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Proper Roman Coloring of some Cycle related Graphs

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Abstract: ***Suresh Kumar [12] introduced Roman coloring and Roman Chromatic Number of graphs motivated from the traditional Roman military defence strategy. However, it is not a proper coloring. Suresh Kumar and Preethi K Pillai [13] introduced Proper Roman coloring and Proper Roman Chromatic Number of graphs. In this paper, we investigate the Proper Roman colorings and obtain the Proper Roman Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph.***
Keyword: Graph, Roman coloring, Proper Roman coloring, Proper Roman Chromatic Number, Wheel graph, Helm graph, Gear graph, Flower graph, Friendship graph, Crown graph, Web graph

I. INTRODUCTION

The majority of early graph theory research on graph coloring pays attention only to find some possible solution to the Four Color Conjecture. After Appel and Haken gave a computer verification proof of the Four Color Conjecture, research focus on graph coloring was shifted to vertex coloring that satisfies some specified property for the induced edge coloring [5]. The coloring is also played an important role in combinatorial optimization problems and critical graphs were crucial in the Chromatic number Theory [7, 8, 9, 10, 11].

Jason Robert Lewis [1] suggested several new graph parameters in his Doctoral Thesis. Several studies were made in applying such parameters to Roman defence strategy [2, 3, 4,5, 6]. The basic idea was that in a specified city, if the streets are considered as the edges of a graph and the meeting points of the streets, called the junctions, as the edges of the graph, then we can color each vertex by the number of soldiers deployed at that junction and require that every street (edge) should be guarded by at least one soldier using a strategy that if any street have no soldier, then there must be an adjacent junction with two soldiers so that one among them may be deployed to the former junction in case of emergency. Motivated by this, Suresh Kumar [12] defined a new type of graph coloring, namely, Roman Coloring and a related parameter, Roman Chromatic number. However, it is not a proper coloring. Suresh Kumar and Preethi K Pillai [ ] introduced and studied the Roman colorings, which are proper colorings also. In this paper, we investigate the Proper Roman colorings and obtain the Proper Roman Chromatic number of some cycle related graphs such as the Wheel graph, the Helm graph, the Closed Helm graph, the Gear graph, the Flower graph, the Friendship graph, the Double Wheel graph, the Crown graph, the Double Crown graph and the Web graph. For terms and definitions not explicitly here, refer Harary [13].

We begin by recalling some basic definitions which are useful for the present investigation.

1) Definition.1.1.The Wheel graph, $W_n, n \geq 3$, is the join of the graphs C_n and K_1 .That is, W_n is the (n+1)-vertex graph obtained from the graph C_n by adding a new vertex, v and joining it to each of the n vertices of the cycle, C_n.Here we call the vertices corresponding to C_n as rim vertices and the vertex corresponding to K_1(the newly added vertex) is called the apex vertex.
2) Definition. 1.2. The Helm graph $H_n, n \geq 3$ is the graph obtained from Wheel graph, W_n by adding a pendent edge at each vertex on the rim of the Wheel, W_n.
3) Definition. 1.3. The closed Helm graph, CH_n, is the graph obtained from a Helm graph H_n and adding edges between the pendent vertices.
4) Definition. 1.4. The Gear graph, G_n, is a graph obtained from Wheel graph, W_n by adding an extra vertex between each pair of adjacent vertices on the rim of the Wheel graph W_n.
5) Definition 1.5. The Flower graph FL_n is the graph obtained from a Helm graph by joining each pendant vertex to the central vertex of the Helm.
6) Definition. 1.6. The Friendship graph F_n can be constructed by joining n copies of the cycle Graph, C_3 to a common vertex.

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7) Definition. 1.7. The Double Wheel graph, DW_n of size n is composed of $2C_n + K_1$. It consists of two cycles C_n, where vertices of each of these two cycles are connected to a common vertex.
8) Definition. 1.8. The Crown graph, C_n^+ is obtained from the cycle graph, C_n by adding a pendent edge to each vertex of C_n
9) Definition. 1.9. The Double crown graph, C_n^{++} is the graph obtained from the cycle C_n by adding two pendent edge at each vertex of C_n
10) Definition. 1.10. The Web graph is the graph obtained from a Helm graph by joining the pendent vertices of the Helm to form a cycle and then adding a pendent edge to each vertex of the outer cycle.
11) Definition. 1.11. The floor of a real number x is the largest integer less than or equal to x (that is, the integral part of x) and it is denoted by $\lfloor x \rfloor$. The ceil of a real number x is the smallest integer greater than or equal to x and it is denoted by $\lceil x \rceil$.

