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# Existence results for fractional order pantograph equation with Riemann-Liouville derivative

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***Abstract— In this paper, we study the pantograph equations of order $\alpha \in (0,1)$ with Riemann-Liouville derivative. By means of the Banach fixed-point theorem with Bielecki norms, some results concerning the existence of solutions are obtained.***
***Keywords— Fractional order pantograph equations; Riemann-Liouville fractional derivatives; Banach-fixed point theorem.***

## I. INTRODUCTION

Fractional differential equation (FDE) has been used by many researchers to effectively describe the evolution of a variety of engineering, economical, physical, and biological processes [1-5].Among all the topics, initial value problems for fractional differential equations have attracted considerable attention [7].In fact FDE is considered as an alternative model to nonlinear differential equations [3].Though the concepts and the calculus of fractional derivative are few centuries old, it is recognized only recently that these derivatives form an excellent framework for modelling real world problems. This in turn led to the maintained study of the theory of FDEs [5].In [3,4] the authors have confirmed the existence of solutions of abstract fractional differential equations by using fixed point techniques.

In this paper, we discuss the non-linear fractional order pantograph equation

$$\begin{cases} D^{\alpha} y(t) = f(t, y(t), y(\lambda t)), t \in [0,T], \\ \tilde{y}(0) = r, \end{cases} \tag{1}$$

Where $\alpha, \lambda \in (0,1)$. Let $X$ is a Banach space and $f : [0,T] \times X \times X \to X$ is a continuous function. Furthermore, $J = [0,T], y(0) = t^{1-\alpha} y(t)|_{t=0}$ and $D^{\alpha} y$ represents a Riemann-Liouville fractional derivative.

The pantograph equations is a kind of delay differential equations and arise in many applications such as electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics and cell growth, etc. [6].Pantograph equations are distinguished by the presence of a linear functional argument and play an vital role in describing numerous different phenomena. In particular they turn out to be fundamental when ODEs-based model fail. These equations arise in industrial applications [8] and in studies based on biology, economy, control and electrodynamics [8]. K.Balachandar et.al investigated fractional order pantograph differential equations[2]. The newness of this paper is that existence results for fractional order pantograph equations with Riemann-Liouville derivative.

The paper is organized as follows. In Section 2, preliminaries and notations are given and existence results of pantograph equations of order $\alpha \in (0,1)$ are solved in Section 3, followed by conclusions in Section 4.

## II. PRELIMINARIES

In this section, we introduce definitions and preliminary facts which are used throughout this paper.Let X be a Banach space and $C_{1-\alpha}(J, X) = \{ y \in C((0,T], X) : t^{1-\alpha} y \in C(J, X) \}$.For $y \in C_{1-\alpha}(J, X)$ and $t \in J = [0,T]$.We defined two weighted norms,

$$\|y\|^{*} = \max_{t \in J} t^{1-\alpha} |y(t)| \qquad \text{or} \qquad \|y\|_{*} = \max_{t \in J} t^{1-\alpha} e^{-\mu t} |y(t)|,$$

with a fixed positive constant $\mu$ .

*Definition 1*([1]): The Riemann-Liouville fractional integral operator of order $q$ >0 of a function $f \in L_1(0,T]$ is defined as

$$I^{q}u(t)=\frac{1}{\Gamma(q)}\int_{0}^{t}(t-s)^{q-1}u(s)ds,$$

where $\Gamma(.)$ is the gamma function.

*Definition 2*([2]): The Riemann-Liouville fractional derivative of order $q>0,\ n-1<q<n,\ n\in N,$ is defined as

$$D^{q}u(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-q-1}u(s)ds,$$

where the function $u(s)$ has absolutely continuous derivative upto order $(n-1)$.

## III. EXISTENCE RESULTS

Assume the following conditions:

(A1): There exists a positive constant $L>0$, such that

$$\left|f(t,u_1,v_1)-f(t,u_2,v_2)\right|\le L\left(\left|u_1-u_2\right|+\left|v_1-v_2\right|\right).$$

The Lipchitz condition is noticed that the case when $\alpha\in(\frac{1}{2},1)$, we do not need and conditions on the coefficients.

$$\text{(A2):}\ \rho\equiv\frac{T^{\alpha}L\Gamma(\alpha)}{\Gamma(2\alpha)}\left(1+\lambda^{\alpha-1}\right).$$

*Theorem 1:* If the assumption (A1) is hold, then the fractional pantograph equation (1) has a unique solution.

