1$ of $x_1$ and $y_1$ respectively in $T_1$ such that $B_1$ + $C_1$ Î $A_1$ and neighbourhood $B_2$ and $C_2$ of $x_2$ and $y_2$ respectively in $T_2$ such that $B_2$ + $C_2$ Í $A_2$. Then in $T_1$ × $T_2$ ($B_1$, $B_2$) is a neighbourhood ($B_1$, that ($B_1$, $B_2$) + ($C_1$, $C_2$) = $T_1$ × $T_2$ ($B_1$, $B_2$) Now Let ($A_1$, $A_2$) be a neighborhood of l ($x_1$,$x_2$) $T_1$ × $T_2$ Then A1 is a neighborhood of l$x_1$ in $T_1$ and $T_2$ $A_2$ is a $x_2$ in $T_2$. So there exists two $B_1$ and $B_2$ off $x_1$ and $x_2$ respectively such that l$B_1$ Í $A_1$ and $B_2$ Í $A_2$. This implies that ($B_1$, $B_2$) is a neighbourhood of ($x_1$, $x_2$) such that l($B_1$, $B_2$) Í ($A_1$, $A_2$). Thus $T_1$ × $T_2$ is a binary linear topology.

3) *Proposition:* If ($V_1$. $V_2$, M) is a BLTS, then a(M) = {A Í $V_1$ : (A, B) Î M for some B
Î$V_2$} is a linear topology on $V_1$ and a (M) = {B Í $V_2$) (A, B) Î M for come A Í $V_1$} is a linear topology on $V_2$.

a) *Proof:* By proposition a (M) are both topologies in $V_1$ a $V_2$ respectively. Let $x_1$, $y_1$ Î $V_1$ and A ÎT(M) contains x, + $y_1$. Then for some $x_2$, $y_2$ Î $v_2$ there exsists B Í $V_2$ such that ($x_1$ ++ $y_1$, $x_2$ + $y_2$) Î (A, B) Where (A, B) Î M, since M is a binary linear topology, there exist ($E_1$, $E_2$) and $F_1$, $F_2$) in M such that ($x_1$, $x_2$) ($e_1$, $e_2$), ($y_1$, $y_2$) Î ($f_1$, $f_2$) and ($E_1$, $E_2$) + ($F_1$ + $F_2$) Í (A, B). x, Î $E_1$, $Y_1$ Î F1and ($E_1$, $E_2$) by the definition of binary sets. Also $E_1$ and $F_1$ $Î_1$ (M) by the construction of (T). Similary for lxÎA. Where for lxÎA. Where A ÎT(M) we can find also a linear of x say U such that ÎUÍA. Thus $_T$(M) is linear topology in the same say we can prove that (M) topology.

4) *Definition:* A local base of a binary linear topology ($V_1$, $V_2$, M) is the base Consists of the neighborhood of a binary points (x, y)

5) *Definition:* A set (A, B) Îd ($V_1$) × d ($V_2$) is convese if for all pairs ($x_1$, $x_2$), ($y_1$, $y_2$) Î (A, B) l ($x_1$, $x_2$) + (1 - l) ($y_1$, $y_2$) Î (A, B) lÎ (0, 1).

6) *Definition:* A binary topology is called locally conver if there exist a local base at (0, 0) whose members are conver.

7) *Definition:* A BLTS is locally bounded of (0, 0) as a bounded neighbourhood, i, e, a neighbourhood (E, F) such that (N, M) ÎNo. the set of neighbourhood of (0, 0) there exists S Î R such that t> S, (E, F) Í t (N, M). Let ($V_1$, $V_2$, M) be a BLTS. Then for every ($w_1$, $w_2$) Î No. ' balanced and symmetic sets ($x_1$, $y_1$), ($x_2$, $y_2$) Î No such that ($x_1$, $x_2$) t ($x_2$, $y_2$) c ($w_1$, $w_2$).

a) *Proof :* If ($w_1$, $w_2$) ÎNo then $w_1$ and $w_2$ are neighbourhood of O in ($v_1$, T (M) and ($V_2$, T respectively by the property of linear topologices there exists symmetrie balanced neighbourhood of 0, $x_1$, $x_2$ Î T(M) and y1 + $y_2$ C$W_2$ Now, $x_1$, $y_1$ are Þ aÎR with |a| £ 1, a $x_1$ c $x_1$ and $y_1$ c$y_1$.

