iJRASET
International Journal For Research in
Applied Science and Engineering Technology

# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

**Volume:** 8 **Issue:** III **Month of publication:** March 2020

**DOI:**

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
*ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.429*
*Volume 8 Issue III Mar 2020- Available at www.ijraset.com*

# Fractional Y-Domination in Bipartite Graphs

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***Abstract:** Let $G=(V,E)$ be a bipartite graph with bipartition $X,Y$. A function $f:X\rightarrow[0,1]$ is called a Y-dominating function (Y-DF) of $G$ if $\sum_{x\in N(y)}f(x)\geq 1$, for all $y\in Y$. A Y-DF is minimal (MY-DF) if any function $g:X\rightarrow[0,1]$ with $g\leq f$ and $g(x)<f(x)$ for at least one $x\in X$ is not a Y-DF. The minimum value of $|f|=\sum_{x\in X}f(x)$ taken over all MY-DFs of G is called the fractional Y-domination number of G and is denoted by $\gamma_{Yf}(G)$. In this paper we initiate a study of these parameters. We obtain sharp bounds for $\gamma_{Yf}$ (G) and determine $\gamma_{Yf}$(G) for several families of bipartite graphs. We investigate the behaviour of convex combinations of Y -dominating functions. We prove that the decision problem corresponding to the upper fractional Y-domination number ΓYf(G) is NP-complete.*
***Keywords:** Bipartite Graphs, Y-Domination, Domination, Dominating Function, Fractional Domination, Fractional Y-Domination number, upper fractional Y-domination number.*

## I. INTRODUCTION

For By a graph $G=(V,E)$, we mean a finite, simple undirected and bipartite graph with bipartition $X,Y$. The order and size of $G$ are denoted by $n$ and $m$ respctively. For basic terminology in graphs we refer to Chartrand and Lesniak [1].

Hedetniemi and Laskar[6,7] proposed a bipartite theory of graphs and suggested an equivalent formulation of several concepts in graphs as concepts for bipartite graphs. One such example is the concept of Y-domination in a bipartite graph G = (V, E) with bipartition V = X∪Y. In this chapter we initiate a study of the fractional version of Y -domination.

Let G=(X,Y,E) be a bipartite graph, where V(G)=X∪Y. A subset D of X is a Y-dominating set of G if every y∈Y is adjacent to at least one vertex in D. The minimum order $\gamma_Y(G)$ of a Y-dominating set of G is called the Y-domination umber of G. The maximum cardinality of a minimal Y-dominating set of G is called the upper Y-dominating number of G denoted by $\Gamma_Y(G)$.

If y ∈Y, then the open neighborhood of y is given by N(y)={x∈X:xy∈E(G)}. Thus D ⊆X is a Y –dominating set of G if $N(y)\cap D\neq\emptyset$ for all y∈Y. Two well known bipartite graphs associated with any given graph G=(V,E) are the subdivision graph and the neighborhood graph. The subdivision graph S(G) is the graph obtained from G by subdividing each edge of G exactly once. If we label the vertex subdividing an edge by e itself, then (V(G),E(G)) is a bipartition of S(G). The neighborhood graph N(G) has with vertex set V(G) ∪N where N={N(v):v∈V(G)} and u is joined to N(v) if u∈N(v).

In this paper we initiate a study of the fractional version of Y-domination. For a detailed study of fractional graph theory and fractionalization of various graph parameters, we refer to Scheinerman and Ullman [8]

## II. Y-DOMINATING FUNCTIONS

There is an extensive study of fractionalization of several domination related parameters and for details we refer to the books[5] and [8]. The values of fractional parameters are useful not only for their own applications but also in providing information about the related integer valued parameters. We now proceed to define the fractional version of Y-domination in bipartite graphs.

1) *Definition 2.1.* Let G=(X,Y,E) be a bipartite graph. A function $f:X\rightarrow[0,1]$ is called a Y-dominating function (Y-DF) of G if f(N(y))≥1 for all y ∈Y, $f(N(Y))=\sum_{x\in N(y)}f(x)$. AY-dominating function f is minimal (MY-DF) if any function g:X→[0,1] such that g≤f and $g(x)\neq f(x)$ for at least one x∈X is not a Y-dominating function of G.

