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Numerical solution of fractional-order logistic equations by fractional Euler's method

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Abstract— In this article, fractional descriptions of logistic equations are solved by using fractional Euler's formula. The fractional derivative in this problem is in the Caputo sense. Special attention is focused on the stability, existence and uniqueness of the fractional-order logistic equation. Illustrative examples are included to exhibit the efficiency of the proposed method.
Keywords— Logistic equations, Fractional-order differential equations, Existence and uniqueness, Euler's method.

I. INTRODUCTION

In the midst of 19th century, fractional calculus was initially elaborated as a mathematical model. Later on it has progressively come across in numerous disciplines of science and engineering. Fractional differential equations (FDEs) have been the concentration of plenty of research by the reason of their regular development in several uses in fluid mechanics, biology [12]. Various physical processes seem to show evidence of fractional order behaviour that may possibly vary with time or space. Generally FDEs do not possess exact solutions, thus fairly accurate and numerical procedure can be employed. Various applications of such problems involve a substantial demand for accomplished method for their numerical treatment [14]. In view of the fact that only some of the FDEs come across in practice can be worked out explicitly, it is essential to utilize numerical methods to acquire the approximate solutions [4].

The fractional logistic model can be acquired by using the fractional derivative operator on the logistic equation. The model is firstly circulated by Pierre Verhulst in 1938 [7]. The continuous model is expressed by first order ordinary differential equations. There are ample of variations of the population modelling [5]. The model is expressed the population growth perhaps restricted by some factors like population density. The distinctive uses of logistic curves are in medicine, where the logistic differential is handled to model the growth of tumours. This use can be deemed an augmentation of the above alluded use in the structure of environmental science [3].

The present paper is structured as follows: in section 2, we provide some necessary definitions. Section 3 and 4 contain equilibrium and stability investigation; and fractional-order logistic equation is furnished in section 4. In section 5, we present the numerical algorithm. After all, in section 6, numerical illustrations are elucidated.

II. BASIC CONCEPTS

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 2.1 [12] A real function $f(t), t>0$, is said to be in the space $C_\mu, \mu \in R$ if there exist a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C(0,\infty)$, and it said to in the space C_μ^n if and only if $f^{(n)} \in C_\mu, n\in N$.

Definition 2.2 [12] The Riemann-Lioville fractional integral operator I_a^α of order $\alpha>0$, with $a\geq 0$ of a function $f\in C_\mu, \mu\geq -1$ is defined as

$$I_a^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)ds, (t>a) \quad (1)$$

$$I_a^0 f(t)=f(t) \text{ , for } \alpha>0 \quad (2)$$

$\Gamma(z)$ is the well known Gamma function. Some of the properties of the operator I^α, which we will need here are in below:

For $f\in C_\mu, \mu\geq -1, \alpha, \beta\geq 0$ and $\gamma\geq -1$:

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1. $I^{\alpha} I^{\beta} f(t) = I^{\alpha+\beta} f(t)$;
2. $I^{\alpha} I^{\beta} f(t) = I^{\beta} I^{\alpha} f(t)$;
3. $I^{\alpha} t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 2.3 [12] The fractional derivative $({}^{c}D_{a}^{\alpha})$ *of* $f(t)$ in the Caputo sense is defined as

$$ {}^{c}D_{a}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds \quad \text{for } n-1 < \alpha < n, n \in N, t \geq a, f \in C_{-1}^{n} \tag{3} $$

The following are two basic properties of the Caputo's fractional derivatives [16];

1. Let $f \in C_{-1}^{n}, n \in N$. Then ${}^{c}D_{a}^{\alpha}$; $0 \leq \alpha \leq n$, is well defined and ${}^{c}D_{a}^{\alpha} f \in C_{-1}$.
2. Let $n-1 < \alpha \leq n, n \in N$ and $f \in C_{\mu}^{n}, \mu \geq -1$. Then

$$ I_{a}^{\alpha} ({}^{c}D_{a}^{\alpha}) f(t) = f(t) - \sum_{k=0}^{n-1} f^{k}(a) \frac{(t-a)^{k}}{k!} \tag{4} $$

