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On the Positive Pell Equation $y^2 = 8x^2 + 49$

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Abstract: The binary quadratic Diophantine equation represented by the positive pellian $y^2 = 8x^2 + 49$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.
Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 8x^2 + 49$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 8x^2 + 49 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 2, y_0 = 9 \quad D = 8$$

The pellian equation is

$$y^2 = 8x^2 + 1 \tag{2}$$

The initial solution of pellian equation is

$$\tilde{x}_0 = 1, \tilde{y}_0 = 3,$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{8}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n$$

Where,

$$f_n = (3+\sqrt{8})^{n+1} + (3-\sqrt{8})^{n+1}$$

$$g_n = (3+\sqrt{8})^{n+1} - (3-\sqrt{8})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = f_n + \frac{9}{2\sqrt{8}} g_n$$

$$y_{n+1} = \frac{9}{2} f_n + \frac{8}{\sqrt{8}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 6x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - 6y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	2	9
1	15	43
2	88	249
3	513	1451
4	2990	8457

From the above table, we observe some interesting relations among the solutions which are presented below.

A. Both x_n *and* y_n values are odd and even.

B. Each of the following expression is a nasty number:

1) $\frac{6}{49}\left[98 + 18y_{2n+2} - 32x_{2n+2}\right]$

2) $\frac{6}{49}\left[98 + 18x_{2n+3} - 86x_{2n+2}\right]$

3) $\frac{6}{147}\left[294 + 9x_{2n+4} - 249x_{2n+2}\right]$

4) $\frac{6}{147}\left[294 + 18y_{2n+3} - 240x_{2n+2}\right]$

5) $\frac{6}{833}\left[1666 + 18y_{2n+4} - 1408x_{2n+2}\right]$

6) $\frac{6}{147}\left[294 + 86y_{2n+2} - 32x_{2n+3}\right]$

7) $\frac{6}{833}\left[1666 + 498y_{2n+2} - 32x_{2n+4}\right]$

8) $\frac{6}{392}\left[784 + 240y_{2n+2} - 32y_{2n+3}\right]$

9) $\frac{6}{2352}\left[4704 + 1408y_{2n+2} - 32y_{2n+4}\right]$

10) $\frac{6}{49}\left[98 + 86y_{2n+3} - 240x_{2n+3}\right]$

11) $\frac{6}{49}\left[98 + 86x_{2n+4} - 498x_{2n+3}\right]$

12) $\frac{6}{147}\left[294 + 86y_{2n+4} - 1408x_{2n+3}\right]$

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13) $\frac{6}{147}\left[294+498y_{2n+3}-240x_{2n+4}\right]$