II. MAIN RESULTS

Let G be a connected graph. Proper Roman coloring of a graph G is an assignment of three colors {0, 1, 2} to the vertices of G such that adjacent vertices must have distinct colors and any vertex with the color, 0 must be adjacent to a vertex with color, 2. The color classes will be denoted as V_0, V_1, V_2 which are the subsets of V(G) with colors 0,1,2 respectively.
Weight of a Roman coloring is defined as the sum of all the vertex colors. Proper Roman Chromatic number of a graph G is defined as the minimum weight of a Roman coloring of G and is denoted by $\chi_R(G)$. A Proper Roman coloring of G with the minimal weight is called a minimal Proper Roman coloringof G.
In this section, we discuss the Proper Roman Coloring and obtain the Proper Roman Chromatic number of the cycle related Graphs mentioned above. For the terms and definitions not explicitly defined here, reader may refer Harary [13].

1) *Theorem.2.1.* The Wheel graph, W_n, n ≥ 3 is not Proper Roman colourable if n is odd. If n is even, the Wheel graph, W_n, n ≥ 3 is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(W_{2n}) = 2 + n$.

Proof. Since the chromatic number of W_n is 4 when n is odd, and only 3 colors, viz {0,1,2}, are availabe in a Proper Roman Coloring, it follows that the Wheel graph, W_n, n ≥ 3 is not Proper Roman colourable if n is odd.
Asssume that n is even. Let the apex vertex of the Wheel graph, W_n be v and the vertices on the rim are $v_1, v_2, \dots v_n$.
Define a coloring function C : V(W_n)→ {0,1,2} as follows:

$C(v) = 2$,

C(v_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$

C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

Then this coloring is a minimal Proper Roman colouring and its chromatic number is given by χ_R (W_n) =$\sum_{v\in V(G)} C(v)$ =2+$\frac{n}{2}$

2) *Theorem. 2.2.* Helm graph, H_n, $n \geq 4$ and n is even is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(H_n)$ = n+2.

Proof: Let the central vertex of the Helm graph H_n be v and the vertices on the rim are $v_1, v_2, \dots v_n$ and the pendent vertices are $w_1, w_2, w_3 \ldots\ldots\ldots w_n$.
Define C : V(H_n)→ {0,1,2} as follows:

$C(v) = 2$

C(v_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$

C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

C(w_{2i})= 1 if 1≤ $i \leq \frac{n}{2}$

C(w_{2i-1})= 0 if 1≤ $i \leq \frac{n}{2}$.

Then this coloring is a minimal Proper Roman colouring and its chromatic number is given by $\chi_R(H_n)$ =$\sum_{v\in V(G)} C(v)$ = n+2.

3) *Theorem. 2.3.* The Closed Helm graph, CH_n $n \geq 4$ and n is even is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(CH_n)$ = n+2.

Proof:Let the central vertex of the Helm graph H_n be v and the vertices on the rim are $v_1, v_2, \dots v_n$ and the pendent vertices are $w_1, w_2, w_3 \ldots\ldots\ldots w_n$.
Define C : V(CH_n)→ {0,1,2} as follows:

$C(v) = 2$

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C(v_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$
C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$
C(w_{2i})= 1 if 1≤ $i \leq \frac{n}{2}$
C(w_{2i-1})= 0 if 1≤ $i \leq \frac{n}{2}$
Then this coloring is a minimal Roman colouring and its chromatic number is given by $\chi_R(CH_n) = \sum_{v\in V(G)} C(v)$ = n+2.

