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Perfect Degree Support Product Graphs

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Abstract--For a graph G(V,E), the support s(v) of a vertex v is defined as the sum of degrees of its neighbours. A graph G is said to be balanced (highly unbalanced), if the support of all the vertices are same (distinct). Let k be any positive integer. A graph G is said to be a k – perfect degree support graph (k – pds graph) if for any vertex v in G, the ratio of its support and its degree is the constant k. A graph G is called a k – linear degree support graph (k – lds graph) if, for any two vertices in V with distinct degrees, the ratio of difference between their supports and the difference between their degrees is the constant k. The properties of k – lds graphs and k – pds graphs in various product graphs have been studied in this paper.
Keywords--Support, balanced graphs, highly unbalanced graphs, k – perfect degree support graphs, k – linear degree support graphs.
AMS Subject Classification code (2000): 05C (Primary)

I. INTRODUCTION

Throughout this paper, we consider only finite, simple, undirected graphs. For notations and terminology we follow [3]. A graph G is said to be *r – regular,* if every vertex of G has degree r. Let G_1 and G_2 be any two graphs. The graph $G_1 \circ G_2$ obtained from one copy of G_1 and $|V(G_1)|$ copies of G_2 by joining each vertex in the i^{th} copy of G_2 to the i^{th} vertex of G_1 is called the *corona* of G_1 and G_2. The *cartesian product* of G and H is denoted by G X H and their join is denoted by $G \vee H$. The *composition graph* of G_1 to G_2 is denoted by $G_1[G_2]$. The concepts of support, balanced graphs, highly unbalanced graphs have been introduced and studied by Selvam Avadayappan and G. Mahadevan [1]. The *support* s(v) of a vertex v is the sum of degrees of its neighbours. That is, $s(v) = \sum_{u \in N(v)} d(u)$. Note that the support of any vertex in an r – regular graph is r^2.

A graph G is said to be a *balanced graph,* if the support of every vertex in G is equal. It is easy to observe that the complete bipartite graphs $K_{m,n}$ and the regular graphs are balanced graphs. A graph G is said to be *highly unbalanced*, if distinct vertices of G have distinct supports. For example, a highly unbalanced graph is shown in Figure 1.

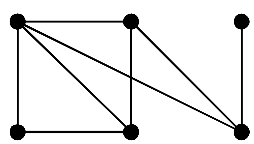


Figure 1

The following results have been proved in [1]:

Result 1 $\sum_{v \in V} s(v) = \sum_{v \in V} d(v)^2$.

Result 2 $s(v) = (n-1)^2$ for every $v \in G$ if and only if $G \cong K_n$.

Result 3 For any balanced graph G, $\delta(G) = 1$ if and only if $G \cong K_2$ or $K_{1,n}$.

Result 4 For any $n \geq 6$, there is a highly unbalanced graph of order n.

Consequently the concepts of k – perfect degree support graph and k – linear degree support graph have been defined in [2].

A graph G is said to be a *k – perfect degree support graph* (or simply a *k – pds graph*), if for any vertex v in G, $\frac{s(v)}{d(v)} = k$. For example, the graph shown in Figure 2 is a 3 – pds graph. In general, $C_n \circ K_2$ is a 3 – pds graph for any n > 2.

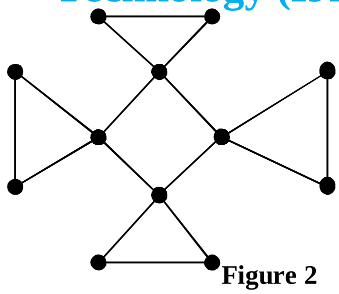


Figure 2

A graph G is said to be a *k – linear degree support graph* (or simply a *k – lds graph*), if for any two vertices u and v in G with d(u) ≠ d(v) in G, $\frac{s(u)-s(v)}{d(u)-d(v)}$ = k, for some integer k. Or equivalently, a graph in which for any vertex v, s(v) = k d(v) + c for some constant c is called a k – lds graph. Note that in a k – lds graph, s(u) – k d(u) = s(v) – k d(v). For example, the graph shown in Figure 3 is a 3 – lds graph.