So a($x_1$, $y_1$) = (a $x_1$, $y_1$) C ($x_1$, $y_1$) thus (x, $y_1$) and ($x_2$, $y_2$) are balanced by the symmetry of x1 and y1 we get x1 = -$x_1$, $y_1$ = -$y_1$ Þ ($x_1$, $y_1$) = (-$x_1$, $y_1$) thus ($x_1$, $y_1$) is symmetric and similarly ($x_2$, $y_2$) is also symmetric. ($x_1$ $y_1$) + ($x_2$, $y_2$) = ($x_1$ + $x_2$, y1 +$y_2$) C ($w_1$, $w_2$).

8) Proposition: Let $V_1$ and $V_2$ be real vector space and $U_1$ be a convex set in $V_1$ and $U_1$ be a convex set in $V_2$ then ($U_1$, $U_2$) is convex d ($V_1$) × d ($V_2$).

a) *Proof :* Let ($x_1$, $y_1$) Î ($U_1$, $U_2$) for i = 1, 2 then x1Î $U_1$ Î $U_2$ for i = 1, 2 Þ l$x_1$ + (1 - l) $x_2$ Î
U1 for 0 £ > Í 1. So (l$x_1$ + ($x_1$, $y_1$) + (1 - l) $y_2$) ($U_1$, $U_2$). Consider l($x_1$ , $y_1$) + (1 - l) ($x_2$, $y_2$)
= (l$x_1$, l$y_1$) + (1 - l) $y_2$) ¬ $U_1$, $U_2$) for ) £ > Í 1. Thus ($U_1$, $U_2$) is convex.

9) *Corollary:* If ($V_1$, $T_1$) and ($V_2$, $T_2$) are both locally convex topological vector spaces then their binary product ($V_1$, $V_2$, $T_1$ × $T_2$) is locally convex BLTS.

10) *Proposition:* Let $U_1$ and $U_2$ be bounded sets in two uleal vecotor spaces $V_1$ and $V_2$ respectively then bounded.

a) *Proof:* Since $U_1$ is bounded for every neighbourhood e1Î No. ($V_1$). 's1 ÎR such that t> $E_2$ ÎNo. ($V_2$), ' $s_2$ ÎR such that t > S1, V1 C tE1. Similarly for every neighbourhood $E_2$ ÎNo. ($V_2$), ' s2 Î R such that t > $S_2$, $U_2$ C t$E_2$. Let $T_1$ Î R correspound to E and $T_2$ Î to f then t > $t_1$, $U_1$ C tE and t > $t_2$. $U_2$ c tf. So t> S, where S = max ($t_1$, $t_2$), U1 c tE and $U_2$ ctf i.e ($U_1$ $U_2$) Ì t (E, F), t >S. Thus ($U_1$, $U_2$) is bounded.

11) Corollary : If ($V_1$ $T_1$) and ($V_2$ $T_2$) are both locally bounded topological vector spaces, then their binary product ($V_1$, $V_2$ T1 × $T_2$) is a locally bounded BLTS.

12) Proposition: Let ($V_1$, $T_1$) be a topological vector space and $V_2$ be another vector space such that map T : $V_1$®$V_2$ is an isomo rphism. Then $T_2$ = {T(A) : A Î$T_1$} is a linear topology in $V_2$ and hence $T_1$ × $T_2$ is a binary linear topology from $V_1$ to V2.

a) *Proof:* Since T is an isomorphism, T (f) = f and T($V_1$) = $V_2$ and So f $V_2$ and So f $V_2$ Î$T_2$. Let A. B Î$T_2$. Then A = T(A) and B = T(B) for some A' and B' Î $T_1$. So A' Ç B' Î $T_1$ (A' Ç B') Î $T_2$ T (A' Ç B') = T (A') Ç (B') = A Ç B Thus. A Ç B Î $T_2$. Now Let {Aa} . . $_T$ $T_2$ for some index set. T then exists (Ba). . $_T$ e$T_1$ Such that Aa = T (Ba) for each aeT Then U . . $_T$ Î$T_2$ for each aeT. So $x_1$ + $y_1$ ¬ A and U. . $_T$ Aa = U. . $_T$ T (Ba). Then $B_1$, $B_2$ Î $T_2$ and $x_1$ Î$A_1$ Þ $x_2$ = T ($x_1$) ÎT ($A_1$) = $B_1$ $y_1$.

