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# $\gamma$ - Splitting Graphs

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**Abstract**--Let  $G(V,E)$  be a graph. A dominating set is a subset  $S$  of  $V$  such that every vertex not in  $S$  is adjacent to at least one vertex in  $S$ . The cardinality of a minimum dominating set is called the domination number,  $\gamma(G)$ . A dominating set with  $\gamma$  vertices is called a  $\gamma$ -set. Let  $\eta$  denote the number of  $\gamma$ -sets in  $G$ . For a graph  $G$ , the splitting graph  $S(G)$ , is obtained by adding a new vertex  $v'$  corresponding to each vertex  $v$  of  $G$  and joining  $v'$  to all vertices which are adjacent to  $v$  in  $G$ . Here we introduce a new type of graphs called minimum domination splitting graphs or simply  $\gamma$ -splitting graphs. Let  $G$  be a graph and let  $S_1, S_2, \dots, S_\eta$  be the  $\gamma$ -sets in  $G$ . The  $\gamma$ -splitting graph,  $S_\gamma(G)$ , of a graph  $G$  is the graph obtained from  $G$  by adding new vertices  $w_1, w_2, \dots, w_\eta$  and joining  $w_i$  to each vertex in  $S_i$  where  $1 \leq i \leq \eta$ . In this paper, we establish some results on  $\gamma$ -splitting graphs.

**Keywords:** Dominating set, domination number, splitting graph,  $\gamma$ -splitting graph.

**AMS Subject Classification Code(2010):** 05C(Primary)

## I. INTRODUCTION

Throughout this paper, we consider only finite, simple, undirected graphs. For notations and terminology we follow [3]. Let  $G(V,E)$  be a graph of order  $n$ . We denote the cycle on  $n$  vertices by  $C_n$ , the path of  $n$  vertices by  $P_n$ , and the complete graph on  $n$  vertices by  $K_n$ . The complete bipartite graph is denoted by  $K_{m,n}$ . In a graph  $G$ , degree of a vertex  $v$  is denoted by  $d(v)$ . If  $S$  is a subset of  $V$ , then  $\langle S \rangle$  denotes the vertex induced subgraph of  $G$  induced by  $S$ . For any vertex  $v \in V(G)$ , the open neighbourhood  $N(v)$  of  $V(G)$  is the set of all vertices adjacent to  $v$ , that is,  $N(v) = \{u \in V(G) / uv \in E(G)\}$ , and the closed neighbourhood of  $v$  is defined by  $N[v] = N(v) \cup \{v\}$ .  $N^c(v) = V - N(v)$  is called the neighbourhood complement. For any set  $S$ ,  $N(S) = \bigcup_{v \in S} N(v)$ .

A full vertex of  $G$  is a vertex in  $G$  which is adjacent to all other vertices of  $G$ . A graph  $G$  is said to be  $r$ -regular if every vertex in  $G$  is of degree  $r$ . For any two integers  $k$  and  $d$ ,  $k \neq d$ , a  $(k,d)$ - biregular graph is a graph in which every vertex is of degree either  $k$  or  $d$ . For any three integers  $x$ ,  $a$ , and  $b$ ,  $x \neq a \neq b$ , a  $(x,a,b)$ - triregular graph is a graph in which every vertex is of degree either  $x$  or  $a$  or  $b$ . For example, a  $(2,3)$ -biregular and a  $(1,2,6)$ - triregular graphs are shown in Figure 1.

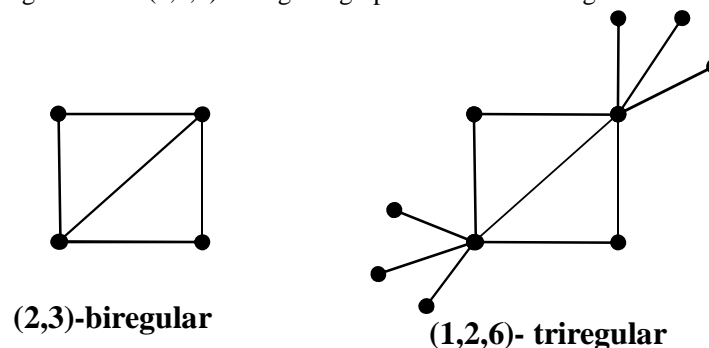


Figure 1

The distance  $d(u,v)$  in  $G$  between two vertices  $u$  and  $v$  is the length of a shortest  $u$ - $v$  path in  $G$ . The eccentricity  $e(u)$ , of a vertex  $u$  is the distance of a farthest vertex from  $u$ , and radius  $\text{rad}(G)$  of  $G$  is the minimum eccentricity. The maximum distance between any two vertices in  $G$  is the diameter of  $G$ , denoted by  $\text{diam}(G)$ , that is,  $\text{diam}(G) = \max_{u,v \in V(G)} \{d(u,v)\}$ . A vertex  $u$  with  $e(u) = \text{rad}(G)$  is

called a central vertex. A graph  $G$  for which  $\text{rad}(G) = \text{diam}(G)$  is called a self-centered graph of radius  $\text{rad}(G)$ . Or equivalently, a graph is self-centered if all of its vertices are central vertices. For further basic definitions on distance in graphs one can refer [4].

Let  $H_{n,n}$  denote the graph with vertex set  $\{v_1, v_2, \dots, v_n; u_1, u_2, \dots, u_n\}$  and edge set  $\{v_i u_j / 1 \leq i \leq n, n-i+1 \leq j \leq n\}$ . The graph  $B_{m,n}$  is the bistar obtained from the stars  $K_{1,m}$  and  $K_{1,n}$  by joining their central vertices by means of an edge. For example, the graph  $H_{4,4}$

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and the bistar  $B_{4,5}$  are shown in Figure 2.