   The fractional Y-domination number $\gamma_{Yf}$(G) and the upper fractional Y-domination number $\Gamma_{Yf}$(G) are defined by

   $\gamma_{Yf}(G)$=min{|f|:f is a MY-DF of G}and

$\Gamma_{Yf}$(G)=max {|f|:f is a MY-DF of G},where |f| = f(X)= $\sum_{x\in X}f(x)$.

2) *Remark 2.2.* The characteristic function of a $\gamma_Y$-set and that of a $\Gamma_Y$-set of a bipartite graph G are MY-DFs of G. Hence it follows that $1\leq\gamma_{Yf}(G)\leq\gamma_Y(G)\leq\Gamma_Y(G)\leq\Gamma_{Yf}(G)\leq|X|$.
3) *Observation 2.3.* The problem of finding the fractional Y-domination number $\gamma_{Yf}(G)$ of a bipartite graph G= (X,Y,E) is equivalent to finding the optimal solution of the following linear programming problem

Minimize z=$\sum_{x\in X}f(x)$ Subject to $\sum_{x\in N(y)}f(x)\geq 1$ for all $y\in Y$ and $0\leq f(x)\leq 1$ for all $x\in X$.

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4) *Definition 2.4.*Let G=(X,Y,E) be a bipartite graph. A function $f:Y\to[0,1]$is called a Y-packing function (Y-PF) of G if $f(N(x))\leq 1$ for all $x\in X$, where $N(x)=\{y\in Y:xy\in E(G)\}$. AY-packing function f is maximal (MY-PF) if any function $g:Y\to[0,1]$ such that $g\geq f$ and $g(y)\neq f(y)$ for at least one $y\in Y$ is not a Y-packing function of G. The fractional Y-packing number $p_{Yf}(G)$ and the fractional Y-packing number $P_{Yf}(G)$ are defined by $p_{Yf}(G)=\min\{|f|:$ f is a MY-PF of G $\}$ and $P_{Yf}(G)=\max\{|f|:$ f is a MY-PF of G$\}$ where $|f|=f(Y)=\sum_{y\in Y} f(y)$.
5) *Observation 2.5.* The problem of finding the fractional Y-packing number $P_{Yf}(G)$ of a bipartite graph G= (X,Y,E) is equivalent to finding the optimal solution of the linear programming problem.

Minimize $z=\sum_{y\in Y} f(y)$ Subject to $\sum_{y\in N(x)} f(y)\leq 1$ for all $x\in X$ and $0\leq f(y)\leq 1$ for all $y\in Y$.

6) *Remark 2.6.* The L.P.P given in Observation 2.3 and the L.P.P given in Observation 2.5 are duals of each other. Hence it follows from the strong duality theorem that $P_{Yf}(G)=\gamma_{Yf}(G)$. Thus if there exists a minimal Y-dominating function f and a maximal Y-packing function g with $|f|=|g|$, then $P_{Yf}(G)=|g|=|f|=\gamma_{Yf}(G)$. This is very useful in determining the fractional Y -domination number $\gamma_{Yf}(G)$, of a bipartite graph G.
7) *Example 2.7.*Consider the graph G given in Figure 3.1