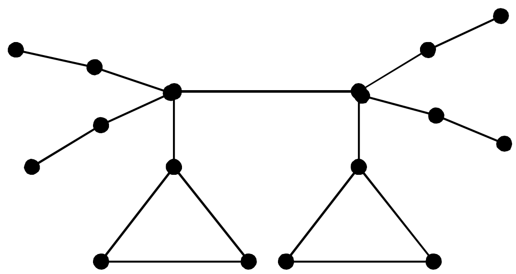


Figure 3

In general any k – pds graph is a k – lds graph but not the converse. For example, the graph shown in Figure 3 is a 3 – lds graph but it is not a 3 – pds graph. In fact k – pds graphs are k – lds graphs with c = 0. All balanced graphs are 0 – lds graphs and any r – regular graphs are r – pds graphs and hence r – lds graphs.

Note that in a k – lds graph, two vertices of same degree have the same support. That is, d(u) = d(v) implies that s(u) = s(v). A few families of k – pds graphs and k – lds graphs with some constraints have been constructed in [2]. Also k – pds trees have been characterized in [2].

In this paper, many new k – lds and k – pds graphs have been generated using various graph products.

k – pds and k – lds product graphs

Regarding the properties of k – lds and k – pds graphs in product graphs, the following facts can be easily verified:

Fact 2.1 A disconnected graph is a k – lds graph if and only if each of its components is a k – lds graph.

Fact 2.2 Let G_1 be a k_1 – lds graph and G_2 be a k_2 – lds graph. Then $G_1 \cup G_2$ is a k – lds graph if and only if $k_1 = k_2$.

By a (p, q) – graph, we mean a graph with p vertices and q edges.

Fact 2.3 Let G_1 be a (n_1,m_1) - graph and G_2 be a (n_2,m_2) - graph. Then we can easily verify the following:

(i) For any $v \in V(G_1 \cup G_2)$, we have

$d_{G_1 \cup G_2}(v) = d_{G_1}(v)$ and $s_{G_1 \cup G_2}(v) = s_{G_1}(v)$, if $v \in V(G_1)$ and

$d_{G_1 \cup G_2}(v) = d_{G_2}(v)$ and $s_{G_1 \cup G_2}(v) = s_{G_2}(v)$, if $v \in V(G_2)$

(ii) For any $v \in V(G_1 \vee G_2)$, we have

$d_{G_1 \vee G_2}(v) = d_{G_1}(v) + n_2$; $s_{G_1 \vee G_2}(v) = s_{G_1}(v) + n_2 d_{G_1}(v) + 2m_2$, if $v \in V(G_1)$; $d_{G_1 \vee G_2}(v) = d_{G_2}(v) + n_1$; $s_{G_1 \vee G_2}(v) = s_{G_2}(v) + n_1 d_{G_2}(v) + 2m_1$, if $v \in V(G_2)$

(iii) For any $(u,v) \in V(G_1 \times G_2)$, we have

$d_{G_1 \times G_2}((u,v)) = d_{G_1}(u) + d_{G_2}(v)$ and

$s_{G_1 \times G_2}((u,v)) = 2d_{G_1}(u)\, d_{G_2}(v) + s_{G_2}(v) + s_{G_1}(u)$,

(iv) For any $(u,v) \in V(G_1[G_2])$, we have

$d_{G_1[G_2]}((u,v)) = n_2 d_{G_1}(u) + d_{G_2}(v)$.