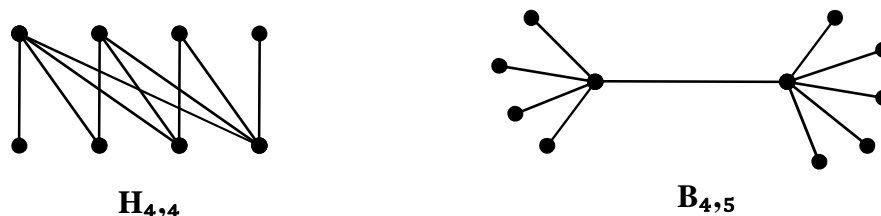


Figure 2

The *join*  $G \vee H$  of the graph  $G$  and  $H$  is the graph obtained from  $G \cup H$  by joining every vertex of  $G$  to each vertex of  $H$  by means of an edge. The graph  $W_n = C_{n-1} \vee K_1$  is called the *wheel* graph on  $n$  vertices. The *corona*  $G \circ H$  of two graphs  $G$  and  $H$  is obtained by taking one copy of  $G$  and  $|V(G)|$  copies of  $H$ , and by joining each vertex in the  $i^{\text{th}}$  copy of  $H$  to the  $i^{\text{th}}$  vertex of  $G$ , where  $1 \leq i \leq |V(G)|$ . The corona graph  $C_5 \circ K_2$  is depicted in Figure 3, for reference,

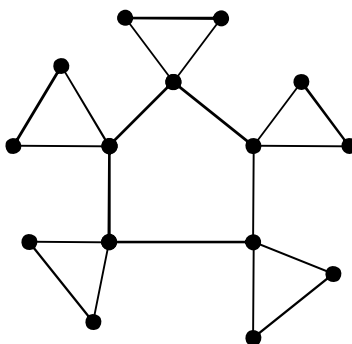


Figure 3

In a graph  $G$ , the process of deleting an edge  $uv$  and introducing a new vertex  $w$  and the edges  $uw$  and  $vw$  is called the *subdivision of the edge*  $uv$ . A *spider* is a tree on  $2n + 1$  vertices obtained by subdividing each edge of a star  $K_{1,n}$ . In other words, spider is nothing but  $K_{1,n} \circ K_1$ . A *wounded spider* is a graph obtained from subdividing at most  $n - 1$  edges of a star  $K_{1,n}$ . The wounded spider includes  $K_1$ , the star  $K_{1,n-1}$ . For example, a wounded spider  $G$  the graph shown in Figure 4. The *cartesian product* of two graphs  $G_1$  and  $G_2$  is denoted by  $G_1 \times G_2$ . The graph  $K_{1,m} \times P_2$  is called the *m-book* graph and it is denoted by  $B_m$ . For example, the book graph  $B_4$  is shown in Figure 5.

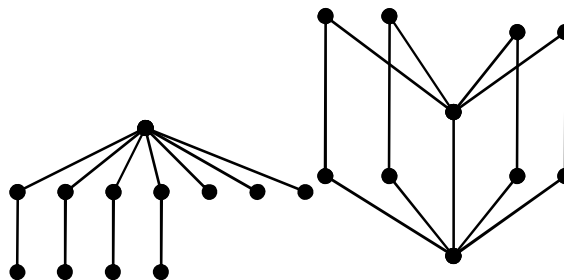
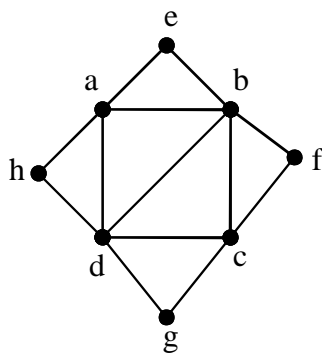


Figure 4

Figure 5

A *dominating set* is a subset  $S$  of the vertex set  $V$  such that every vertex is either in  $S$  or adjacent to a vertex in  $S$ , that is, such that every vertex in  $V - S$  is adjacent to at least one vertex in  $S$ . The *domination number* is the number of vertices in a smallest dominating set of  $G$ , it is denoted by  $\gamma(G)$ . A dominating set with  $\gamma$  elements is called a  $\gamma$ -set. For example,  $S_1 = \{b, d\}$  and  $S_2 = \{a, c\}$  are the minimum dominating sets of the graph  $G$  can be verified in Figure 6. For further results on domination in graphs, one can refer [5].

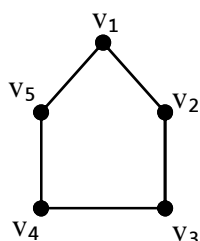
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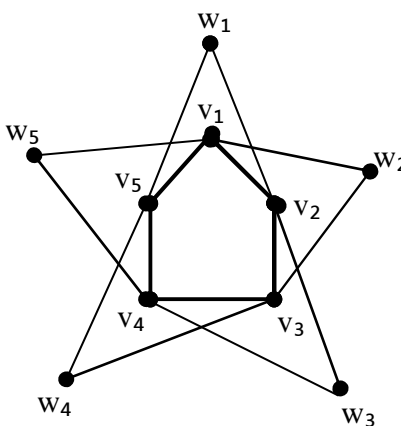
**G**

**Figure 6**

Note that  $S_3 = \{a, b, c, d, e, f, g, h\}$  and  $S_4 = \{a, b, c, d\}$ , etc., are also dominating sets in  $G$ . The concept of splitting graph was introduced by Sampath Kumar and Walikar [6]. The *splitting graph*  $S(G)$ , is the graph obtained from  $G$ , by adding a new vertex  $w$  for every vertex  $v \in V(G)$ , and joining  $w$  to all vertices of  $G$  adjacent to  $v$ . For example, a graph  $G$  and its splitting graph  $S(G)$  are shown in Figure 7.