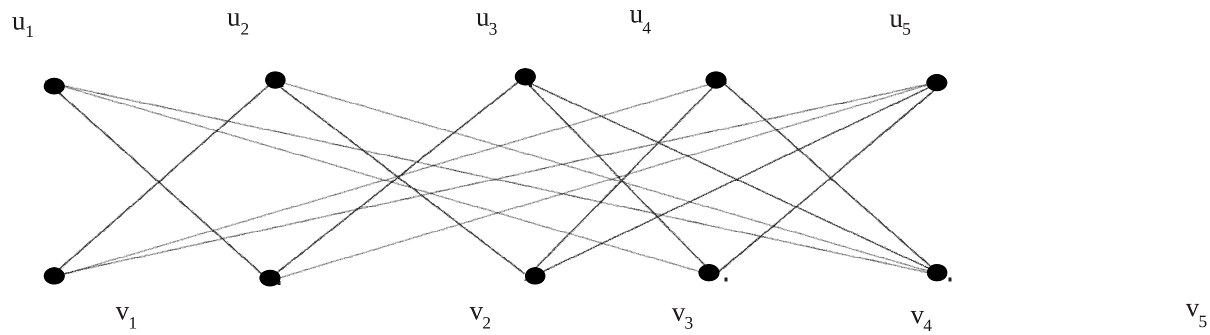


Figure 2.1

Let $X=\{u_1,u_2,u_3,u_4,u_5\}$ and $Y=\{v_1,v_2,v_3,v_4,v_5\}$. Then $S=\{u_1,u_5\}$ is a $\gamma_Y$-set of G and hence $\gamma_Y(G)=2$. We have $N(v_1)=\{u_2,u_4,u_5\}=N(v_3)$, $N(v_2)=\{u_1,u_3,u_5\}=N(v_4)$ and $N(v_5)=\{u_1,u_2,u_3,u_4\}$. Now the function f: $X\to[0,1]$ defined by $f(x)=\begin{cases}\frac{1}{2} & \text{if } x\in\{u_1,u_2,u_5\}\\ 0 & \text{otherwise}\end{cases}$ is a Y-dominating function of G with $|f|=\frac{3}{2}$ and hence $\gamma_{Yf}(G)\leq\frac{3}{2}$.

Also we have $N(u_1)=\{v_2,v_4,v_5\}=N(u_3)$, $N(u_2)=\{v_1,v_3,v_5\}=N(u_4)$ and $N(u_5)=\{v_1,v_2,v_3,v_4\}$.

Now the function $g:Y\to[0,1]$ defined by $g(y)=\begin{cases}\frac{1}{2} & \text{if } y\in\{v_1,v_2,v_5\}\\ 0 & \text{otherwise}\end{cases}$ is a Y -packing function of G with $|g|=\frac{3}{2}$ and hence $P_{Yf}(G)\leq\frac{3}{2}$. Hence it follows from Remark 2.6 that $\gamma_{Yf}(G)=\frac{3}{2}$.

8) *Theorem 2.8.* For the path $P_n$, $n\geq 2$, with $|X|\leq|Y|$, we have $\gamma_{Yf}(P_n)=\gamma_Y(P_n)=\left\lceil\frac{n}{4}\right\rceil$.

Proof. Let $P_n=(u_1,u_2,\ldots,u_n)$, $n\geq 2$.

Case 1. n is odd.

Let $n=2r+1$, $r\geq 1$. Then $X=\{x_1,x_2,\ldots,x_r\}$where $x_i=u_{2i}$, $i=1,2,\ldots,r$ and $Y=\{y_1,y_2,\ldots,y_{r+1}\}$ where $y_i=u_{2i-1}$, $i=1,2,\ldots,(r+1)$ is the bipartition of G. Now $D=\begin{cases}\{x_i: i\equiv 1 \pmod 2\} & \text{if } r \text{ is odd}\\ \{x_i: i\equiv 1 \pmod 2\}\cup\{x_n\} & \text{if } r \text{ is even}\end{cases}$ is a $\gamma_Y$–set of G with $|D|=\left\lceil\frac{n}{4}\right\rceil$. Hence $\gamma_{Yf}(G)\leq\left\lceil\frac{n}{4}\right\rceil$.

Let $h(y_i)=\begin{cases}1 & \text{if } i\equiv 1 \pmod 2\\ 0 & \text{otherwise}\end{cases}$. Define the function $g:Y\to[0,1]$ by $g(y_i)=\begin{cases}\frac{1}{2}\ \forall i & \text{if } r \text{ is odd}\\ h(y_i)\ \forall i & \text{if } r \text{ is even}\end{cases}$. Then h is a Y-packing function of G with $|g|=\left\lceil\frac{n}{4}\right\rceil$ and hence it follows from Remark 2.6 that $\gamma_{Yf}(G)=\left\lceil\frac{n}{4}\right\rceil$.

Case2. n is even.

Let $n=2r$, $r\geq 1$. Then $X=\{x_1,x_2,\ldots,x_r\}$ where $x_i=u_{2i-1}$, $i=1,2,\ldots,r$ and $Y=\{y_1,y_2,\ldots,y_r\}$ where $y_i=u_{2i}$, $i=1,2,\ldots,r$ is a bipartition of G. Now $D=\begin{cases}\{x_i: i\equiv 1 \pmod 2\} & \text{if } r \text{ is odd}\\ \{x_i: i\equiv 0 \pmod 2\} & \text{if } r \text{ is even}\end{cases}$ is a $\gamma_Y$ - set of G with $|D|=\left\lceil\frac{n}{4}\right\rceil$ and hence $\gamma_{Yf}(G)\leq\left\lceil\frac{n}{4}\right\rceil$. Also the function g :Y→ [0, 1] defined by $g(y_i) = \begin{cases} 1 \text{ if } i \equiv (1 \bmod 2) \\ 0 \quad \text{otherwise} \end{cases}$ is a Y -packing function of G with $|g|=\left\lceil \frac{n}{4} \right\rceil$ and hence it follows from Remark 2.6 that $\gamma_{Yf}(G)=\left\lceil \frac{n}{4} \right\rceil$.