*Proof:* Consider the problem $y=Fy$, where operator $F$ is defined as

$$Fy(t)=rt^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}f(s,y(s),y(\lambda s))ds$$

Now, we have to show that $F$ has a fixed point. We shall show that $F$ is a contraction map. We consider two cases.

Case1: Let $0<\alpha<\frac{1}{2}$ then, assumption (A1), we have

$$\begin{aligned}\left\|Fy_1-Fy_2\right\|^{*}&\le\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}\left[\left\|f(s,y_1(s),y_2(\lambda s))-f(s,y_2(s),y_2(\lambda s))\right\|\right]ds\\&\le\frac{1}{\Gamma(\alpha)}\max_{t\in J}t^{1-\alpha}\int_{0}^{t}(t-s)^{\alpha-1}\left[L\left|y_1(s)-y_2(s)\right|+L\left|y_1(\lambda s)-y_2(\lambda s)\right|\right]ds\\&\le\frac{1}{\Gamma(\alpha)}\max_{t\in J}t^{1-\alpha}\left\|y_1-y_2\right\|^{*}\int_{0}^{t}(t-s)^{\alpha-1}\left[Ls^{\alpha-1}+L(\lambda s)^{\alpha-1}\right]ds\\&\le\left\|y_1-y_2\right\|^{*}\frac{T^{\alpha}L\Gamma(\alpha)}{\Gamma(2\alpha)}\left(1+\lambda^{\alpha-1}\right)\\&=\rho\left\|y_1-y_2\right\|^{*}.\end{aligned}$$

Hence, operator $F$ has a unique fixed point by the Banach fixed point theorem.

Case2: Assume that $\frac{1}{2}<\alpha<1.$ Now, we use the norm $\|\cdot\|_{*}$ with a positive $\mu$ such that:

$$\sqrt{\mu} > \rho_1 \equiv \frac{L\left(1+\lambda^{\alpha-1}\right)}{\Gamma(\alpha)} \frac{\Gamma(2\alpha-1)}{\sqrt{2\Gamma(2(2\alpha-1))}} \sqrt{T^{2\alpha-1}} .$$

We will use the Schwarz inequality for integrals

$$\int_0^t |x(s)||y(s)|ds \le \sqrt{\int_0^t x^2(s)ds}\sqrt{\int_0^t y^2(s)ds} .$$

Using assumption (A1), the Schwarz inequality, we have

$$\|Fy_1 - Fy_2\|_* \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\mu t} \left[\| f(s, y_1(s), y_2(\lambda s)) - f(s, y_2(s), y_2(\lambda s))\|\right] ds$$

$$\le \frac{1}{\Gamma(\alpha)} \max_{t\in J} t^{1-\alpha} \|y_1 - y_2\|_* e^{-\mu t} \int_0^t (t-s)^{\alpha-1} \left[Ls^{\alpha-1} e^{\mu s} + L(\lambda s)^{\alpha-1} e^{\mu s}\right] ds$$

$$\le \left(\frac{L + L\lambda^{\alpha-1}}{\Gamma(\alpha)}\right) \max_{t\in J} t^{1-\alpha} \|y_1 - y_2\|_* e^{-\mu t} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} e^{\mu s} ds$$

$$\le \left(\frac{L + L\lambda^{\alpha-1}}{\Gamma(\alpha)}\right) \max_{t\in J} t^{1-\alpha} \|y_1 - y_2\|_* e^{-\mu t} \sqrt{\int_0^t (t-s)^{2(\alpha-1)} s^{2(\alpha-1)} ds} \sqrt{\int_0^t e^{2\mu s} ds}$$

$$\le \frac{\rho_1}{\sqrt{\mu}} \|y_1 - y_2\|_* .$$

It proves that problem (1) has unique solution. This ends the proof.