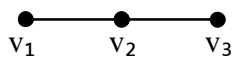
**G**



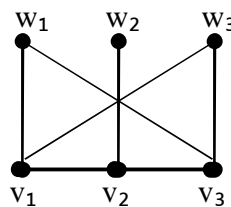
**S(G)**

**Figure 7**

The concept of cosplitting graphs has been recently introduced by Selvam Avadayappan and M. Bhuvaneshwari [1]. Let  $G$  be a graph with vertex set  $\{v_1, v_2, \dots, v_n\}$ . The *cosplitting graph*  $CS(G)$  is the graph obtained from  $G$ , by adding a new vertex  $w_i$  for each vertex  $v_i$  and joining  $w_i$  to all vertices which are not adjacent to  $v_i$  in  $G$ . As an illustration, a graph  $G$  and its cosplitting graph  $CS(G)$  are shown in Figure 8.



**G**



**CS(G)**

**Figure 8**

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The concept of  $\beta$ -splitting graph has been introduced by Selvam Avadayappan, M. Bhuvaneshwari and B. Vijaya Lakshmi [2]. Let  $S_1, S_2, \dots, S_\rho$  be the maximum independent sets of  $G$ . The  $\beta$ -splitting graph  $S_\beta(G)$  of a graph  $G$  is a graph obtained from  $G$  by adding new vertices  $w_1, w_2, \dots, w_\rho$  such that each  $w_i$  is adjacent to each vertex in  $S_i$ , for  $1 \leq i \leq \rho$ . For example, a graph  $G$  and its  $\beta$ -splitting graph  $S_\beta(G)$  are shown in Figure 9.

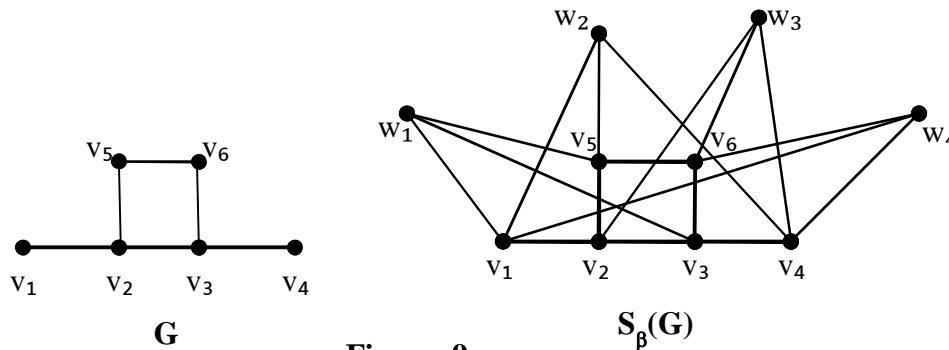


Figure 9

In this paper, we introduce a new type of splitting graphs called  $\gamma$ -splitting graphs. Let  $G$  be a graph and let  $\eta$  be the number of  $\gamma$ -sets in  $G$ . Let  $S_1, S_2, \dots, S_\eta$  be the minimum dominating sets in  $G$ . The  $\gamma$ -splitting graph,  $S_\gamma(G)$ , of a graph  $G$  is the graph obtained from  $G$  by adding new vertices  $w_1, w_2, \dots, w_\eta$  and joining  $w_i$  to each vertex in  $S_i$  where  $1 \leq i \leq \eta$ . For example, the  $\gamma$ -splitting graph of  $P_4$  is shown in Figure 10.

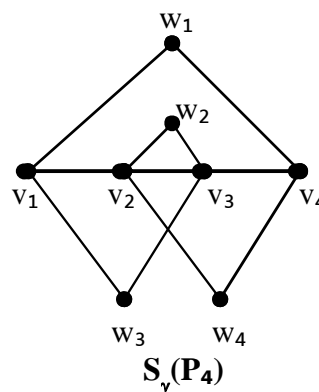


Figure 10

Clearly,  $S_1 = \{v_1, v_4\}$ ,  $S_2 = \{v_2, v_3\}$ ,  $S_3 = \{v_1, v_3\}$ ,  $S_4 = \{v_2, v_4\}$  are the  $\gamma$ -sets in  $P_4$ , also  $w_1, w_2, w_3, w_4$  are newly added vertices in  $S_\gamma(P_4)$ . Here, we discuss a few results on  $\gamma$ -splitting graphs. In this paper, we independently characterise graphs for which  $S_\gamma(G)$  is a regular, biregular, tree, unicyclic graph. We attain bounds for the maximum and minimum degree of a vertex in  $S_\gamma(G)$ . Finally we study the distance properties of  $\gamma$ -splitting graphs.

## II. CHARACTERISATION OF $\gamma$ -SPLITTING GRAPHS

The following facts can be easily verified for  $\gamma$ -splitting graphs. For a vertex  $v$  in  $S_\gamma(G)$ , let  $d^*(v)$  denote the degree of  $v$  in  $S_\gamma(G)$ .

Fact 2.1 The newly added vertices  $\{w_1, w_2, \dots, w_\eta\}$  are independent in  $S_\gamma(G)$ , that is,  $d(w_i, w_j) \geq 2$ , for any  $i, j$ ,  $1 \leq i, j \leq \eta$ .

Fact 2.2  $d^*(w_i) = \gamma(G)$ , for  $i$ ,  $1 \leq i \leq \eta$ .

Fact 2.3 For any vertex  $v \in V(G)$ ,  $d(v) \leq d^*(v)$ .

Fact 2.4 Every graph  $G$  is an induced subgraph of  $S_\gamma(G)$ . Even more  $G$  is a proper subgraph of  $S_\gamma(G)$ , since every graph contains at least one  $\gamma$ -set.

Fact 2.5 The graph having only one full vertex, bistar graph, the graph  $H_{n,n}$ , the path  $P_{3k}$ ,  $k \geq 1$  and the book graph  $B_m$  are some

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graphs whose  $\gamma$ -splitting graphs contain exactly one newly added vertex.

Fact 2.6  $S_\gamma(K_n) \cong K_n \circ K_1$  for any  $n \geq 1$ .

