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Fixed Point Results in Quasimetric Spaces

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Abstract: In the setting of quasimetric spaces, we prove some new results on the existence of fixed points for contractive type maps with respect to Q-function. Our results either improve or generalize many known results in the literature.
Keywords: Quasimetric space, contractive maps, fixed point, Q-function.

I. INTRODUCTION AND PRELIMINARIES

Let X be a metric space with metric *d*. We use $S(X)$ to denote the collection of all nonempty subsets of $X, cl(X)$ for the collection of all nonempty closed subsets of $X, CB(X)$ for the collection of all nonempty closed bounded subsets of *X* and *H* for the Hausdorff metric on $CB(X)$ that is,

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, A,B \in CB(X),$$

Where $d(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from the point *a* to the subset *B*.

For a multivalued map $T : X \to CB(X)$, we say

(a) *T* is *contraction* [1] if there exists a constant $\lambda\epsilon(0,1)$, such that for all $x, y\epsilon X$,

$$H(T(x),T(y)) \le \lambda d(x,y),$$

(b) *T* is *weakly contractive* [2] if there exist constants $h,b \in (0,1), h < b,$ such that for any $x\epsilon X$, there is $y \in I_b^x$ satisfying

$$d(y,T(y)) \le hd(x,y),$$

where $I_b^x = \{y \in T(x) : bd(x,y) \le d(x,T(x))\}$.

A point $x \in X$ is called a *fixed point* of a multivalued map $T : X \to S(X)$ if $x \in T(X)$. We denote $Fix(T) = \{x \in X : x \in T(X)\}$.

A sequence $\{x_n\}$ in X is called an orbit of *T* at $x_0\epsilon X$ if $x_n\epsilon T(x_{n-1})$ for all integer $n\ge1$. A real valued function f on X is called *lower semi continuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x\epsilon X$ implies that $f(x) \le \liminf_{n\to\infty} f(x_n)$.

Using the Hausdorff metric, Nadler Jr. [1] has established a multivalued version of the well-known Banach contraction principle in the setting of metric spaces as follows.

A. Theorem

Let (X, d) be a complete metric space, then each contraction map T:X→CB(X) has a fixed point.

Without using the Hausdorff metric, Feng and Liu [2] generalized Nadler's contraction principle as follows.

B. Theorem

Let (X, d) be a complete metric space and let $T : X \to cl(X)$ be a weakly contractive map, then T has a fixed point in X provided the real valued function f(x) = d(x, T(x)) on X is a lower semicontinuous.

In [3], Kada et al. introduced the concept of ω-distance in the setting of metric spaces as follows.

A function $\omega : X \times X \to [0,\infty)$ is called a ω-distance on X if it satisfies the following:

(w1) $\omega(x,z) \le \omega(x,y) + \omega(y,z)$, for all x, y, zϵX

(w2) ω is lower semicontinuous in its second variable;

(w3) for any ϵ>0, there exists δ >0, such that ω(z, x)≤ δ and ω(z, y)≤ δ imply d(x, y) ≤ ϵ.

Note that in general for $x, y \in X, \omega(x,y) \ne \omega(y,x)$, and not either of the implications $\omega(x,y) = 0 \Leftrightarrow x = y$ necessarily

holds. Clearly, the metric d is a ω-distance on X . Many other examples and properties of ω-distances are given in [3].
In [4], Suzuki and Takahashi improved Nadler contraction principle (Theorem 1.1) as follows.

C. Theorem

Let (X, d) be a complete metric space and let $T : X \to cl(X)$. If there exist a ω-distance ω on X and a constant λϵ(0, 1), such that for each x, yϵX and uϵT(x), there is vϵT(y) satisfying