4) *Theorem. 2.4.* The Gear graph, G_n is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(G_n) = n+2$

Proof: Let the central vertex of the Gear graph, G_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the newly added vertices are $v_1', v_2', v_3', \ldots\ldots\ldots\ldots v_n'$.
Define C : V(G_n)→ {0,1,2} as follows:
$C(v) = 2$
C(v_i)=1 , 1≤ $i \leq n$
C(v_j')= 0 if 1≤ $j \leq n$
This coloring is a minimal Proper Roman colouring and its chromatic number is given by $\chi_R(G_n) = \sum_{v\in V(G)} C(v) = n+2$

5) *Theorem. 2.5.* Flower graph, $FL_n, n \geq 4$ and n is even is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(FL_n) = n+2$

Proof: Let the central vertex of the Helm graph H_n be v and the vertices on the rim are $v_1, v_2, \ldots v_n$ and the pendent vertices corresponding to the cycle are $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.
Define C : V(FL_n)→ {0,1,2} as follows:
$C(v) = 2$
C(v_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$
C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.
C(w_{2i})= 1 if 1≤ $i \leq \frac{n}{2}$
C(w_{2i-1})= 0 if 1≤ $i \leq \frac{n}{2}$.
Then this coloring is a minimal Proper Roman colouring and its chromatic number is given by $\chi_R(FL_n) = \sum_{v\in V(G)} C(v)$ =n+2.

6) *Theorem. 2.6.* The Friendship graph F_n is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(F_n)$ = n+2 .

Proof: Let the central vertex of the Friendship graph F_n be v and let v_{11}, v_{12} be the vertices of the first copy of C_3, v_{21}, v_{22} be the vertices of the second copy of C_3, v_{31}, v_{32} be the vertices of the third copy of C_3 and so on. Let v_{n1}, v_{n2} be the vertices of the nth copy of C_3.
Define C : V(F_n)→ {0,1,2} as follows.
$C(v) = 2$
C(v_{i1})= 0 if 1≤ $i \leq n$
C(v_{i2})= 1 if 1≤ $i \leq n$
Then this coloring is a minimal Proper Roman coloring and its chromatic number is given by $\chi_R(F_n) = \sum_{v\in V(G)} C(v)$ = n+2.

7) *Theorem.2.7.* The Double Wheel graph, DW_n, $n \geq 4$ and n is even is proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R(DW_n)$ = n+2.

Proof : Let v be the central vertex. Let $\{v_1, v_2, v_3 \ldots\ldots\ldots\ldots.. v_n\}$ and $\{u_1, u_2, u_3 \ldots\ldots\ldots\ldots.. u_n\}$ be vertices of inner and outer cycles of C_n respectively.
Define C : V(DW_n)→ {0,1,2} as follows.
$C(v) = 2$
C(u_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$
C(u_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$
C(w_{2i})= 1 if 1≤ $i \leq \frac{n}{2}$

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C(w_{2i-1})= 0 if $1\leq i \leq \frac{n}{2}$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by $\chi_R(DW_n) = \sum_{v\in V(G)} C(v)$ =n+2

8) *Theorem. 2.8.* The Crown graph C_n^+ is Proper Roman colourable, $n \geq 4$ and its Proper Roman chromatic number is given by χ_R

$$(C_n^+) = \begin{cases} 3\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 3\left\lceil\frac{n}{2}\right\rceil & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let the vertices on the cycle be $v_1, v_2, v_3 \ldots\ldots\ldots\ldots v_n$ and the pendent vertices corresponding to the cycle be $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.

a) *Case.1.* $n \geq 4$ and n is even

Define C : V(C_n^+)→ {0,1,2} as follows.

C(v_{2i})= 2 if $1\leq i \leq \frac{n}{2}$

C(v_{2i-1})= 0 if $1\leq i \leq \frac{n}{2}$.

C(w_{2i})= 0 if $1\leq i \leq \frac{n}{2}$.

C(w_{2i-1})= 1 if $1\leq i \leq \frac{n}{2}$.

This coloring is a minimal Proper Roman coloring and its chromatic number is given by$\chi_R(C_n^+) = \sum_{v\in V(G)} C(v) = 3\left(\frac{n}{2}\right)$.

b) *Case.2.* $n \geq 4$and n is odd

Define C : VC_n^+)→ {0,1,2} as follows.

C(v_{2i})= 2 if $1\leq i \leq \frac{n-1}{2}$.