Fact 2.7  $S_\gamma(K_{1,n}) \cong K_{1,n+1}$  for any  $n \geq 2$ .

Fact 2.8  $S_\gamma(K_n^c) \cong K_{1,n}$  for any  $n \geq 1$ .

The following theorems establish some properties of  $\gamma$ -splitting graphs.

Proposition 2.9 For any  $m \geq 1$  and  $n \geq 1$ ,  $\eta(K_{m,n}) = \begin{cases} 1 & \text{if } m = 1, n \geq 2 \\ 2 & \text{if } m = n = 1 \\ 6 & \text{if } m = n = 2 \\ mn + 1 & \text{if } m = 2, n > 2 \\ mn & \text{if } m \geq 3, n \geq 3. \end{cases}$

Proof Let  $V = \{u_1, u_2, \dots, u_m; v_1, v_2, \dots, v_n\}$  be the vertex set of  $K_{m,n}$ .

Case (i) Suppose  $m = n = 1$ , then clearly  $\{u_1\}$  and  $\{v_1\}$  are only the  $\gamma$ -sets and hence  $\eta(K_{m,n}) = 2$ .

Case (ii) If  $m = n = 2$ , then clearly  $\{u_1, v_1\}, \{u_2, v_2\}, \{u_1, v_2\}, \{u_2, v_1\}, \{u_1, u_2\}$  and  $\{v_1, v_2\}$  are the only  $\gamma$ -sets in  $K_{2,2}$  and hence  $\eta(K_{m,n}) = 6$ .

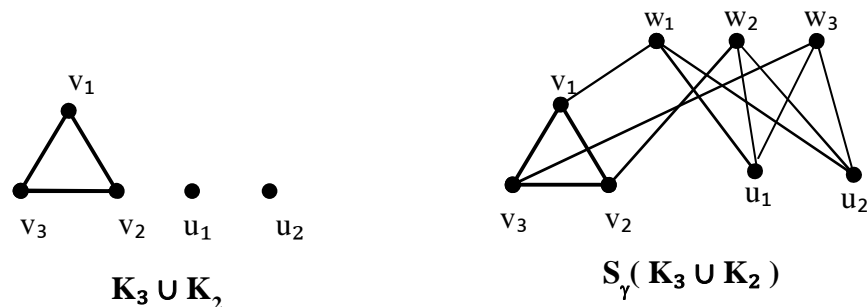
Case (iii) If  $m = 1$  and  $n \geq 2$ , then  $G \cong K_{1,n}$ , and therefore  $\{u_1\}$  is the only  $\gamma$ -set. That is,  $\eta(G) = 1$ .

Case (iv) Suppose  $m = 2$  and  $n > 2$ . Then  $\{u_1, u_2\}$  and  $\{u_j, v_k\} \ 1 \leq j \leq 2, 1 \leq k \leq n$  are the  $\gamma$ -sets of  $G$ . Thus  $\eta(K_{m,n}) = mn + 1$ .

Case (v) If  $m \geq 3$  and  $n \geq 3$ , then clearly  $\{u_i, v_k\} \ 1 \leq i \leq m, 1 \leq k \leq n$ . Thus  $\eta(K_{m,n}) = mn$ . ■

Theorem 2.10 For any  $n \geq 1$ , there exists a graph  $G$  of order  $n$ , such that  $S_\gamma(G)$  is  $n$ -regular.

Proof When  $n = 1, G \cong K_1$ , for which  $S_\gamma(G) \cong K_2$  is the required graph. Therefore assume that  $n \geq 2$ , consider the graph  $G \cong K_n \cup K_{n-1}^c$  with vertex set  $\{v_1, v_2, \dots, v_n; u_1, u_2, \dots, u_{n-1}\}$  with edge set  $\{v_i v_j / 1 \leq i, j \leq n\}$ . For any  $i, 1 \leq i \leq n$ , clearly  $\{v_i, u_1, u_2, \dots, u_{n-1}\}$  is a  $\gamma$ -set of  $G$ , that is,  $\gamma(G) = n$ . Hence there are  $n$  such  $\gamma$ -sets in  $G$ . Let  $w_1, w_2, \dots, w_n$  be the newly added vertices in  $S_\gamma(G)$ . Now for any  $i, j, 1 \leq i \leq n, 1 \leq j \leq n-1$ . Thus  $d^*(v_i) = d^*(w_i) = d^*(u_j) = n$ . Hence  $S_\gamma(G)$  is  $n$ -regular. Thus  $G$  is the required graph. For example, the graph  $K_3 \cup K_2^c$  and  $S_\gamma(K_3 \cup K_2^c)$  which is a 3-regular graph are shown in Figure 11.



Figure

Now, consider the star graph  $K_{1,n-1}$ ,  $n \geq 3$ , which is biregular. In addition  $S_\gamma(K_{1,n-1})$  is also biregular. This shows that there are biregular graphs  $G$  whose  $S_\gamma(G)$  are also biregular. Some examples are listed below:



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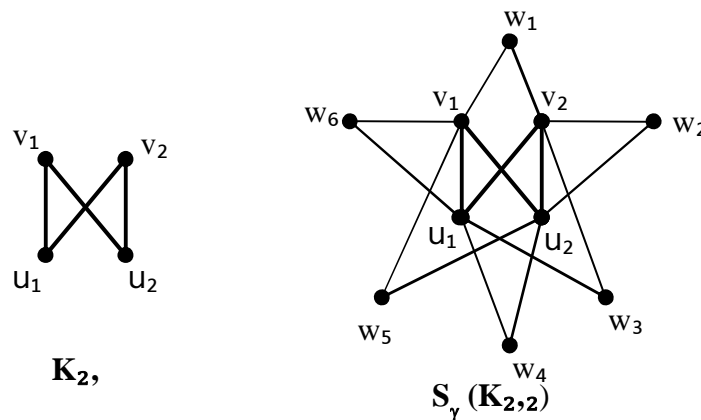
Graph G	Degree set of G	$S_\gamma(G)$	Degree set of $S_\gamma(G)$
$K_{1,n-1}, n \geq 3$	$\{1, n-2\}$	$K_{1,n}$	$\{1, \Delta(G)+1\}$
$P_5$	$\{1, 2\}$	$S_\gamma(P_5)$	$\{2, \Delta(G)+2\}$
$B_m$	$\{2, m+1\}$	$S_\gamma(B_m)$	$\{2, \Delta(G)+1\}$

**Theorem 2.11** The graph  $S_\gamma(K_{m,n})$  is biregular if  $m = n$  and  $S_\gamma(K_{m,n})$  is triregular if  $m \neq n$  for  $m \geq 2$ .