14) $\frac{6}{49}\left[98+498y_{2n+4}-1408x_{2n+4}\right]$

15) $\frac{6}{392}\left[784+1408y_{2n+3}-240y_{2n+4}\right]$

C. Each of the following expressions is a cubical integer.

1) $\frac{1}{49}\left[18y_{3n+3}-32x_{3n+3}+54y_{n+1}-96x_{n+1}\right]$

2) $\frac{1}{49}\left[18x_{3n+4}-86x_{3n+3}+54x_{n+2}-258x_{n+1}\right]$

3) $\frac{1}{147}\left[9x_{3n+5}-249x_{3n+3}+27x_{n+3}-747x_{n+1}\right]$

4) $\frac{1}{147}\left[18y_{3n+4}-240x_{3n+3}+54y_{n+2}-720x_{n+1}\right]$

5) $\frac{1}{833}\left[18y_{3n+5}-1408x_{3n+3}+54y_{n+3}-4224x_{n+1}\right]$

6) $\frac{1}{147}\left[86y_{3n+3}-32x_{3n+4}+258y_{n+1}-96x_{n+2}\right]$

7) $\frac{1}{833}\left[498y_{3n+3}-32x_{3n+5}+1494y_{n+1}-96x_{n+3}\right]$

8) $\frac{1}{392}\left[240y_{3n+3}-32y_{3n+4}+720y_{n+1}-96y_{n+2}\right]$

9) $\frac{1}{2352}\left[1408y_{3n+3}-32y_{3n+5}+4224y_{n+1}-96y_{n+3}\right]$

10) $\frac{1}{49}\left[86x_{3n+5}-498x_{3n+4}+258x_{n+3}-1494x_{n+2}\right]$

11) $\frac{1}{49}\left[86y_{3n+4}-240x_{3n+4}+258y_{n+2}-720x_{n+2}\right]$

12) $\frac{1}{147}\left[86y_{3n+5}-1408x_{3n+4}+258y_{n+3}-4224x_{n+2}\right]$

13) $\frac{1}{147}\left[498y_{3n+4\backslash}-240x_{3n+5}+1494y_{n+2}-720x_{n+3}\right]$

14) $\frac{1}{49}\left[498y_{3n+5\backslash}-1408x_{3n+5}+1494y_{n+3}-4224x_{n+3}\right]$

15) $\frac{1}{392}\left[1408y_{3n+4\backslash}-240y_{3n+5}+4224y_{n+2}-720y_{n+3}\right]$

D. Each of the following expressions is a biquadratic integer.

16) $\frac{1}{49}\left[18y_{4n+4}-32x_{4n+4}+72y_{2n+2}-128x_{2n+2}+294\right]$

17) $\frac{1}{49}\left[18x_{4n+5}-86x_{4n+4}+72x_{2n+3}-344x_{2n+2}+294\right]$

18) $\frac{1}{147}\left[9x_{4n+6}-249x_{4n+4}+36x_{2n+4}-996x_{2n+2}+882\right]$

19) $\frac{1}{147}\left[18y_{4n+5}-240x_{4n+4}+72y_{2n+3}-960x_{2n+2}+882\right]$

20) $\frac{1}{833}\left[18y_{4n+6}-1408x_{4n+4}+72y_{2n+4}-5632x_{2n+2}+4998\right]$

21) $\frac{1}{147}\left[86y_{4n+4}-32x_{4n+5}+344y_{2n+2}-128x_{2n+3}+882\right]$

22) $\frac{1}{833}\left[498y_{4n+4}-32x_{4n+6}+1992y_{2n+2}-128x_{2n+4}+4998\right]$

23) $\frac{1}{392}\left[240y_{4n+4}-32y_{4n+5}+960y_{2n+2}-128y_{2n+3}+2352\right]$

24) $\frac{1}{2352}\left[1408y_{4n+4}-32y_{4n+6}+5632y_{2n+2}-128y_{2n+4}+14112\right]$

25) $\frac{1}{49}\left[86x_{4n+6}-498x_{4n+5}+344x_{2n+4}-1992x_{2n+3}+294\right]$

26) $\frac{1}{49}\left[86y_{4n+5}-240x_{4n+5}+344y_{2n+3}-960x_{2n+3}+294\right]$

27) $\frac{1}{147}\left[86y_{4n+6}-1408x_{4n+5}+344y_{2n+4}-5632x_{2n+3}+882\right]$

28) $\frac{1}{147}\left[498y_{4n+5}-240x_{4n+6}+1992y_{2n+3}-960x_{2n+4}+882\right]$

29) $\frac{1}{49}\left[498y_{4n+6}-1408x_{4n+6}+1992y_{2n+4}-5632x_{2n+4}+294\right]$

30) $\frac{1}{392}\left[1408y_{4n+5}-240y_{4n+6}+5632y_{2n+3}-960y_{2n+4}+2352\right]$

E. Each of the following expression is a quintic integer:

1) $\frac{1}{49}\left[18y_{5n+5}-32x_{5n+5}+90y_{3n+3}-160x_{3n+3}+180y_{n+1}-320x_{n+1}\right]$