III.EQUILIBRIUM AND STABILITY

Let $q \in (0,1]$ and consider the initial value problem [8]

$$ {}^{c}D^{q} u(t) = f(u(t)), t > 0 \tag{5} $$

$$ u(0) = u_{0}. \tag{6} $$

Calculating the equilibrium points of (1) and (2) let

$$ {}^{c}D^{q} u(t) = 0, $$

Then

$$ f(u_{eq}) = 0. $$

Calculating the asymptotic stability, let

$$ u(t) = u_{eq} + \xi(t), $$

then

$$ {}^{c}D^{q} (u_{eq} + \xi) = f(u_{eq} + \xi) $$

which means that

$$ {}^{c}D^{q} \xi(t) = f(u_{eq} + \xi) \quad , $$

but

$$ f(u_{eq} + \xi) \approx f(u_{eq}) + f^{'}(u_{eq}) \xi + \ldots $$

$$ f(u_{eq} + \xi) \approx f^{'}(u_{eq}) \xi $$

Where $f(u_{eq} = 0)$, and then

$$ {}^{c}D^{q} \xi(t) \approx f^{'}(u_{eq}) \xi(t), t > 0 \quad \text{and} \quad \xi(0) = u_{0} - u_{eq} \quad . \tag{7} $$

Now let the solution $\xi(t)$ of (3) exist. So if $\xi(t)$ is increasing, then the equilibrium point u_{eq} is unstable and if $\xi(t)$ is decreasing, then the equilibrium point u_{eq} is locally asymptotically stable.

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It must be jot down that these results are the same out comes as for analysing the stability of the initial value problem (IVP) of the ODEs.

$$u'(t) = f(u(t)) \quad , t > 0 \qquad \text{and } u(0) = u_0 .$$

Also these results concur with [1, 2, and 11].

IV. FRACTIONAL-ORDER LOGISTIC EQUATION

Let $q \in (0,1], r > 0$ and $x_0 > 0$; the IVP of the fractional-order logistic equation is presented in [8] by

$$^{c}D^{q}u(t) = ru(t)(1-u(t)),\ t > 0 \text{ and } x(0) = x_0 , \tag{8}$$

and calculating the equilibrium points, let

$$^{c}D^{q}u(t) = 0;$$

then $x = 0,1$ are the equilibrium points.

Now, we study the stability of the equilibrium points, we have

$$^{c}D^{q}\xi(t) = f'(u_{eq} = 0)\xi(t) = r\xi(t), t > 0 \text{ and } \xi(0) = u_0$$

is given by [9].

$$\xi(t) = \sum_{n=0}^{\infty} \frac{r^n t^{nq}}{\Gamma(nq+1)} u_0 \tag{9}$$

and then the equilibrium point $x = 0$ is unstable.

Also for the equilibrium point $x = 1$ we have the IVP

$$^{c}D^{q}\xi(t) = f'(u_{eq} = 1)\xi(t) = -r\xi(t), t > 0 \text{ and } \xi(0) = u_0 - 1 \tag{10}$$

which is (if $u_0 > 1$) the fractional-order logistic equation and has the solution [10]

$$\xi(t) = \sum_{n=0}^{\infty} \frac{(-r)^n t^{nq}}{\Gamma(nq+1)} (u_0 - 1) \tag{11}$$

and then the equilibrium point $x = 1$ is asymptotically stable.

V. EXISTENCE AND UNIQUENESS

Let $I = [0,T], T < \infty$ and $C(I)$ be the class of all continuous functions defined on I, with norm

$$\|u\| = \sup_t \left|e^{-Nt}u(t)\right|, N > 0 \tag{12}$$

Which is equivalent to the sup- norm $\|u\| = \sup_t |u(t)|$. When $t > r \geq 0$ we write $C(I_r)$.

Consider the initial value problem of the fractional-order logistic equation (4).

Definition 5.1 [8] we will define $u(t)$ to be a solution of the IVP (4) if

1. $(t, u(t)) \in D, t \in I$ where $D = I \times B, B = \{u \in R : |u| \leq b\}$.
2. $u(t)$ satisfies (4).