**Proof** Let  $V = \{v_1, v_2, \dots, v_m; u_1, u_2, \dots, u_n\}$  be the vertex set of  $K_{m,n}$ .

Case (i) Suppose  $m = n$ , and  $m \geq 3$ . The graph  $S_\gamma(K_{m,m})$ , then  $d^*(w_i) = 2$ . Also, by Proposition 1,  $\eta = m^2$ . Each  $u_i$  or  $v_i$  belongs to exactly  $m$   $\gamma$ -sets. Hence  $d^*(u_i) = d^*(v_i) = 2m$ . Then  $S_\gamma(K_{m,m})$  is a  $(2m, 2)$ -biregular graph when  $m = n$ .

Case (ii) Let  $m \neq n$ . The graph  $S_\gamma(K_{m,n})$ , then  $d^*(w_i) = 2$ , and  $\eta = mn$ . Each  $u_i$  belongs to  $n$   $\gamma$ -sets and each  $v_i$  belongs to  $m$   $\gamma$ -sets. Then  $d^*(u_i) = 2n$  and  $d^*(v_i) = 2m$ . Hence  $S_\gamma(K_{m,n})$  is a  $(2m, 2n, 2)$ -triregular graph when  $m \neq n$ . Hence the proof. For example, the graph  $K_{2,2}$  and  $S_\gamma(K_{2,2})$  are shown in Figure 12.



**Figure 12**

**Theorem 2.12** The graph  $S_\gamma(G)$  is a tree if and only if  $G$  is one among the following graphs  $K_n^c, P_2, \left(\bigcup_{i=1}^k K_{1,n_i}\right) \cup K_m^c, k \geq 1, n_i$

$\geq 2, m \geq 1$ , or  $\bigcup_{i=1}^k K_{1,n_i}, k \geq 1, n_i \geq 2$ .

**Proof** Consider a graph  $G$  for which  $S_\gamma(G)$  is a tree. Since  $G$  is an induced subgraph of  $S_\gamma(G)$ ,  $G$  is acyclic. If  $G$  contains only two vertices, then obviously  $G \cong K_2$  or  $K_2^c$  for which  $S_\gamma(G) \cong P_4$  or  $P_3$  respectively. So we assume that  $G$  contains at least three vertices.

Case (i) Suppose  $G$  is a tree. Then  $G$  contains at most one full vertex. If  $G$  contains only one full vertex, then  $G \cong K_{1,n}$  for which  $S_\gamma(G) \cong K_{1,n+1}$ . If  $G$  contains no full vertex, then  $\gamma(G) > 1$  and thus  $G$  contains at least two vertices  $u$  and  $v$  in any  $\gamma$ -set  $S$  of  $G$ . Let  $w$  be the newly added vertex in  $S_\gamma(G)$ , corresponding to  $S$ . Now the  $u$ - $v$  path together with the edges  $uw$  and  $wv$  forms a cycle in  $S_\gamma(G)$ , which is a contradiction to our assumption that  $S_\gamma(G)$  is a tree. Therefore, this case does not arise.

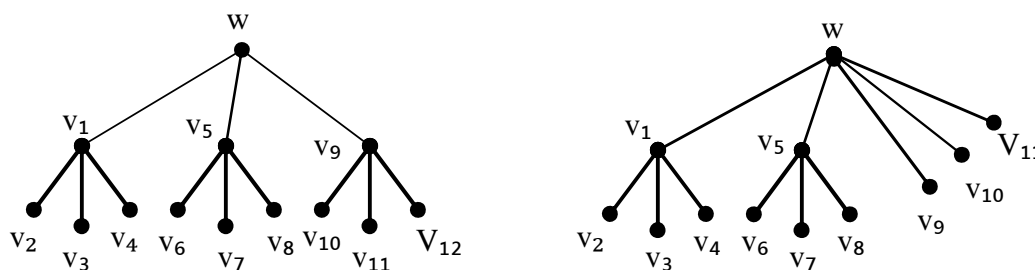
Case (ii) Let  $G$  be a forest. If a  $\gamma$ -set contains at least two vertices in the same component, then  $S_\gamma(G)$  contains a cycle, which is a contradiction. Therefore every component must contain exactly one vertex of each  $\gamma$ -set of  $G$ , which is possible when each

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component is a star or a trivial graph and hence  $G \cong \left( \bigcup_{i=1}^k K_{1,n_i} \right) \cup K_m^c$ ,  $k \geq 1$ ,  $n_i \geq 2$  and  $m \geq 1$  or  $G \cong \bigcup_{i=1}^k K_{1,n_i}$ ,  $k \geq 1$ ,  $n_i \geq 2$ .