2) $\frac{1}{49}\left[18x_{5n+6}-86x_{5n+5}+90x_{3n+4}-430x_{3n+3}+180x_{n+2}-860x_{n+1}\right]$

3) $\frac{1}{147}\left[9x_{5n+7}-249x_{5n+5}-45x_{3n+5}-1245x_{3n+3}+90x_{n+3}-2490x_{n+1}\right]$

4) $\frac{1}{147}\left[18y_{5n+6}-240x_{5n+5}+90y_{3n+4}-1200x_{3n+3}+180y_{n+2}-2400x_{n+1}\right]$

5) $\frac{1}{833}\left[18y_{5n+7}-1408x_{5n+5}+90y_{3n+5}-7040x_{3n+3}+180y_{n+3}\right]$

6) $\frac{1}{147}\left[86y_{5n+5}-32x_{5n+6}+430y_{3n+3}-160x_{3n+4}+860y_{n+1}-320x_{n+2}\right]$

7) $\frac{1}{833}\left[498y_{5n+5}-32x_{5n+7}+2490y_{3n+3}-160x_{3n+5}+4980y_{n+1}-320x_{n+3}\right]$

8) $\frac{1}{392}\left[240y_{5n+5}-32y_{5n+6}+1200y_{3n+3}-160y_{3n+4}+2400y_{n+1}-320y_{n+2}\right]$

9) $\frac{1}{2352}\left[1408y_{5n+5}-32y_{5n+7}+7040y_{3n+3}-160y_{3n+5}+14080y_{n+1}-320y_{n+3}\right]$

10) $\frac{1}{49}\left[86x_{5n+7}-498x_{5n+6}+430x_{3n+5}-2490x_{3n+4}+860x_{n+3}-4980x_{n+2}\right]$

11) $\frac{1}{49}\left[86y_{5n+6}-240x_{5n+6}+430y_{3n+4}-1200x_{3n+4}+860y_{n+2}-2400x_{n+2}\right]$

12) $\frac{1}{147}\left[86y_{5n+7}-1408x_{5n+6}+430y_{3n+5}-7040x_{3n+4}+860y_{n+3}-14080x_{n+2}\right]$

13) $\frac{1}{147}\left[498y_{5n+6}-240x_{5n+7}+2490y_{3n+4}-1200x_{3n+5}+4980y_{n+2}-2400x_{n+3}\right]$

14) $\frac{1}{49}\left[498y_{5n+7}-1408x_{5n+7}+2490y_{3n+5}-7040x_{3n+5}+4980y_{n+3}-14080x_{n+3}\right]$

15) $\frac{1}{392}\left[1408y_{5n+6}-240y_{5n+7}+7040y_{3n+4}-1200y_{3n+5}+14080y_{n+2}-24840y_{n+3}\right]$

F. Relations among the solutions are given below:

1) $x_{n+2}=y_{n+1}+3x_{n+1}$
2) $x_{n+3}=6y_{n+1}+17x_{n+1}$
3) $y_{n+2}=3y_{n+1}+8x_{n+1}$
4) $y_{n+3}=17y_{n+1}+48x_{n+1}$
5) $x_{n+3}=6x_{n+2}-x_{n+1}$
6) $y_{n+2}=3x_{n+2}-x_{n+1}$
7) $y_{n+3}=17x_{n+2}-3x_{n+1}$
8) $2y_{n+2}=x_{n+3}-x_{n+1}$
9) $6y_{n+3}=17x_{n+3}-x_{n+1}$
10) $3y_{n+3}=17y_{n+2}+8x_{n+1}$
11) $3x_{n+3}=y_{n+1}+17x_{n+2}$
12) $3y_{n+2}=y_{n+1}+8x_{n+2}$