(v) Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$.In $V(G_1 \circ G_2)$, let u_{ij} denote the vertex corresponding to u_j in the i^{th} copy of G_2. Then we have,

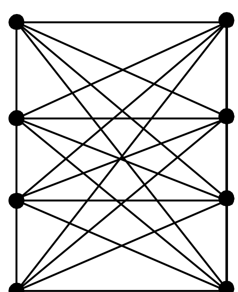
$d_{G_1 \circ G_2}(v_i) = d_{G_1}(v_i) + n_2$ and $s_{G_1 \circ G_2}(v_i) = s_{G_1}(v_i) + n_2 d_{G_1}(v) + 2m_2 + n_2$,

$d_{G_1 \circ G_2}(u_{ij}) = d_{G_2}(u_{ij}) + 1$ and $s_{G_1 \circ G_2}(u_{ij}) = s_{G_2}(u_{ij}) + d_{G_2}(u_{ij}) + d_{G_1}(v_i) + n_2$.

Now let us prove some results on k – lds and k – pds product graphs.

Theorem 2.4 If G is a k – lds graph of order n, then $G \vee G$ is a (k + n) – lds graph.

Proof Let G be any k – dsl graph of order n. Then for any two vertices v_i and v_j of distinct degrees, $\frac{s_G(v_i) - s_G(v_j)}{d_G(v_i) - d_G(v_j)} = k$. Now consider $G \vee G$. Then the degree of every vertex gets increased by n and the support of any vertex gets increased by its degree times n. For any two vertices of distinct degrees, $\frac{s_{G\vee G}(v_i) - s_{G\vee G}(v_j)}{d_{G\vee G}(v_i) - d_{G\vee G}(v_j)} = \frac{s_G(v_i) + n\, d_G(v_i) - s_G(v_j) - n\, d_G(v_j)}{d_G(v_i) - d_G(v_j)} = k + n$. Hence, $G \vee G$ is a (k + n) – lds graph. For example, a 1 – lds graph P_4 and the corresponding 5 – lds graph $P_4 \vee P_4$ are shown in Figure 3.



$P_4 \vee P_4$

Figure 3

Theorem 2.5 Let G be any (n, m) – graph. If $G \vee K_1$ is a k – pds graph, then $\frac{2m}{n} = k - 1$.

Proof Let G be a graph with n vertices and m edges such that $G \vee K_1$ is a k – pds graph. Let v be the vertex of K_1. Then clearly v is a full vertex with support $s_{G\vee K_1}(v) = \frac{\sum_{u\in V(G)} d_{G\vee K_1}(u)}{n} = \frac{\sum_{u\in V(G)} d_G(u) + n}{n} = \frac{2m+n}{n} = k$. Hence the proof.

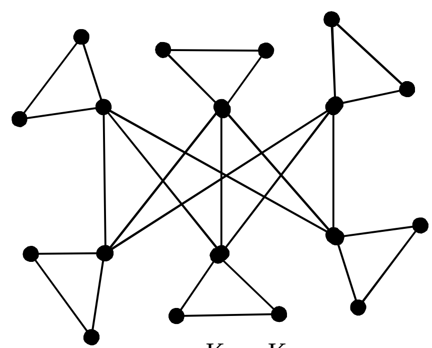
Theorem 2.6 There does not exist a k – pds graph G such that $G \vee K_1$ is also k – pds.

Proof Suppose G is a k – pds graph such that $G \vee K_1$ is a k – pds graph. Then for any vertex v in G, $s_G(v) / d_G(v) = s_{G\vee K_1}(v) /$

$d_{G\vee K_1}(v) = k$. Now by using Fact 5.8 (ii), $\frac{s_G(v)+d_G(v)+n}{d_G(v)+1} = k$. That is, $\frac{kd_G(v)+d_G(v)+n}{d_G(v)+1} = k$, which implies $d_G(v) = k - n$. Since $G \vee K_1$ is connected, we have $k > n$. In other words, $s_G(v) > n\, d_G(v)$ which is impossible. Therefore such a graph G does not exist ■

Theorem 2.7 For an r – regular graph G of order n_1 and a k – regular graph H of order n_2, $G \circ H$ is a $(r + k)$ – pds graph.