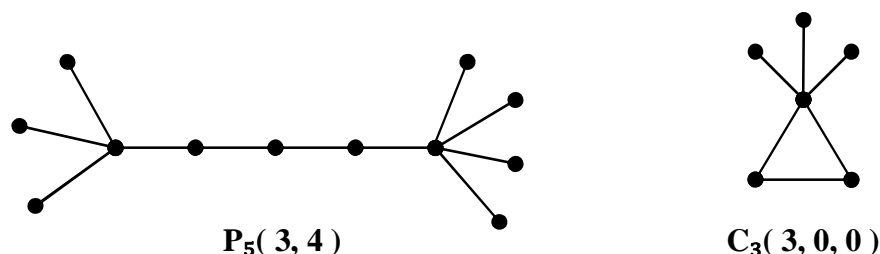
And the converse is obvious. ■

For example, the graph  $S_7\left(\bigcup_{i=1}^3 K_{1,3}\right)$  and  $S_7\left(\left(\bigcup_{i=1}^2 K_{1,3}\right) \cup K_3^c\right)$  are shown in Figure 13.



**Figure 13**

Let  $P_k(m,n)$ , where  $k \geq 2$  and  $m,n \geq 1$ , be the graph obtained by identifying the centre vertices of the stars  $K_{1,m}$  and  $K_{1,n}$  at the ends of  $P_k$  respectively. The graph  $C_3(m_1, m_2, m_3)$ , where  $m_i \geq 0$ , is obtained from the cycle  $C_3 = v_1 v_2 v_3 v_1$  by identifying the centre of the star  $K_{1,m_i}$ , at  $v_i$  of  $C_3$ , for  $1 \leq i \leq 3$ . For example, the graph  $P_5(3, 4)$  and  $C_3(3, 0, 0)$  are shown in Figure 14.



**Figure 14**

**Theorem 2.13** The graph  $S_\gamma(G)$  is unicyclic if and only if  $G$  is isomorphic to any one of the following graphs: (i)  $P_2 \cup K_1$ , (ii)  $K_3$ , (iii)  $B_{m,n}$ ,  $m > 1$ ,  $n > 1$ , (iv)  $P_k(m,n)$ ,  $k = 3, 4$  and  $m,n \geq 1$ , (v)  $B_{m,n} \cup K_t^c$ ,  $m > 1$ ,  $n > 1$ ,  $t \geq 1$ , (vi)  $P_k(m,n) \cup K_t$ ,  $k = 3, 4$  and  $m, n \geq 1$ ,  $t \geq 1$ , (vii)  $C_3(m_1, 0, 0) \bigcup_{p=0}^r pK_{1,n} \bigcup_{q=0}^s qK_n^c$  where  $m_1 \geq 1$ .

**Proof** Consider the graph  $G$  for which  $S_\gamma(G)$  is unicyclic. Then there arise two cases.

**Case (i)** Suppose  $G$  is acyclic. Then clearly the cycle contains a newly added vertex  $w$  in  $S_\gamma(G)$ . Therefore,  $\gamma(G) \neq 1$ . Let  $G$  be a connected graph. Then  $\eta = 1$ , that is,  $G$  contains exactly one  $\gamma$ -set, since every newly added vertex forms a new cycle. In particular,  $\gamma(G) = 2$  with the  $\gamma$ -set  $\{u, v\}$ . Let  $w$  be the newly added vertex in  $S_\gamma(G)$ . Then the  $(u, v)$ -path in  $G$  together with the newly added edges  $wu$  and  $wv$  forms the unique cycle in  $S_\gamma(G)$ , this is possible only when  $G \cong B_{m,n}$ ,  $m > 1$ ,  $n > 1$ ,  $P_k(m,n)$ ,  $k = 3, 4$  and  $m, n \geq 1$ .

Let  $G$  be disconnected. If  $G$  has more than one component, with at least one edge, then  $S_\gamma(G)$  has more cycles, which is a contradiction to our assumption that  $S_\gamma(G)$  is unicyclic. Hence only one component  $G_1$  of  $G$  can contain edges and the others are isolated vertices. If  $G_1$  contains only one edge, then  $G$  must be  $P_2 \cup K_1$ . If  $G_1$  contains more than one edge, then  $G_1$  is isomorphic to  $B_{m,n}$ ,  $m > 1$ ,  $n > 1$ ,  $P_k(m,n)$ ,  $k = 3, 4$  and  $m,n \geq 1$  and hence  $G \cong B_{m,n} \cup K_t^c$ ,  $m > 1$ ,  $n > 1$ ,  $t \geq 1$ ,  $P_k(m,n) \cup K_t^c$ ,  $k = 3, 4$  and  $m, n \geq 1$ ,  $t \geq 1$ .

**Case (ii)** Suppose  $G$  is unicyclic. Let  $G$  be a connected graph. Then newly added edges cannot be in a cycle. This is possible only



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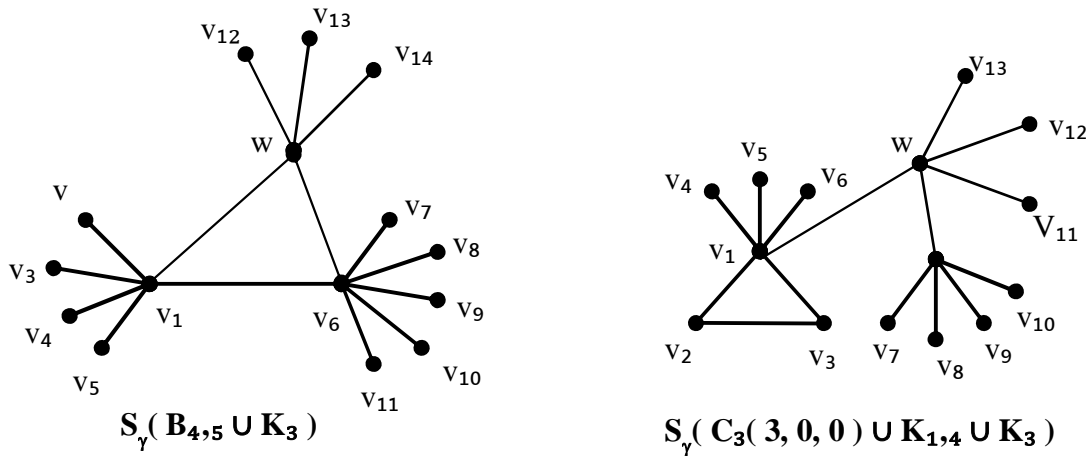
when  $\gamma(G) = 1$ . This forces that  $G \cong K_3$  or  $C_3(m_1, 0, 0)$  where  $m_1 \geq 1$ .