## IV. CONCLUSION

In this paper we consider the pantograph equations with Riemann-Liouville derivative, and we studied the existence of it's solutions by Banach-fixed point theorem.

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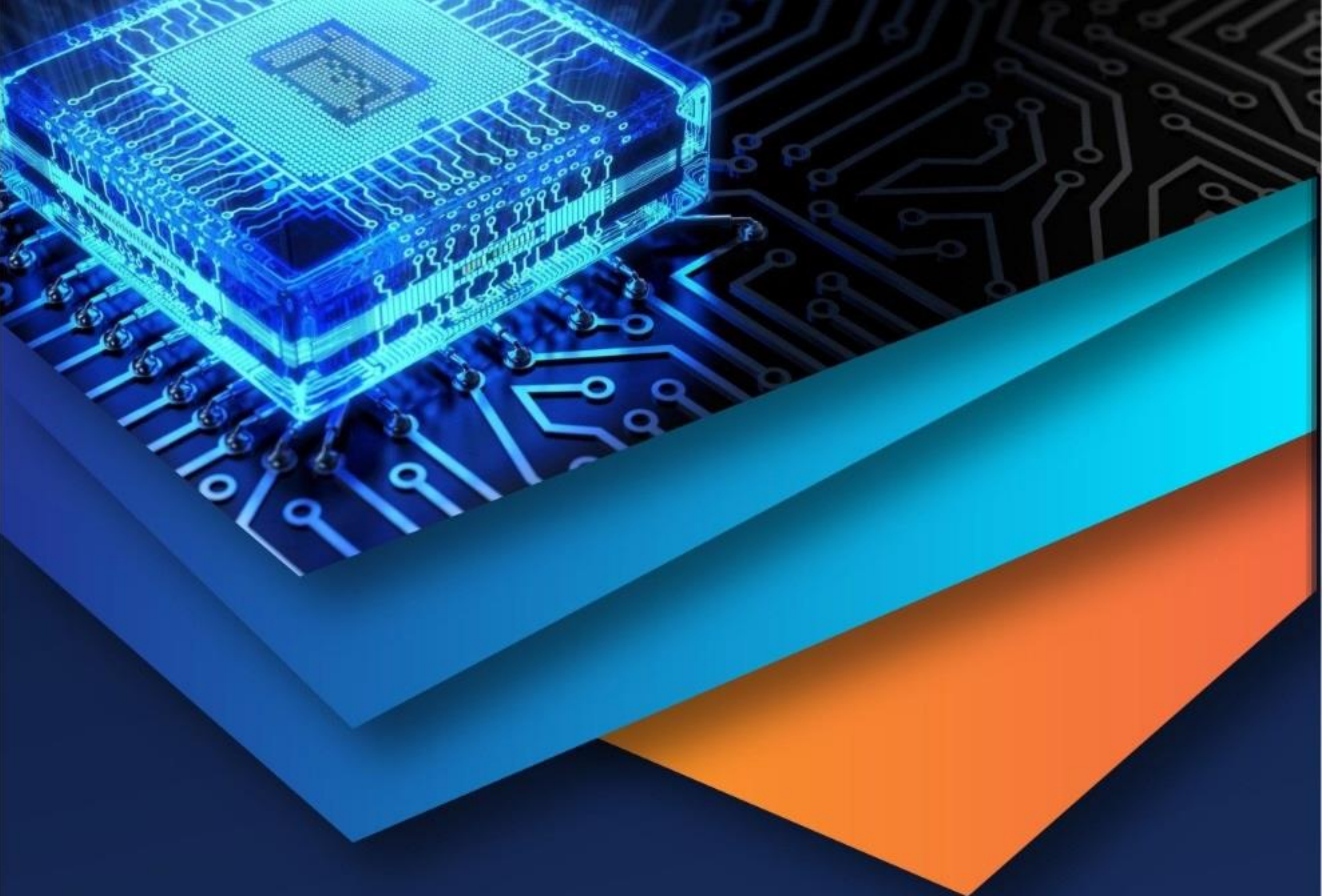


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