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## IV. BINARY MERITABLE AND BINARY NORMABLE BLTS

1) *Definition:* A binary metric on two sets $V_1$ and $V_2$ is a map d : $(V_1 \times V_2) \times (V_1 \times V_2)$ ® R satisfying the following axioms : If $(x_1, x_2)$ $(y_1, y_2)$ Î $V_1 \times V_2$ then.

df $[(x_1, x_2), (y_1, y_2)]$ ³ 0 d $[(x_1, x_2), (y_1, y_2)]$ = d $[(y_1, y_2), (x_1, x_2)]$ and.

d $[(x_1, x_2), (y_1, y_2)]$ £ d $[(x_1, x_2), (z_1, z_2)]$ + d $[(z_1, z_2), (y_1, y_2)]$ for every $(z_1, z_2)$ Î $v_1 \times v_2$

d $[(x_1, x_2), (y_1, y_2)] = 0$ Û $x_1 = x_2$ and $y_1 = y_2$.

2) *Definition:* Let $(v_1, v_2, M)$ be a BLTS. A binary topology M is metrizable with a binary metric d if for any (x, y) in some binary open set (A, B) Î M, ' g > 0 Such that B, (x, y)

CB where pi is the projection map to $V_1$ for i = 1, 2.

3) *Proposition:* If $(V_1, T_1)$ and $(V_2, T_2)$ are two linear topological space such that $T_1$ and $T_2$ are both metrizable with metrics $d_1$ and $d_2$ respectively then $T_1 \times T_2$ are both metrizable with metrics $d_1$ and $d_2$ respectively then $T_1 \times T_2$ is binary metrizable.

a) *Prof:* Consider the map d : $(V_1 \times V_2) \times (V_1 \times V_2)$ R defined by

$$d((x_1, x_2), (y_1, y_2)) = \frac{d_1 (x_1, y_1) + d_2 (x_2, y_2)}{2}, \ (x_1, x_2), (y_1, y_2) ¬ (v_1 \times v_2)$$

If $(x_1, x_2), (y_1, y_2)$ Î $v_1 \times v_2$ then

(1) $$d [(x_1, x_2) (y_1, y_2)] = \frac{d_1 (x_1, y_1) + d_2 (x_2, y_2)}{2}$$ ³ 0, since $d_1$.

$(x_1, y_1)$ and $d_2 (x_2, y_2)$ are both non-negative.

(2) $$d [(x_1, x_2), (y_1, y_2)] = \frac{d_1 (x_1, y_1) + d_2 (x_2, y_2)}{2}$$ 0 Û $d_1 (x_1 y_1) = 0$ and $d_2 (x_2, y_2) = 0$.

This happens if and only x1 = $x_2$ and $y_1 = y_2$ i. e. when $(x_1, y_1) = (x_2, y_2)$

(3) $$d[(x_1, x_2) = d (y_1, y_2)] = \frac{d_1 (x_1, y_1) + d_2 (x_2, y_2)}{2} = \frac{d_1 (y_1 + y_1) + d_2}{2}$$

$(y_2, x_2) = d (y_1, y_2), x_1 x_2))$ and if $(z_1, z_2)$ Î $v_1 \times v_2$

(4) $$d[(x_1, x_2), (y_1, y_2)] = \frac{(x_1, y_1) + d_2 (x_2, y_2)}{2} = \frac{£ (d_1 (x_1 , z_1) + d_1}{2}$$

$$(z_1, y_1)] + [d_2 (x_2, z_2) + d_2 (z_2, y_2)]^2 = \frac{d_1 (x_1, z_1) + d_2 (x_2, z_2)}{2} + \frac{d_1 (z_1 , y_1) +}{2}$$

$d_2 (x_2, y_2) = d[(x_1, x_2), (z_1, z_2)] + d [(z_1, z_2), (y_1, y_2)]$.

Thus dis a binary metric let (A, B) Î $T_1 \times T_2$ and (x, y) Î (A, B) Then x Î A Î $T_1$ and y Î B Î $T_2$ since $T_1$ and $T_2$ are metrizable. '$G_1, G_2$ >0 with respect to $d_1$ and $d_2$ respectively such that $Br_1$ (x) CA and $Br_2$ (y) c B. i. e if $d_1(x_1\ x_1) < n$ then $x_1$ Î $Br_1$ (x) and if $d_2 (y_1\ y_1) < r_2$ then $y_1$ Î$Br_2$ (y) Þ (x, y,) ¬ (A, B) let r = min $(r_1, r_2)$ and (u, v) Î B6/2 (x, y) then (x, y), (u, v) < r/2, i, e, d1 $(x_1\ y) + d_2 (y, v)$ < r/2. So $d_1 (x, u) + d_2 (y, v)$ < r/2, i, e $d_1\ d_1 (x_1, u_1) + d_2 (y, v)$

2 <r/2. So d1 (x, u) + $d_2$ (y, v) <r<$r_1$ and d (y, v) <r<$r_2$ Hence u Î $Br_1$

(x) c and ut $Br_2$ (y) CB. Thus (u, v) Î (A, B)showing that Br/2 (xy) C (A, B).