Let  $G$  be disconnected graph. Then  $\omega(G) \geq 2$ . Clearly, one of the component of  $G$  is unicyclic and the remaining are trees. Since every component is connected, by the above argument exactly one vertex of each component belongs to  $\gamma$ -set of  $G$ . Also, the  $\gamma$ -set

must be unique to avoid cycles formed by newly added vertices. Such a graph is isomorphic to  $C_3(m_1, 0, 0) \bigcup_{p=0}^r pK_{1,n} \bigcup_{q=0}^s qK_n^c$

where  $m_1 \geq 1$ . And the converse is obvious. ■

For example, the graphs  $S_\gamma(B_{4,5} \cup K_3^c)$  and  $S_\gamma(C_3(3, 0, 0) \cup K_{1,4} \cup K_3^c)$  are shown in Figure 15.



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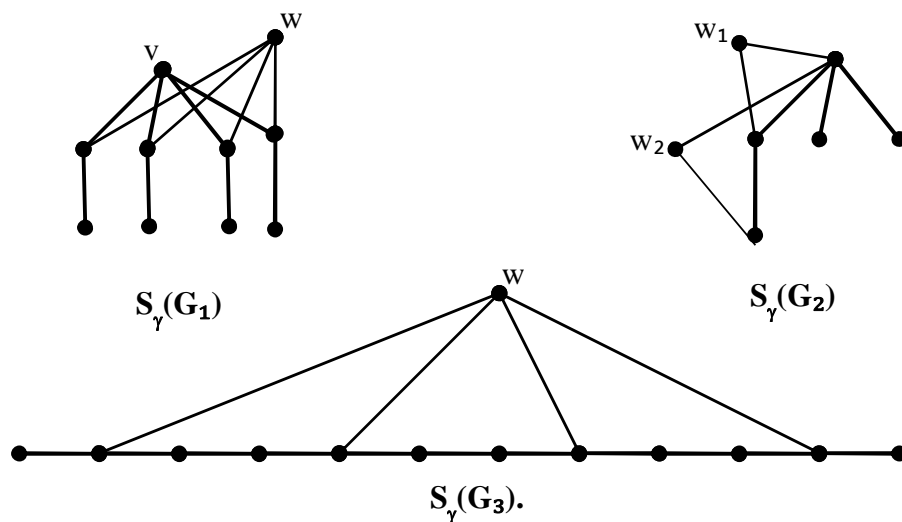


Figure 16

## III. DISTANCE PROPERTIES OF $\gamma$ -SPLITTING GRAPHS

Here we are interested in studying about the distance properties in  $S_\gamma$ -graphs. Also normally we expect  $\text{diam}(S_\gamma(G)) < \text{diam}(G)$ . But there are graphs with  $\text{diam}(S_\gamma(G)) \geq \text{diam}(G)$ . This behaviour gives rise to following three definitions  $S_\gamma^+$ -graphs,  $S_\gamma^-$ -graphs, and  $S_\gamma^*$ -graphs as given below:

A graph  $G$  is called a  $S_\gamma^+$ -graph if  $\text{diam}(G) < \text{diam}(S_\gamma(G))$ .

It is called a  $S_\gamma^-$ -graph if  $\text{diam}(G) > \text{diam}(S_\gamma(G))$ .

Finally, it is said to be a  $S_\gamma^*$ -graph if  $\text{diam}(G) = \text{diam}(S_\gamma(G))$ . For example,  $S_\gamma^+$ ,  $S_\gamma^-$ , and  $S_\gamma^*$ -graphs are shown in Figure 17.

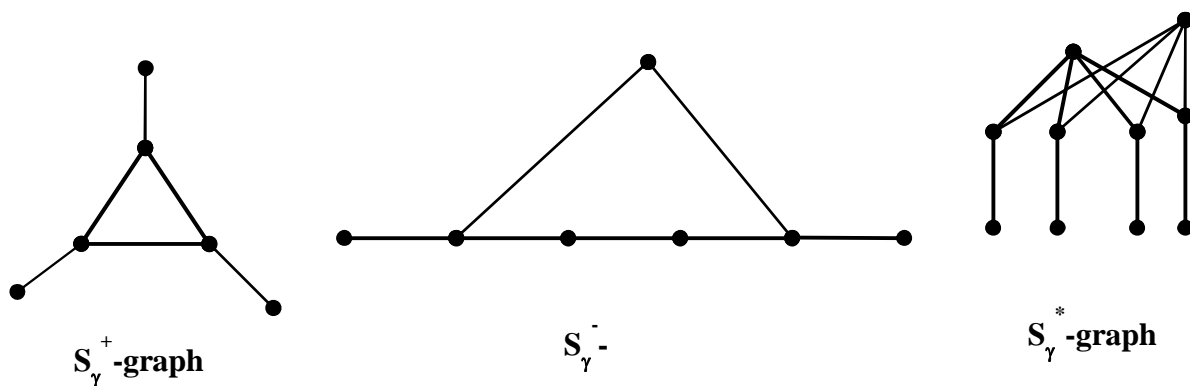


Figure 17

Some standard graphs with their diameters and corresponding families are listed below:

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Graph G	diam(G)	diam( $S_\gamma(G)$ )	Type
$K_n$	1	3	$S_\gamma^+$ -graph
$K_{m,n}$	2	3	$S_\gamma^+$ -graph
$W_3$	1	2	$S_\gamma^+$ -graph
Graph with exactly one full vertex	2	2	$S_\gamma^*$ -graph
$H_{n,n}$	3	3	$S_\gamma^*$ -graph
$B_m$	3	3	$S_\gamma^*$ -graph
$B_{m,n}$	3	3	$S_\gamma^*$ -graph
Spider	4	4	$S_\gamma^*$ -graph

**Theorem 3.1** For any graph G, the distance between newly added vertices in  $S_\gamma(G)$  is 2 or 3.

**Proof** Let G be any graph of order n, and  $w_1$  and  $w_2$  be any two newly added vertices in  $S_\gamma(G)$ . We know that  $d^*(w_i) = \gamma(G)$ ,  $1 \leq i \leq \eta$  and  $d(w_1, w_2) \geq 2$  (Fact 2.1).