C(v_{2i-1})= 0 if $1\leq i \leq \frac{n-1}{2}$.

C(v_n)= 1

C(w_{2i})= 0 if $1\leq i \leq \frac{n-1}{2}$

C(w_{2i-1})= 1 if $1\leq i \leq \frac{n-1}{2}$.

C(w_n)= 2

This coloring is a minimal Proper Roman coloring and its chromatic number is given by$\chi_R(C_n^+) = \sum_{v\in V(G)} C(v) = 3\left\lceil\frac{n}{2}\right\rceil$.

9) *Theorem. 2.9.* The Double Crown graph, C_n^{++}, $n > 3$ is Proper Roman colourable and its chromatic number is given by χ_R

$$(C_n^{++}) == \begin{cases} 2n & \text{if } n \text{ is even} \\ 2\left\lceil\frac{n}{2}\right\rceil + n + 2. & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let us label$v_1, v_2, v_3 \ldots\ldots\ldots\ldots v_n$ as the vertices of the cycle C_n. Let the pendent edges corresponding to each vertex v_i *be labeled as* v_{i1}, v_{i2}

a) *Case.1.* $n > 3$ and n is even

Define C : V(C_n^{++})→ {0,1,2}as follows.

C(v_{2i})= 2 if $1\leq i \leq \frac{n}{2}$.

C(v_{2i-1})= 0 if $1\leq i \leq \frac{n}{2}$.

C($v_{(2i-1)1}$)= 1 if $1\leq i \leq \frac{n}{2}$.

C($v_{(2i-1)2}$)= 1 if $1\leq i \leq \frac{n}{2}$.

C($v_{(2i)1}$)= 0 if $1\leq i \leq \frac{n}{2}$.

C($v_{(2i)2}$)= 0 if $1\leq i \leq \frac{n}{2}$.

This coloring is a minimal Proper Roman coloring and its chromatic number is given by

$$\chi_R\,(C_n^{++}) = \sum_{v\in V(G)} C(v) = 2\,\frac{n}{2} + n = 2n.$$

b) *Case.2.* $n > 3$ and n is odd

Define C : V(C_n^{++})→ {0,1,2} as follows.

C(v_{2i})= 2 if $1\leq i \leq \frac{n-1}{2}$.

C(v_{2i-1})= 0 if 1≤ $i \leq \frac{n-1}{2}$.
C(v_n)= 1.
C($v_{(2i-1)1}$)= 1 if 1≤ $i \leq \frac{n-1}{2}$.
C($v_{(2i-1)2}$)= 1 if 1≤ $i \leq \frac{n-1}{2}$.
C($v_{(2i)1}$)= 0 if 1≤ $i \leq \frac{n-1}{2}$.
C($v_{(2i)2}$)= 0 if 1≤ $i \leq \frac{n-1}{2}$.
$C(v_{n1}) = C(v_{n2}) = 2$

This coloring is a minimal Proper Roman coloring and its chromatic number is given by $\chi_R(C_n^{++}) = \sum_{v \in V(G)} C(v) = 2\left\lceil\frac{n}{2}\right\rceil + n + 2$.

10) Theorem. 2.10. The Web graph, Wb_n, n ≥ 4 and n is even is Proper Roman colourable and its Proper Roman chromatic number is given by $\chi_R.(Wb_n) = 2n + 2$

Proof: Let n ≥ 4 and n is even. Let the central vertex of the Web graph, Wb_n be v. Let the vertices on the innercycle be $v_1, v_2, v_3, \ldots . v_n$ be the vertices on the outercycle be $u_1, u_2, u_3 \ldots \ldots \ldots \ldots . . u_n$ and the pendent vertices be $w_1, w_2, w_3 \ldots\ldots\ldots\ldots w_n$.

Define C : V(Wb_n)→ { 0,1,2} as follows:
C(v)=2
C(v_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$
C(v_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$
C(u_{2i})= 2 if 1≤ $i \leq \frac{n}{2}$
C(u_{2i-1})= 0 if 1≤ $i \leq \frac{n}{2}$.
C(w_{2i})= 0 if 1≤ $i \leq \frac{n}{2}$.
C(w_{2i-1})= 1 if 1≤ $i \leq \frac{n}{2}$.