Theorem 5.1 The IVP (4) has a unique solution $u \in C(I)$,

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$$u' \in U = \{u \in L_1[0,T], \|u\| = \|e^{-Nt}x(t)\|_{L_1}\}.$$

Proof. In depth proof can be found in [8].

VI. NUMERICAL ALGORITHM

In this section we shall obtain the fractional Euler's algorithm (generalization of classical Euler's method) [15] that we have developed for the numerical solution of fractional-order logistic equations in Caputo sense.

Consider the IVP

$$^{c}D^{q}u(t) = f(u(t)), \tag{13}$$

$$u(0) = u_0, 0 < q \le 1, t > 0. \tag{14}$$

Let $[0,a]$ be the interval over which we want to get the solution of the problem (9)-(10). In fact, we shall not find a function $y(t)$ that suits the IVP (9). In its place, a set of points $\{(t_j, y(t_j))\}$ is produced, and the points are employed for our approximation.

For convenience we subdivide the interval $[0,a]$ into k subintervals $[t_j, t_{j+1}]$ of equal width $h = a/k$ by using the nodes $t_j = jh$, for $j = 0,1,2,\ldots k$. Assume the $u(t), {}^{c}D^{q}u(t)$ and ${}^{c}D^{2q}u(t)$ are continuous on $[0,a]$, and use the generalized Taylor's formula to expand $u(t)$ about $t = t_0 = 0$. For each value t there is a value c_1 so that

$$u(t) = u(t_0) + ({}^{c}D^{\alpha}u(t))(t_0)\frac{t^q}{\Gamma(q+1)} + ({}^{c}D^{2q}u(t))(c_1)\frac{t^{2q}}{\Gamma(2q+1)} \tag{15}$$

When $({}^{c}D^{q}u(t))(t_0) = f(u(t_0))$ and $h = t_1$ are substituted into equation (11), the result is an expression for $u(t_1)$:

$$u(t) = u(t_0) + f(u(t_0))\frac{t^q}{\Gamma(q+1)} + ({}^{c}D^{2q}u(t))(c_1)\frac{t^{2q}}{\Gamma(2q+1)}. \tag{16}$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving h^{2q}) and get

$$u(t_1) = u(t_0) + \frac{h^q}{\Gamma(q+1)} f(u(t_0)). \tag{17}$$

The process is repeated and generates a sequence of points that approximates the solution $u(t)$. The general formula for Euler's method of fractional-order logistic equation is

$$u(t_{j+1}) = u(t_j) + \frac{h^q}{\Gamma(q+1)} f(u(t_j)) \tag{18}$$

$t_{j+1} = t_j + h$, for $j = 0,1,2,\ldots,k-1$. It is clear that if $q = 1$, then the fractional-order Euler's method (18) reduces to the classical Euler's method.

VII. NUMERICAL EXAMPLES

Example 6.1 Consider a fractional-order logistic equation

$$^{c}D^{q}u(t) = \frac{1}{2}u(t)(1-u(t)), t > 0, 0 < q \le 1 \tag{19}$$

$$u(0) = 0.85 \tag{20}$$

Now we apply the Euler's method (18) and obtain approximate solutions which are plotted in Figure 1 for $h = 0.05$ and different values of q.

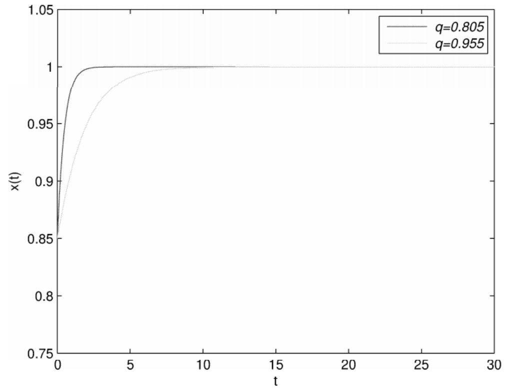

Fig. 1 Solutions of the equation

Example 6.2 Consider a fractional version of logistic equation

$$ {}^{c}D^{0.98}u(t) = ru(t)(1-u(t)), t > 0, 0 < q \leq 1 \tag{21}$$

$$ u(0) = 0.85 \tag{22}$$

We take $r = 1$. Then, the solution of equation (21) is drawn in Figure 2 for step size 0.05.
Again we take $r = 0$, and then the solution is plotted in Figure 3 for same step size.