The following lemma gives an upper bound for $\gamma_{Yf}(G)$.

9) *Lemma 2.9.* Let G=(X,Y,E) be a bipartite graph and let k = min {|N(y)|: y ∈Y}. Then $\gamma_{Yf}(G) \le \frac{|X|}{k}$.

Proof. The constant function f:X→[0,1] defined by f(x) $=\frac{1}{k}$ for all $x \in X$, is a Y- dominating function of G with $|f| = \frac{|X|}{k}$ and hence $\gamma_{Yf}(G) \le \frac{|X|}{k}$.

10) *Corollary 2.10.* Let G be any connected graph of order n. Then for the subdivision graph S(G) with X = V(G) and Y = E(G) as bipartite sets, we have $\gamma_{Yf}(S(G)) \le \frac{|X|}{2}$.

11) *Theorem 2.11.* Let G = (X,Y,E) be a bipartite graph. If deg(y)≥2 for all $y \in Y$ then $\gamma_{Yf}(G) \le \frac{|X|}{2}$. Further, if there exists a bijection α:X→X such that for each x∈X, α(x)≠x and there exists a $y_x \in Y$ with $N(y_x) = \{x, \alpha(x)\}$ then $\gamma_{Yf}(G) \le \frac{|X|}{2}$.

Proof. The inequality follows from Lemma 2.9. Now suppose there exists a bijection α:X→X satisfying the conditions of the theorem. Let h be any Y-dominating function of G. Then $h(N(y_x))=h(x)+h(\alpha(x))\ge 1$ for each x∈X. Adding these |X| inequalities we get $\sum_{x\in X}(h(x) + h(\alpha(x))) \ge |X|$. This implies 2|h|≥|X| and hence $\gamma_{Yf}(G) \ge \frac{|X|}{2}$. Thus $\gamma_{Yf}(G)= \frac{|X|}{2}$.

12) *Corollary 2.12.* For each of the following graphs, $\gamma_{Yf}(S(G)) = \frac{|V(G)|}{2}$

a) The complete graph $G = K_n, n \ge 3$.

b) The Petersen graph $G = P$.

c) The cycle $G = C_n, n \ge 3$.

Proof. It follows from Corollary 2.10 that $\gamma_{Yf}(S(G)) \le \frac{n}{2}$. To prove the equality, we will exhibit a bijection α :X = V (G) →V (G) satisfying the conditions stated in Theorem 2.11.

If $G = K_n$ and V(G) ={$u_1, u_2,\ldots,u_n$}, then α :V (G) →V (G) defined by $\alpha(u_i) = \begin{cases} u_{i+1} & \text{if } i \le n-1 \\ u_1 & \text{if } i = n \end{cases}$ is the required condition. If $G = P$ and V(P) ={$u_1,u_2,\ldots,u_5,v_1,v_2,\ldots,v_5$}, $N(u_i)=\{v_i,u_{i+2},u_{i+3}\}$ and $N(v_i)=\{u_i,v_{i+1},v_{i-1}\}$, addition in the suffix is taken modulo 5, then the function α :V(G)→V(G) defined by $\alpha(u_i) = u_i + 2, u_6 = u_1$ and $\alpha(v_i) = v_i + 2, v_6 = v_1$ is the required bijection.

If $G = C_n = (u_1 u_2 \ldots u_n u_1)$, then the function $\alpha: V(G) \to V(G)$ defined by $\alpha(u_i) = \begin{cases} u_{i+1} & \text{if } i \le n-1 \\ u_1 & \text{if } i = n \end{cases}$ is the required bijection.

13) *Theorem 2.13* LetG=(V,E) be an r-regular graph of order n. Then for the neighborhood graph N(G), we have $\gamma_{Yf}(N(G))=\frac{n}{r}$, where X=V and $Y = \{N(v): v \in V\}$ is the bipartition of N(G).