13) $y_{n+3} = y_{n+1} + 16x_{n+2}$
14) $17y_{n+2} = 3y_{n+1} + 8x_{n+3}$
15) $17y_{n+3} = y_{n+1} + 48x_{n+3}$
16) $y_{n+3} = 6y_{n+2} - y_{n+1}$
17) $y_{n+2} = x_{n+3} - 3x_{n+2}$
18) $y_{n+3} = 3x_{n+3} - x_{n+2}$
19) $y_{n+3} = 3y_{n+2} + 8x_{n+2}$
20) $3y_{n+3} = y_{n+2} + 8x_{n+3}$

III.REMARKABLE OBSERVATION

A. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$Y^2 - 8X^2 = 9604$	$(18x_{n+1} - 4y_{n+1}, 18y_{n+1} - 32x_{n+1})$
2	$Y^2 - 8X^2 = 9604$	$(30x_{n+1} - 4x_{n+2}, 18x_{n+2} - 86x_{n+1})$
3	$Y^2 - 8X^2 = 86436$	$(88x_{n+1} - 2x_{n+3}, 9x_{n+3} - 249x_{n+1})$
4	$Y^2 - 8X^2 = 86436$	$(86x_{n+1} - 4y_{n+2}, 18y_{n+2} - 240x_{n+1})$
5	$Y^2 - 8X^2 = 2775556$	$(498x_{n+1} - 4y_{n+3}, 18y_{n+3} - 1408x_{n+1})$
6	$Y^2 - 8X^2 = 86436$	$(18x_{n+2} - 30y_{n+1}, 86y_{n+1} - 32x_{n+2})$
7	$Y^2 - 8X^2 = 2775556$	$(18x_{n+3} - 176y_{n+1}, 498y_{n+1} - 32x_{n+3})$
8	$Y^2 - 8X^2 = 614656$	$(18y_{n+2} - 86y_{n+1}, 240y_{n+1} - 32y_{n+2})$
9	$Y^2 - 8X^2 = 22127616$	$(18y_{n+3} - 498y_{n+1}, 1408y_{n+1} - 32y_{n+3})$
10	$Y^2 - 8X^2 = 9604$	$(176x_{n+2} - 30x_{n+3}, 86x_{n+3} - 498x_{n+2})$
11	$Y^2 - 8X^2 = 9604$	$(86x_{n+2} - 30y_{n+2}, 86y_{n+2} - 240x_{n+2})$
12	$Y^2 - 8X^2 = 86436$	$(498x_{n+2} - 30y_{n+3}, 86y_{n+3} - 1408x_{n+2})$
13	$Y^2 - 8X^2 = 86436$	$(86x_{n+3} - 176y_{n+2}, 498y_{n+2} - 240x_{n+3})$
14	$Y^2 - 8X^2 = 9604$	$(498x_{n+3} - 176y_{n+3}, 498y_{n+3} - 1408x_{n+3})$
15	$Y^2 - 8X^2 = 614656$	$(86y_{n+3} - 498y_{n+2}, 1408y_{n+2} - 240y_{n+3})$