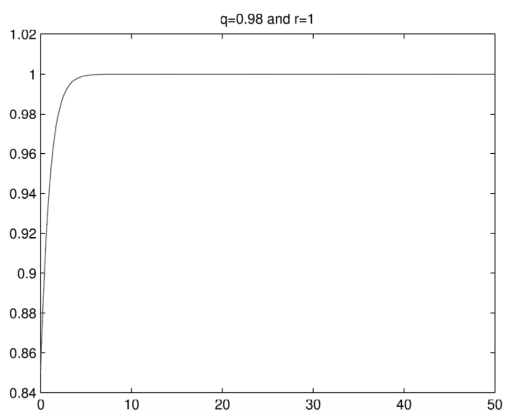

Fig. 2 Solutions of the equation

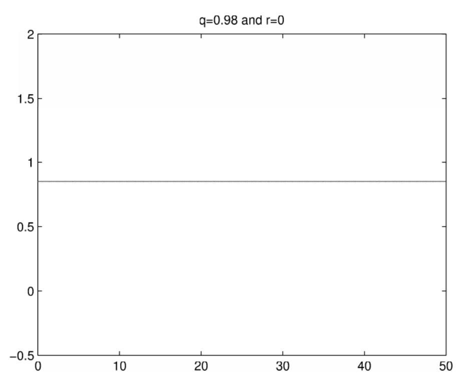

Fig. 3 Solutions of the equation

VIII. CONCLUSION

In this paper, Euler's method of fractional order is employed to solve fractional-order logistic equations numerically. We have studied the equilibrium and stability properties to the value of fractional order. The proposed method has proved its efficiency in

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solving fractional-order logistic equations.

REFERENCES

[1] Ahmed. E., EI-syed. A.M.A., EI-saka. H.A.A.: On some Routh-Hurwitz conditions for fractional order differential equations and their applications in lorenz, Rossler, Chua and Chen systems, Phys.,Lett A 35,(1) (2006).

[2] Ahmed. E., EI-syed. A.M.A., EI-saka. H.A.A.: Equilibrium points, stability and numerical solutions of fractional order predator-prey and rabies model, journal of mathematical Analysis and applied, 325, 542-553,(2007).

[3] Alligood. K.T., sauerr. T.D., Yorke. J.A.: An introduction to dynamical systems, springer (1996).

[4] Alwas. M.A.M.: polynomial differential equations with piecewise linear coefficient, Differential equations and dynanamical systems, 19(3), 267-281 (2011).

[5] Ausloos. M.: the logistic map and the route to chaos: from the beginnings to modern apllications, XV1, 411,(2006).

[6] Burden. R.L., Faires. J.D.: numerical analysis, PWS,Boston (1993).

[7] Cushing. J.M.: An Introduction to structured population Dynamics, Society for industrial and Applied mathematics, (1998).

[8] EI-Sayed. A.M.A., EI-mesiry., El-Saka. H.A.A.: On the fractional order logistic equation, Applied mathematics letters, 20, 817-823, (2007).

[9] EI-Sayed. A.M.A.: Fractional diferential difference equations, Journal of fractional calculus, 10, 101-106, (1996).

[10] EI-Sayed. A.M.A., Gaafar. F.M.: Fractional order differential equations with memory and fractional order relaxiation-oscillation model, pure mathematical and Applications, 12, (2001).

[11] Matignon. D.: Stability results for fractional differential equations with applications to control processing, in: Computational Engineering in System Applications, 2, lille, France, p.963, (1996).

[12] podlubny. I.: Fractional differential equations, Academic press, New york(1991).

[13] Richtmyer. R.D., morton. K.W.: Difference methods for initial value problems, Inter-science publishers, New york (1967).

[14] Smith. G.D.: Numerical solutions of partial differential equations, Oxford university press, New york,(1965).

[15] Zaid. M., Shaher. M.: An algorithm for the numerical solution of differential equations of fractional order, 26, 15-27, (2008).