This coloring is a minimal Proper Roman coloring and its chromatic number is given by $\chi_R(Wb_n) = \sum_{v \in V(G)} C(v) = 2n + 2$.

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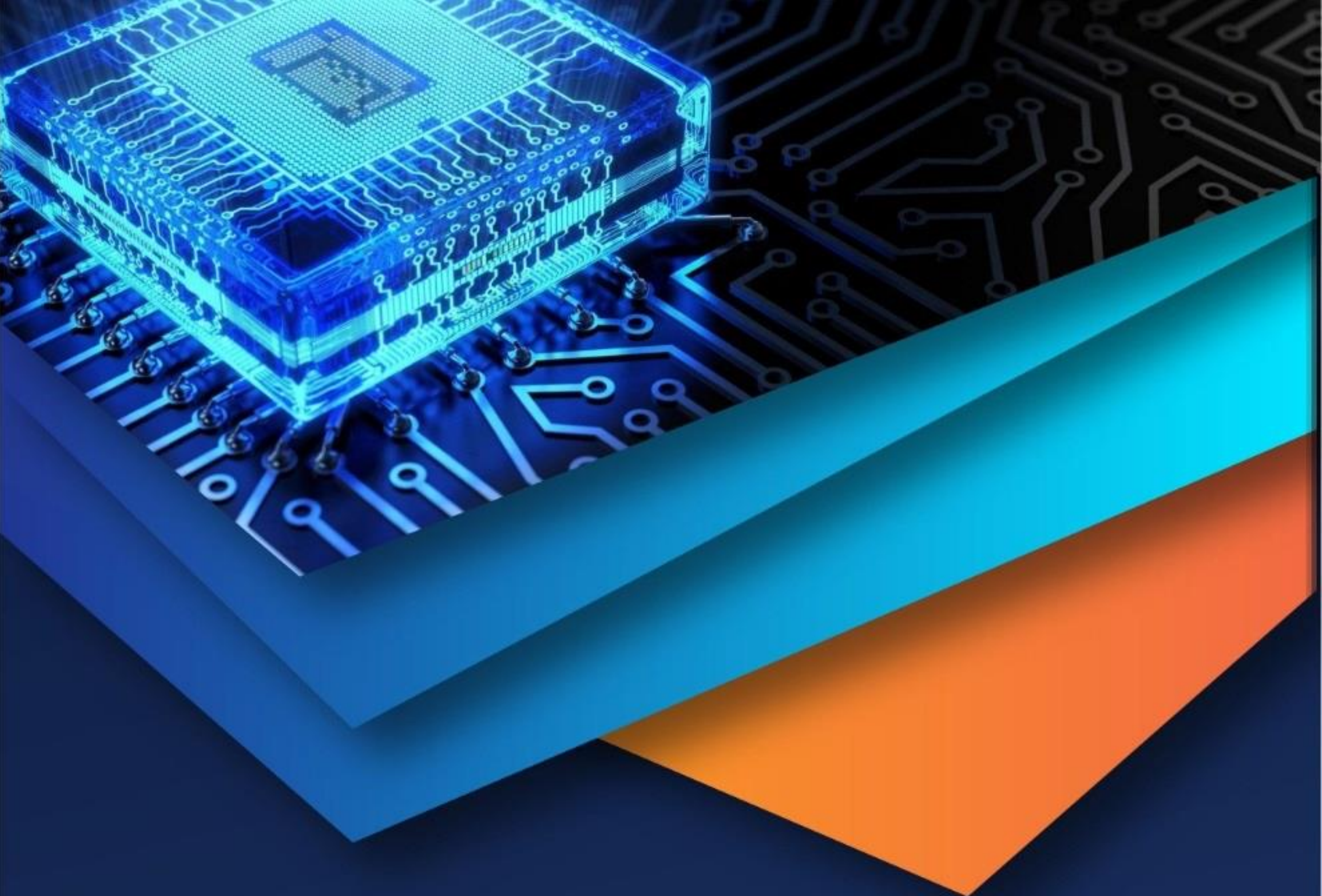

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