Proof. Clearly, |N (y)| = r for all y ∈Y and |N (x)| = r for all x ∈X. Now the constant function f : X→ [0, 1] defined by f (x) $=\frac{1}{r}$ for all x ∈X, is a Y -dominating function of N(G) with $|f| = \frac{|X|}{r} = \frac{n}{r}$. Also the constant function $g: Y \to [0,1]$ defined by $g(y) = \frac{1}{r}$ for all $y \in Y$, is a $Y$-packing function of $N(G)$ with $|g| = \frac{|Y|}{r} = \frac{n}{r}$. Hence by Remark 2.6, $\gamma_{Yf}(N(G))=\frac{n}{r}$.

## III.CONVEXITY OF Y-DOMINATING FUNCTIONS

In the study of fractional domination Cockayne et al. [2] have obtained several results about the convexity of the set of all minimal dominating functions of a graph. In this we give, similar results regarding the convexity of minimal $Y$-dominating functions of a bipartite graph $G = (X, Y, E)$. The following theorem gives a necessary and sufficient condition for a Y-dominating function to be a minimal Y-dominating function.

1) *Theorem 3.1.* Let f be an Y -DF of a bipartite graph G = (X,Y,E). Then f is an MY-DF if and only if for every x∈X with f(x)>0 there exists y∈Y such that x∈N(y) and f(N(y))=1.

Proof. Let f:X→[0,1] be an MY-DF of G. Let x∈X andf(x)=a>0. Suppose f(N(y))>1 for all y∈Y with x∈N(y). Let δ = min{f(N(y))−1:y∈Y,x∈N(y)}. Then δ>0.

It can be easily verified that the function g:X → [0, 1] defined by $g(u) = \begin{cases} f(u) & \text{if } u \ne x \\ \max\{0, a-\delta\} & \text{if } u = x \end{cases}$ is a Y-dominating function of G with g < f, which is a contradiction. Hence there exists y ∈Y with f (N (y)) = 1. The converse is obvious.

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Volume 8 Issue III Mar 2020- Available at www.ijraset.com

2) *Definition 3.2.* Let G=(X,Y,E) be a bipartite graph. Let $x \in X$, $S \subseteq X$ and $D \subseteq Y$. We say that x Y-dominates D, if there exists an element $y \in D$ such that $x \in N(y)$ and write $x \underset{Y}{\rightarrow} D$. Also we say that S, Y-dominate D, if $x \underset{Y}{\rightarrow} D$ for all $x \in S$, and write $S \underset{Y}{\rightarrow} D$.
3) *Definition3.3* Let f be an Y-dominating function of a bipartite graph G=(X,Y,E). The boundary set $B_f$ and the positive set $P_f$ of f are defined by $B_f = \{y \in Y : f(N(y))=1\}$ and $P_f = \{x \in X : f(x) > 0\}$.

The following theorem is an immediate consequence of Theorem 3.1.

4) *Theorem 3.4* A Y-dominating function f of a bipartite graph G is a minimal Y -dominating function of G if and only if $P_f \rightarrow B_f$. We now proceed to investigate the behaviour of convex combinations of Y- dominating functions. The proof of the following lemma is obvious.
5) *Lemma 3.5.* Let f and g be two Y-DFs of a bipartite graph G and let $0 < \lambda < 1$. Then a convex combination $h_\lambda = \lambda f + (1-\lambda)g$ is again an Y-dominating function of G.

However, a convex combination of two MY-DFs need not be an MY-DF of G.
Consider the following example.

6) *Example 3.6.* Consider the graph G given in Figure3.1. Now the function $f : X \rightarrow [0,1]$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in \{u_1, u_5\} \\ 0 & \text{otherwise} \end{cases}$ is an MY -DF of G with $P_f = \{u_1,u_5\}$ and $B_f = \{v_1,v_3,v_5\}$. Also the function $g : X \rightarrow [0, 1]$ defined by $g(x) = \begin{cases} 1 & \text{if } x \in \{u_4, u_5\} \\ 0 & \text{otherwise} \end{cases}$ is an MY-DF of G with $P_g = \{u_4,u_5\}$ and $B_g = \{v_2, v_4,v_5\}$. Now let $h = \frac{1}{2}f + \frac{1}{2}g$. Then $h(u_1) = \frac{1}{2} = h(u_4)$, $h(u_5) = 1$ and $h(u_2) = 0 = h(u_3)$. Clearly $P_h = \{u_1,u_4,u_5\}$ and $B_h = \{v_5\}$. Since $u_5$ does not Y-dominate $B_h$, it follows from Theorem 3.4 that h is not minimal.