$$\omega(u,v) \le \lambda\omega(x, y),$$

then T has a fixed point.
Recently, Latif and Albar [5] generalized Theorem 1.2 with respect to ω-distance (see, Theorem 3.3 in [5]), and Latif [6] proved a fixed point result with respect to ω-distance (see, Theorem 2.2 in [6]) which contains Theorem 1.3 as a special case.
A nonempty set X together with a quasimetric d (i.e., not necessarily symmetric) is called a quasimetric space. In the setting of a quasimetric spaces, Al-Homidan et al. [7] introduced the concept of a q-function on quasimetric spaces which generalizes the notion of a ω-distance.
A function $q : X \times X \to [0,\infty)$ is called a q-*function* on X if it satisfies the following conditions:
(Q1) $q(x, z) \le q(x, y) + q(y, z),$ for all x, y, zϵX,
(Q2) If $\{y_n\}$ is a sequence in X such that $y_n \to y\epsilon X$ and for xϵX, $q(x, y_n) \le M$ for some M=M(x)>0, then q(x, y)≤M,
(Q3) for any ϵ>0, there exists δ>0, such that q(x, y)≤δ and q(x, z)≤δ imply d(y, z)≤ε.
Note that every ω-distance is a q-function, but the converse is not true in general [7]. Now, we state some useful properties of q-function as given in [7].

D. Lemma

Let (X, d) be a complete quasimetric space and let q be a Q-function on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X. Let $\{\alpha_n\}$ *and* $\{\beta_n\}$ be sequences in {0, ∞) converging to 0, then the following hold for any x, y, zϵX.
if $q(x_n, y) \le \alpha_n$ *and* $q(x_n, z) \le \beta_n$ for all n≥1, then y=z in particular, if $q(x, y) = 0$ *and* $q(x, z) = 0$, then y=z,if $q(x_n, y_n) \le \alpha_n$ *and* $q(x_n, z) \le \beta_n$ for all n≥1, then $\{y_n\}$ converges to z;
if $q(x_n, x_m) \le \alpha_n$ for any n,m≥1 with m>n, then $\{x_n\}$ is a Cauchy sequence;
if $q(y, x_n) \le \alpha_n$ for any n≥1, then $\{x_n\}$ is a Cauchy sequence.
Using the concept Q-function, Al-Homidan et al. [7] recently studied an equilibrium version of the Ekeland-type variational principle. They also generalized Nadler's fixed point theorem (Theorem 1.1) in the setting of quasimetric spaces as follows.

E. Theorem

Let (X, d) be a complete quasimetric space and let $T : X \to cl(X)$. If there exist Q-function q on X and a constant λϵ(0, 1),such that for each x, yϵX and uϵT(x), there is vϵT(y) satisfying
q(u, v)≤λq(x, y),
then T has a fixed point.
In the sequel, we consider X as a quasimetric space with quasimetric d.
Considering a multivalued map $T : X \to S(X)$, we say
T is *weakly* q-*contractive* if there exist Q- function q *on* X and constants h, bϵ(0, 1), h<b, such that for any xϵX, there is $y \in J_b^x$ satisfying

$$q(y,T(y)) \le hq(x, y),$$

where $J_b^x = \{y \in T(x) : bq(x, y) \le q(x,T(x))\}$ *and* $q(x,T(x)) = \inf\{q(x, y) : y \in T(x)\}$,
T is *generalized* q-*contractive* if there exists a Q-function q on X, such that for each x, yϵX and uϵT(x), there is yϵT(y) satisfying

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$q(u,v) \le kq(x,y),$

where k is a function of [0, ∞) to [0, 1), such that $\lim Sup_{r\to t^+} k(r) < 1$ for all t≥0.

Clearly, the class of *weakly* q- contractive maps contains the class of weakly contractive maps, and the class of generalized q-contractive maps contains the classes of generalized ω-contraction maps [6], ω-contractive maps [4], and q-contractive maps [7].
In this paper, we prove some new fixed point results in the setting of quasimetric spaces for weakly q-*contractive and* generalized q-contractive multivalued maps. Consequently, our results either improve or generalize many known results including the above stated fixed point results.

II. THE MAIN RESULTS

First, we prove a fixed point theorem for weakly q-contractive maps in the setting of quasimetric spaces.

A. Theorem

Let X be a complete quasimetric space and let $T: X \to cl(X)$ be a weakly q-contractive map. If a real valued function $f(x) = q(x, T(x))$ on X is lower semicontinuous, then there exists $y_0 \epsilon X$, such that $q(y_0, T(y_0))=0$ Further, if $q(x_0, y_0)=0$, then y_0 is a fixed point of T.