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B. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$49Y-8X^2=9604$	$(18x_{n+1}-4y_{n+1},18y_{2n+2}-32x_{2n+2}+98)$
2	$49Y-8X^2=9604$	$(30x_{n+1}-4x_{n+2},18x_{2n+3}-86x_{2n+2}+98)$
3	$147Y-8X^2=86436$	$(88x_{n+1}-2x_{n+3},9x_{2n+4}-249x_{2n+2}+294)$
4	$147Y-8X^2=86436$	$(86x_{n+1}-4y_{n+2},18y_{2n+3}-240x_{2n+2}+294)$
5	$833Y-8X^2=2775556$	$(498x_{n+1}-4y_{n+3},18y_{2n+4}-1408x_{2n+2}+1666)$
6	$147Y-8X^2=86436$	$(18x_{n+2}-30y_{n+1},86y_{2n+2}-32x_{2n+3}+294)$
7	$833Y-8X^2=2775556$	$(18x_{n+3}-176y_{n+1},498y_{2n+2}-32x_{2n+4}+1666)$
8	$392Y-8X^2=614656$	$(18y_{n+2}-86y_{n+1},240y_{2n+2}-32y_{2n+3}+784)$
9	$2352Y-8X^2=22127616$	$(18y_{n+3}-498y_{n+1},1408y_{2n+2}-32y_{2n+4}+4704)$
10	$49Y-8X^2=9604$	$(176x_{n+2}-30x_{n+3},86x_{2n+4}-498x_{2n+3}+98)$
11	$49Y-8X^2=9604$	$(86x_{n+2}-30y_{n+2},86y_{2n+3}-240x_{2n+3}+98)$
12	$147Y-8X^2=86436$	$(498x_{n+2}-30y_{n+3},86y_{2n+4}-1408x_{2n+3}+294)$
13	$147Y-8X^2=86436$	$(86x_{n+3}-176y_{n+2},498y_{2n+3}-240x_{2n+4}+294)$
14	$49Y-8X^2=9604$	$(498x_{n+3}-176y_{n+3},498y_{2n+4}-1408x_{2n+4}+98)$
15	$392Y-8X^2=614656$	$(86y_{n+3}-498y_{n+2},1408y_{2n+3}-240y_{2n+4}+784)$

1) Special Cases

a) $P_y^{10}(t_{3,x+1})^2=153P_x^6(t_{3,y})^2+8(t_{3,y})^2(t_{3,x+1})^2$

b) $9P_y^6(t_{3,x})^2=17P_x^{10}(t_{3,y+1})^2+8(t_{3,x})^2(t_{3,y+1})^2$

c) $P_y^{10}(t_{3,2x-2})^2=17(6P_{x-1}^4)^2(t_{3,y})^2+8(t_{3,y})^2(t_{3,2x-2})^2$

d) $36P_{y-1}^8(t_{3,x})^2=17P_x^{10}(t_{3,2y-2})^2+8(t_{3,x})^2(t_{3,2y-2})^2$

e) $9P_y^6(t_{3,2x-2})^2=17(36P_{x-1}^8)(t_{3,y+1})^2+8(t_{3,2x-2})^2(t_{3,y+1})^2$

f) $(6P_{y-1}^4)^2(t_{3,x+1})^2=17(3P_x^3)^2(t_{3,2y-2})^2+8(t_{3,x+1})^2(t_{3,2y-2})^2$

IV. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $y^2 = 8x^2 + 49$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties

REFERENCES

[1] L.E.Dickson, History of theory of number, Chelsa publishing company, vol-2, (1952) New York.
[2] L.J.Mordel, Diophantine equations, Academic Press, (1969) New York.
[3] S.J.Telang, Number Theory, Tata McGraw Hill Publishing company Limited,(2000) New Delhi.
[4] D.M.Burton, Elementary Number Theory, Tata McGraw Hill Publishing company Limited,(2002) New Delhi.
[5] M.A.Gopolan, s.vidhyalakshmi and A.Kavitha, on the integral solutions of the Binary quadratic Equation $X^2 = 4(K^2+1)Y^2 + 4^t$, Bulletin of Mathematics and Statistics Research, 2(1)(2014)42-46.
[6] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, On the binary quadratic Diophantine equation $x^2 - 3xy + y^2 + 33x = 0$, International Journal of Scientific Engineering and Applied Science, 1(4) (2015) 222-225.
[7] M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, observations on the Hyperbola $10y^2 - 3x^2 = 13$, Archimedes J.Math.,3(1) (2013) 31-34.
[8] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Observations on the Hyperbola $ax^2 - (a+1)y^2 = 3a - 1$, Discovery, 4(10) (2013) 22-24.
[9] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Integral points on the Hyperbola $x^2 - 4xy + y^2 + 15x = 0$, Diophantus J.Maths,1(7) (2014) 338-340.
[10] M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, on the integral solutions Of the binary quadratic equation $x^2 = 15y^2 - 11^t$, scholars journal of Engineering and technology, 2(2A) (2014) 156-158.