4) *Definition:* A binary seminorm on two vector space $V_1$ and $V_2$ is a map ||.|| : $V_1 \times V_2$ ®Rsuch that for each ($x_1$ $x_2$) ($y_1$ $y_2$) $V_1 \times V_2$.

|| ($x_1$, $x_2$) || ³ 0

|| a ($x_1$, $x_2$) || = |a | || ($x_1$, $x_2$) ||

|| ($x_1$, $x_2$) + ($y_1$, $y_2$) || £ || ($x_1$, $x_2$) || + || ($y_1$, $y_2$) ||.

A binary seminarm becomes a binary norm if the following condition holds.

|| ($x_1$, $x_2$) || = 0 Û ($x_1$, $x_2$) = (0, 0)

5) *Proposition:* If ($V_1$, $T_1$) and ($V_2$, $T_2$) are both normable topological vector space, then their ubinary product is binary normable.

a) *Proof:* Let $||.||_1$ and $||.||_2$ be the norms corres ponding to $t_1$ and $t_2$ respectively. Then we get two metrics $d_1$ and $d_2$ defind by $d_1$ (($x_1$, $x_2$) ($y_1$, $y_2$( = ||($x_1$, $x_2$) - ($y_1$, $y_2$) || i, i = 1, 2 and ($x_1$,$x_2$) ($y_1$, $y_2$) $v_1 \times v_2$ with which $t_1$ and $t_2$ are metrizable respectively. So by proposition $T_1 \times T_2$ is metrizable with which $T_1$ and $d_1$ ($x_1$, $y_1$) + $d_2$ ($x_2$, $y_2$) V ($x_1$, $x_2$), ($y_1$, $y_2$) ($v_1 \times v_2$) Hence the binary norm ||.|| defind by || ($x_1$, $x_2$) Î $v_1 \times v_2$ corresponds to the topology $T_1 \times T_2$ but

this norm is same as $\dfrac{||.||_1 + ||.||_2}{2}$ since $||(x_1.\ x_2)|| = d\,(x_1, x_2)\,(0, 0) = \dfrac{d_1 (x_1, 0) + d_2 (x_2, 0)}{2}$

$$= \frac{|| x_1 - 0||_1 + ||x_2 - 0||_2}{2} = \frac{|| x_1 ||_1 + ||x_2||_2}{2}$$

6) *Lemma:* Let $V_1$ and $V_2$ be two vector space and P be a binary seminorm on $V_1 \times V_2$

Then there exists two seminorm $P_1$ and $P_2$ on $V_1$ and $V_2$ respectively.

a) *Proof:* Let $P_1$ : $V_1$ ® R be defined by $P_1$ (x) = infy {P(x, y) : y Î$v_2$} since P (x, y) ³ 0 , (x, y) ¬ $V_1 \times V_2$, P1 (x) ³ 0 x Î$V_1$ for x Î$V_1$ and aÎK, $P_1$ (ax) = inf (P(ax,y) : y Î$v_2$)

inf |a| P (x $^1$ y) : y Î$v_2$) 2y

|a| inf (x $^1$ y) : y Î$v_2$) 2y

= | d | P1 | x |

for x, y Îv1 P1 (x + y) = inf (P (x+y, z) : z Î v2)

= $\inf\limits_{z = z_1 + z_2}$ (P (x+y. $z_1$ + $z_2$) : z = $z_1$ + $z_2$ Î $v_2$)

= $\inf\limits_{z_1\, z_2}$ (P (x, $z_1$) + (y. $z_2$) : $z_1$, $z_2$ Î $v_2$)

= £ $\inf\limits_{z_1\, z_2}$ {(P (x, $z_1$) + P ($y_1$ $z_2$) : $z_1$, $z_2$ Î $v_2$}

Thus P1 (x + y)  P1 (x) + P1 (y)

Hence P1 is a seminors on V1 and family P2 : V2  R defined by P2 (y) = infx {P(x, y): x V1} is a seminorm on V2.