B. Proof

Let $x_0 \epsilon X$, Since T is weakly contractive, there is $x_1 \in J_b^{x_0} \subseteq T(x_0)$, such that

$q(x_1, T(x_1)) \le hq(x_0, x_1),$

where h<b Continuing this process, we can get an orbit $\{x_n\}$ of T at x_0 satisfying $x_{n+1} \in J_b^{x_n}$ and

$q(x_{n+1}, T(x_{n+1})) \le hq(x_n, x_{n+1}),\ n = 0,1,2,...$

Since $bq(x_n, x_{n+1}) \le q(x_n, T(x_n)),$ and h<b<1, thus we get

$q(x_{n+1}, T(x_{n+1})) \le q(x_n, T(x_n)),$

If we put $a = \frac{h}{b}$, then also we have

$q(x_{n+1}, T(x_{n+1})) \le aq(x_n, T(x_n)),$

Thus, we obtain

$q(x_n, T(x_n)) \le a^n q(x_0, T(x_0)),\ n = 0,1,2,...,$

and since 0<a<1, hence the sequence $\{f(x_n)\} = q(x_n, T(x_n)),$ which is decreasing, converges to 0. Now, we show that $\{x_n\}$ is a Cauchy sequence. Note that

$q(x_n, x_{n+1}) \le a^n q(x_0, x_1),\ n = 0,1,2,...,$

Now, for any integer n, m≥1 with m>n, we have

$$\begin{aligned} q(x_n, x_m) &\le q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + ... + q(x_{m-1}, x_m) \\ &\le a^n q(x_0, x_1) + a^{n+1} q(x_0, x_1) + ... + a^{m-1} q(x_0, x_1) \\ &\le \frac{a^n}{1-a} q(x_0, x_1), \end{aligned}$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X, there exists some $y_0 \epsilon X$, such that $\lim_{n\to\infty} x_n = y_0$. Now, since f is lower semicontinuous, we have

$0 \le f(y_0) \le \lim_{n\to\infty} \inf f(x_n) = 0,$

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and thus, $f(y_0) = q(y_0, T(y_0)) = 0$. It follows that there exists a sequence $\{y_n\}$ in $T(y_0)$, such that $q(y_0, y_n) \to 0$. Now, if $q(y_0, y_n) = 0$, then by Lemma 1.4, $y_n \to y_0$. Since $T(y_0)$ is closed, we get $y_0 \in T(y_0)$.

Now, we prove the following useful lemma.

C. Lemma

Let (X, d) be a complete quasimetric space and let $T: X \to cl(X)$ be a generalized q-contractive map, then there exists an orbit $\{x_n\}$ of T at x_0, such that the sequence of nonnegative numbers $\{q(x_n, x_{n+1})\}$ is decreasing to zero and $\{x_n\}$ is a Cauchy sequence.

D. Proof

Let x_0 be an arbitrary but fixed element of x and let $x_1 \epsilon T(x_0)$. Since T is generalized as a q-contractive, there is $x_2 \epsilon T(x_1)$, such that

$$q(x_1, x_2) \le kq(x_0, x_1),$$

Continuing this process, we get a sequence $\{x_n\}$ in X, such that $x_{n+1} \epsilon T(x_n)$ and

$$q(x_n, x_{n+1}) \le kq(x_{n-1}, x_n),$$

Thus, for all $n \ge 1$, we have

$$q(x_n, x_{n+1}) < q(x_{n-1}, x_n),$$

Write $t_n = q(x_n, x_{n+1})$. Suppose that $\lim_{n\to\infty} t_n = \lambda > 0$, then we have

$t_n \le k\, t_{n-1}$

Now, taking limits as $n \to \infty$ on both sides, we get

$$\lambda \le \lim_{n\to\infty} \sup k(t_{n-1})\lambda < \lambda,$$

which is not possible, and hence the sequence of nonnegative numbers $\{t_n\}$ which is decreasing, converges to 0. Finally, we show that $\{x_n\}$ is a Cauchy sequence. Let $a = \lim_{r\to 0^+} \sup k(r) < 1$. There exists real number β such that $\alpha < \beta < 1$. Then for sufficiently large n, $k(t_n) < \beta$, and thus for sufficiently large n, we have $t_n < \beta\, t_{n-1}$. Consequently, we obtain $t_n < \beta^n t_0$, that is,

$$q(x_n, x_{n+1}) < \beta^n q(x_0, x_1),\ n = 0,1,2,...,$$

Now, for any integers $n, m \ge 1$, $m > n$,

$$\begin{aligned} q(x_n, x_m) &\le q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + ... + q(x_{m-1}, x_m) \\ &\le \beta^n q(x_0, x_1) + \beta^{n+1} q(x_0, x_1) + ... + \beta^{m-1} q(x_0, x_1) \\ &\le \frac{\beta^n}{1-\beta} q(x_0, x_1), \end{aligned}$$

and thus by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence.