Case (i) Suppose  $N(w_1) \cap N(w_2) \neq \emptyset$ . Let  $x \in N(w_1) \cap N(w_2)$ . Then x is the common neighbour of  $w_1$  and  $w_2$ , and so  $d(w_1, w_2) = 2$ .

Case (ii) Suppose  $N(w_1) \cap N(w_2) = \emptyset$ . Then let  $x \in N(w_1)$ . Since  $N(w_1)$  is a  $\gamma$ -set, every vertex in  $N^c(w_1)$  is adjacent to at least one vertex in  $N(w_1)$ . But  $N(w_2) \subseteq N^c(w_1)$ . Therefore, there exists a vertex  $y \in N(w_2)$  such that y is adjacent to a vertex x in  $N(w_1)$ . Then  $d(w_1, w_2) = 3$ . ■

**Theorem 3.2** For any graph G,  $\text{diam}(S_\gamma(G)) \leq 4$ .

**Proof** Let G be any graph and  $S_\gamma(G)$  be its corresponding  $\gamma$ -splitting graph. Let u and v be any two vertices in  $S_\gamma(G)$ . We claim that  $d(u, v) \leq 4$  for every  $u, v \in V(G)$ .

Case (i) If u and v are newly added vertices in  $S_\gamma(G)$ . By Theorem 3.1,  $d(u, v) \leq 3$ .

Case (ii) If u is a newly added vertex and  $v \in V(G)$ . Then  $N(u)$  is a dominating set, and therefore  $v \in N(u)$  or v is adjacent to a vertex in  $N(u)$  in  $S_\gamma(G)$ . This forces that  $d(u, v) \leq 2$ .

Case (iii) Suppose  $u, v \in V(G)$ . Then there arise two subcases.

Subcase (i) Let u belong to a  $\gamma$ -set S. Then there is a newly added vertex w corresponding to S. If  $v \in S$ , then  $uwv$  is a u-v path of length 2 in  $S_\gamma(G)$ . Therefore  $d(u, v) \leq 2$ . If  $v \notin S$ , then there is a vertex  $v_1$  in S, adjacent to v. Therefore  $uwv_1v$  is a u-v path of length 3, and so  $d(u, v) \leq 3$ . If v belongs to any other  $\gamma$ -set, then in a similar way we can show that  $d(u, v) \leq 3$ .

Subcase (ii) Neither u nor v belongs to any  $\gamma$ -set. Fix a newly added vertex w. Clearly,  $N(w)$  is a  $\gamma$ -set. So  $V(G) \subseteq N(N(w))$  in  $S_\gamma(G)$ . Therefore,  $d(u, v) \leq 4$ . Hence  $\text{diam}(S_\gamma(G)) \leq 4$ . ■

The inequality stated above is strict. For example,  $\text{diam}(S_\gamma(P_{3k})) = 4$ , for any  $k \geq 2$ . For example,  $\text{diam}(S_\gamma(P_6)) = 4$  can be verified in Figure 18.

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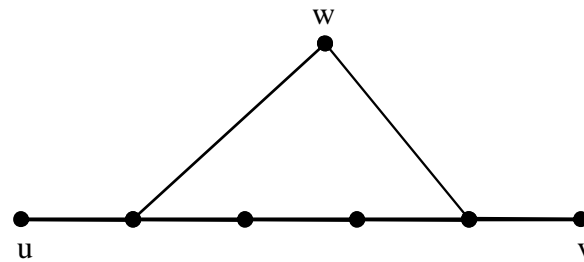


Figure 18

The following corollary gives a characterisation of  $S_\gamma^-$ -graphs.

Corollary 3.3 Any connected graph  $G$  with  $\text{diam}(G) > 4$ , is a  $S_\gamma^-$ -graph.

Proof Suppose  $G$  is a connected graph and  $\text{diam}(G) > 4$ . Let  $S_\gamma(G)$  be its corresponding  $\gamma$ -splitting graph. By Theorem 3.2,  $\text{diam}(S_\gamma(G)) \leq 4$  and the result follows. ■

$\gamma$ -splitting graph. By Theorem 3.2,

$$\text{It has been prove in [7], that } \eta(P_n) = \begin{cases} 1 & \text{if } n = 3k, k \geq 1 \\ \frac{k^2 + 5k + 2}{2} & \text{if } n = 3k+1, k \geq 0 \\ k+2 & \text{if } n = 3k+2, k \geq 0 \end{cases} \quad \text{and}$$

$$\eta(C_n) = \begin{cases} 3 & \text{if } n = 3k, k \geq 1 \\ \frac{(3k+1)(k+2)}{2} & \text{if } n = 3k+1, k \geq 1 \\ 3k+2 & \text{if } n = 3k+2, k \geq 1 \end{cases}$$

Proposition 3.4 The path graph  $P_n$  is  $S_\gamma^+$ -graph if  $n \leq 2$ ,  $S_\gamma^*$ -graph if  $n = 3, 4$ , and  $S_\gamma^-$ -graph if  $n \geq 5$ .

Proposition 3.5 The cycle graph  $C_n$  is  $S_\gamma^+$ -graph if  $n \leq 5$ ,  $S_\gamma^*$ -graph if  $n = 6, 7$ , and  $S_\gamma^-$ -graph if  $n \geq 8$ .

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