Proof Let G and H be r – regular and k – regular graphs respectively. Then by Fact 2.3 (v), $G \circ H$ is a biregular graph. In particular, $s_{G\circ H}(v) = r + n_2$; $s_{G\circ H}(v) = r^2 + n_2(r + k + 1)$, for any $v \in V(G)$ and $d_{G\circ H}(u) = k + 1$; $s_{G\circ H}(u) = k^2 + k + r + n_2$, for all $u \in V(H)$. Therefore we get $\frac{s_{G\circ H}(v) - s_{G\circ H}(u)}{d_{G\circ H}(v) - d_{G\circ H}(u)} = r + k$. Hence $G \circ H$ is a $(r + k)$ – pds graph. For example, $K_{3,3} \circ K_2$ which is a 5 – pds graph is shown in Figure 4. ■



$K_{3,3} \circ K_2$

Figure 4

Theorem 2.8 For any two regular graphs G and H, G[H] is a k – pds graph.

Proof Suppose G is an r – regular graph of order n_1 and H is a k – regular graph of order n_2. Then it is easy to note that G[H] is a $(n_2r + k)$ – regular graph and hence a $(n_2r + k)$ – pds graph. For example, $C_4[K_2]$ which is a 5 – pds graph is shown in Figure 5.

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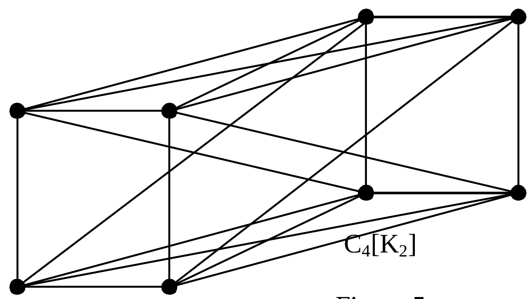

Figure 5

Theorem 2.9 If G is a k – lds graph, then $G \times H$ is a $(k + 2r)$ – lds graph, for any r – regular graph H.

Proof Let G be any k – lds graph and H be any r – regular graph. Then $\frac{s_G(v_i) - s_G(v_j)}{d_G(v_i) - d_G(v_j)} = k$, for any two vertices v_i and v_j of distinct degrees in G and $s_H(w) = r^2$ for any vertex w in H. Using Fact 2.3 (iii), for any two vertices in $G \times H$, we have $s_{G\times G}(v_i,w_k) - s_{G\times G}$

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$(v_j, w_l) = (k + 2r) \{d_{G \times G}(v_i, w_k) - d_{G \times G}(v_j, w_l)\}$, which implies that $G \times H$ is a $(k + 2r)$ – graph.

For example, a 1 – lds graph P_4 and the corresponding 5 – lds graph $P_4 \times C_4$ are shown in Figure 6.

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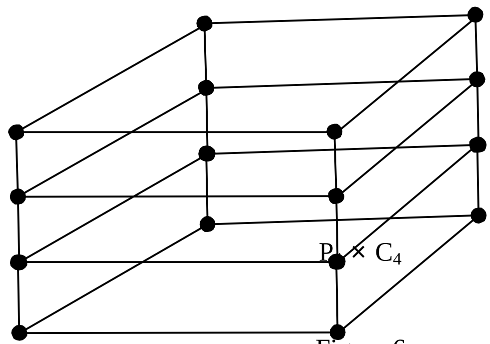

$P_4 \times C_4$

Figure 6

REFERENCES

[1] Selvam Avadayappan and G. Mahadevan, Highly unbalanced graphs, ANJAC Journal of Sciences, 2(1), 23 – 27, 2003.
[2] Selvam Avadayappan and M. Bhuvaneshwari, Perfect degree support graphs, (Preprint).
[3] R.Balakrishnan and K. Ranganathan, A Text Book of graph Theory, Springer-Verlag, New York, Inc(1999).