Applying Lemma 2.2, we prove a fixed point result for generalized q-contractive maps.

E. Theorem

Let (X, d) be a complete quasimetric space then each generalized q-contractive map $T: X \to cl(X)$ has a fixed point.

F. Proof

It follows from Lemma 2.2 that there exists a Cauchy sequence $\{x_n\}$ in X such that the decreasing sequence $\{q(x_n, x_{n+1})\}$ converges to 0. Due to the completeness of X, there exists some $y_0 \epsilon X$ such that $\lim_{n\to\infty} x_n = y_0$. Let n be arbitrary fixed positive integer then for all positive integers m with m > n, we have

$$q(x_n, x_m) \le \frac{\beta^n}{1-\beta} q(x_0, x_1),$$

Let $M = \frac{\beta^n}{1-\beta} q(x_0, x_1)$, then $M \ge 0$. Now, note that

$$q(x_n, x_m) \le M \Rightarrow q(x_n, y_0) \le M,$$

Since n was arbitrary fixed, we have

$$q(x_n, y_0) \le \frac{\beta^n}{1-\beta} q(x_0, x_1), \; forallpositive \operatorname{int} egern \;\; n,$$

Note that $q(x_n, y_0)$ converges to 0 . Now, since $x_n \in T(x_{n-1})$ and T is a generalized q-contractive map, then there is $u_n \in T(y_0)$, such that

$$q(x_n, u_n) \le kq(x_{n-1}, y_0),$$

And for large n, we obtain

$$q(x_n, u_n) \le kq(x_{n-1}, y_0) < \beta\, q(x_{n-1}, y_0),$$

thus, we get

$$q(x_n, u_n) < \beta q(x_{n-1}, y_0) \le \frac{\beta^n}{1-\beta} q(x_0, x_1),$$

Thus, it follows from Lemma 1.4 that $u_n \to y_0$. Since $T(y_0)$ is closed, we get $y_0 \in T(y_0)$.

Corollary 2.4.

Let (X, d) be a complete quasimetric space and q a Q-function on X. Let $T: X \to cl(X)$ be a multivalued map, such that for any x, y$\in$X and u$\in$T(x), there is v$\in$T(y) with

$$q(u, v) \le kq(x, y),$$

Where k is a monotonic increasing function from $(0, \infty)$ to $[0, 1)$, then T has a fixed point.

REFERENCES

[1] Nadler SB Jr.: Multi-valued contraction mappings. Pacific Journal of Mathematics 1969, 30: 475–488.

[2] Feng Y, Liu S: Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings.Journal of Mathematical Analysis and Applications 2006,317(1):103–112. 10.1016/j.jmaa.2005.12.004

[3] Kada O, Suzuki T, Takahashi W: Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Mathematica Japonica 1996,44(2):381–391.

[4] Suzuki T, Takahashi W: Fixed point theorems and characterizations of metric completeness. Topological Methods in Nonlinear Analysis 1997,8(2):371–382.

[5] Latif A, Albar WA: Fixed point results in complete metric spaces. Demonstratio Mathematica 2008,41(1):145–150.

[6] Latif A: A fixed point result in complete metric spaces. JP Journal of Fixed Point Theory and Applications2007,2(2):169–175.

[7] Al-Homidan S, Ansari QH, Yao J-C: Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory. Nonlinear Analysis: Theory, Methods & Applications 2008,69(1):126–139. 10.1016/j.na.2007.05.004

[8] Siddiqi AH, Ansari QH: An iterative method for generalized variational inequalities. Mathematica Japonica1989,34(3):475–481.

[9] Kaneko H: Generalized contractive multivalued mappings and their fixed points. Mathematica Japonica1988,33(1):57–64.