## V. CONCLUSION

In This paper we have introduced the concept of linear topological space to situation in which we have to deal with two vector space and a topology between the spaces. This helps to study both the space simultaneourly. The concept of topological vector space is well used in mathematics engineering and science and particularly is auantum mechains. **Heance** our theory of Binary linear Topological space helps in the future development of such ares.

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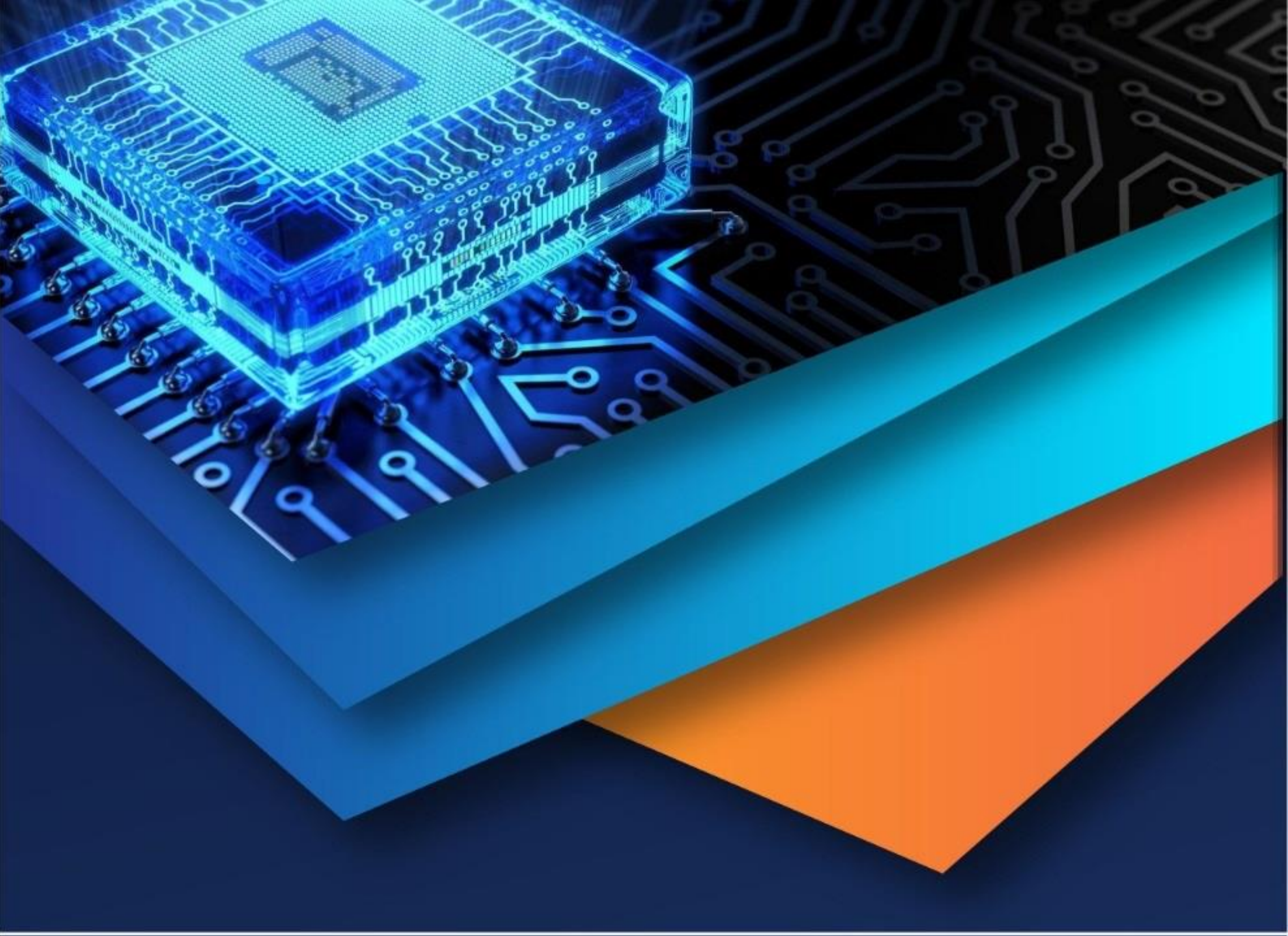

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