The following theorem gives a necessary and a sufficient condition for a convex combination of two MY-DFs of a bipartite graph G to be an MY-DF of G.

7) *Theorem 3.7.* Let f and g be two MY-DFs of a bipartite graph G. Then $h_\lambda = \lambda f + (1-\lambda)g$, $0<\lambda<1$, is an MY-DF of G if and only if $P_f \cup P_g \underset{Y}{\rightarrow} B_f \cap B_g$.

Proof. Since $B_{h_\lambda} = B_f \cap B_g$ and $P_{h_\lambda} = P_f \cup P_g$, the result follows from Theorem 3.4.

8) *Remark 3.8.* It follows from Theorem 3.7 that if f and g are MY-DFs of a bipartite graph G, then either all convex combinations of f and g are MY-DFs or none of them is an MY-DF.

## IV. COMPUTATIONAL COMPLEXITY $\Gamma_{YF}(G)$

In this section we prove that for any bipartite graph G, the decision problem corresponding to the upper fractional Y-domination number $\Gamma_{YF}(G)$ is NP-complete.

The decision problem corresponding o $\Gamma_{Yf}(G)$ is given below.

UPPER FRACTIONAL Y-DOMINATION (UFY-D)
INSTANCE: A bipartite graph G and a positive number r;
QUESTION: Is $\Gamma_{Yf}(G) \geq r$?.

We need the following definition and succeeding known NP-complete problem.

1) *Definition 4.1.* [15] Let G = (V, E) be a graph with no isolated vertices. A set $S \subseteq V$ is said to be a total dominating set of G if N (S) = V. A function $f : V \rightarrow [0, 1]$ is called a total dominating function (TDF ) of G if $f(N(v)) \geq 1$ for all $v \in V$. The upper fractional total domination number is defined as $\Gamma^0(G)=\max\{|f|: f \text{ is a minimal TDF of } G\}$.

Fricke et al. [15] have shown that the following decision problem is NP-complete even for bipartite graphs.

UPPER FRACTIONAL TOTAL DOMINATION (UFTD)
INSTANCE: A graph G without isolated vertices and a positive number r;
QUESTION: Is $\Gamma^0_f(G) \geq r$?.

2) *Theorem 4.2.* UFY-D is NP-complete.

Proof. The proof is by reduction from UFTD. Let (G,r) be an instance of UFTD. Consider the bipartite graph N(G) with bipartition X = V(G) and Y={N(u):u ∈V(G)}. It can be easily verified that $f:V(G)\rightarrow[0,1]$ is a minimal total dominating function of G if and only if f is a minimal Y-dominating function of G. Hence $\Gamma^0_f(G)=\Gamma_{Yf}(N(G))$ and the result follows.

The following are some interesting problems for further investigation.

*a)* Characterize bipartite graphs G for which $\gamma_{Yf}((G)) = \frac{|x|}{2}$.

*b)* Characterize bipartite graphs G for which $\gamma_{Yf}(G) = \gamma_{Yf}(G)$.

*c)* Characterize graphs G for which $\gamma_{Yf}(S(G)) = \frac{n}{2}$.

## V. CONCLUSION AND SCOPE

In this paper a study of fractional Y-domination number of a bipartite graph is initiated. We have determined this parameter for the subdivision graph and neighborhood graph of a graph G. The study of this parameter for other bipartite graphs is interesting. Hedetniemi et al. [6,7] have defined the concepts hyper-independent set, hyper-dominating set, X-matching and hyper-coloring of a bipartite graph G = (X,Y,E). The study of the fractional version of these parameters remains open. The following are some interesting problems for further investigation.

*A.* Characterize bipartite graphs G for which$\gamma_{Yf}((G)) = \frac{|x|}{2}$.

*B.* Characterize bipartite graphs G for which $\gamma_{Yf}(G) = \gamma_{Yf}(G)$.

*C.* Characterize graphs G for which $\gamma_{Yf}(S(G)) = \frac{n}{2}$.

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