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Variational Formulation of Mindlin Plate Equation, And Solution for Deflections of Clamped Mindlin Plates

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Abstract:** In this work, the governing equations for the flexure of linearly elastic, isotropic, homogeneous thick circular plates under static loading are formulated using the methods of the calculus of variations. The total potential energy functional for the thick circular plate, which is the sum of the strain energy functional and the load potential functional was obtained using the equations of material constitutive law, which accounts for the shear deformation of the circular plate and the axi-symmetrical nature of loading and plate. Euler-Lagrange differential equations of equilibrium of the plate were obtained using Euler-Lagrange conditions. The total potential energy functional Π was extremized in accordance with the principle of minimization of the total potential energy functional to obtain the governing equations, and the boundary conditions. The governing partial differential equations were then integrated to obtain the deflection of the thick circular plate. The deflection function w(r) obtained was found to be a sum of expressions for deflection due to pure bending and deflection due to shear deformation only. It was observed that when the shear stress coefficient $k = 4/3$, which corresponds to the parabolic distribution of the shear stress over the plate thickness, the solution for deflection becomes identical with the deflection obtained using the rigourous methods of theory of elasticity, and presented by Love. For the case of thick circular plates, with clamped edges, subject to uniformly distributed load, the maximum deflection was found to occur at the plate center. The effect of the plate thickness h on the maximum deflection was investigated by computing the maximum deflection for varying ratios of h/r_0 ranging from 0.001 to 1.00 for a Poisson's ratio, μ of 0.30. It was observed that for h/r_0 less than 0.05, corresponding to thin circular plates, shear deformation has insignificant contribution to the deflection, while when h/r_0 is greater than 0.05, shear deformation has a significant effect on the maximum deflection, in agreement with literature.
***Keywords:** Mindlin plate theory, total potential energy functional, thick circular plate, axi-symmetrical loading, shear stress coefficient, Euler-Lagrange differential equation of equilibrium.*

I. INTRODUCTION / LITERATURE REVIEW

Plates are three dimensional structural elements that have one dimension called the thickness much smaller than the other two dimensions. They have extensive applications in civil, mechanical, aeronautical, spacecraft, and naval engineering due to their effectiveness and efficiency in carrying loads. They are classified according to their ratio of the largest dimension *L*, to the thickness *t* as: thin plates $L/t > 100$, moderately thin/thick plates $20 > L/t < 100$ and thick plates if $L/t < 3$. (Steele and Balach (2013), Timoshenko and Woinowsky-Krieger (1959), Ugural (1999). They are also classified according to their material properties as laminated plates, anisotropic plates, homogeneous plates, heterogeneous plates, and isotropic plates (Ventsel and Krauthammer (2004), Mansfield (1964), Szilard (1974)). They can also be classified according to their shapes as rectangular, circular, elliptical, polygonal, skewed plates. The focus of this work is circular thick plates. Circular plates find extensive applications in aerospace, civil, geotechnical, naval and mechanical engineering. (Civalek and Ozturk (2009)). They are used in offshore platforms. The determination of displacement and deflection functions in elastic circular plates subjected to symmetrical distribution of loading is a

problem that is frequently encountered in the analysis and design of structural elements and systems (Kirstein and Wooley (1967)).
The fundamental governing equations which describe the structural behaviour of plates usually take advantage of the plate's planar character. This approach is a generalization of the one-dimensional Euler-Bernoulli beam flexure theory which makes use of the slender shape of a beam, and models it to behave in a one-dimensional manner. Two dimensional plate theories have been developed in the technical literature to use the inplane coordinates as independent spatial variables. Plate problems are actually three dimensional problems, in general and the three spatial coordinates are the independent variables in a three dimensional model of plate behaviour. Three dimensional plate theories are based on the three dimensional equations of elasticity, and the solution for such three dimensional problems are generally difficult, and time consuming and in most cases, closed form mathematical solutions do not exist except in very simple cases of loading, support and plate material/geometry (Szilard, (1974)).
Three dimensional theory of elasticity is the basis and the foundation for all approximate theories of plates (Steele and Balch (2016)). The equations for three dimensional elasticity theory are a set of fifteen equations of material constitutive laws, strain-displacement and differential equations of equilibrium, and these are solved subject to the boundary conditions of particular problems (Timoshenko and Goodier (1970), Sokolnikoff (1956)).
The following theories have been used in the literature to describe the structural behaviour of circular plates: Kirchhoff (1876) or classical plate theory, Mindlin (2007) plate theory, Reissner (1945, 1954) plate theory, Von-Karman (1970) plate theory, Refined plate theories (Chandrashekhara (2001)), Levinson's plate theory (Lekhnitskii (1968)), Reddy's plate theory (Chandrashekhara (2001)) etc. The Kirchhoff-Love plate theory is based on the following assumptions: (Gujan and Ladhand (2015))
straight line initially orthogonal to the middle surface of the plate remains straight and normal to the deformed middle surface and unchanged in length.
displacement is very small, hence the slope of deflected surface is small and the square of the slope is negligible compared to unity.
the normal stresses and in plane shear stresses τ are assumed to be zero at the middle surface of the plate ($w << h$).
The middle surface remains unstrained after bending. Transverse normal stress is small as compared to other stress components and may be neglected in the stress-strain laws. These assumptions of the classical plate theory reduce the three dimensional plate problem to two dimensions. Kirchoff-Love's classical plate theory is satisfactory for thin circular plates, but unsatisfactory as the plate thickness increases. The major limitation of the Kirchoff classical plate theory is the disregard for shear deformation which renders the theory incapable of adequately describing the behaviour of moderately thick and thick plates. Von Karman's theory for the axi-symmetric bending of circular plates which is either simply supported or clamped at the boundary are represented by a system of coupled non linear partial differential equations, expressed in terms of two unknown functions: deflections $w(r)$ and the Airy's stress function $\phi(r)$ (Dickey (1976)). Those equations describe large deflection behaviour of plates and are difficult to solve in closed form, except for simple cases of loading, plate supports and plate material properties. Circular plates have been studied by other researchers. (Stanislaw et al (2004), Rao and Rao (2009), Al-Nasra and Daoud (2015)).
The solution of the governing partial differential equations for circular plates have been obtained in closed analytical form using methods for solving partial differential equations, and in approximate numerical form using numerical methods of mathematical analysis like the Finite element method, Finite Difference method, Ritz variational method, Galerkin variational method, Boundary element method, Collocation methods, and weighted residual methods. Numerical methods are useful when the circular plate problem cannot be solved by mathematical methods. In this paper, the problem of static flexure of elastic thick circular plates is formulated using the method of the calculus of variations, and solved by direct integration of the governing partial differential equations obtained. The advantage of the variational formulation of the circular thick plate problem is that the formulation yields both the governing partial differential equations of equilibrium of the plate as well as the boundary conditions, unlike the formulation using the equilibrium approach.

II. RESEARCH AIMS AND OBJECTIVES

The general aim and objective of this research is to apply the method of the calculus of variations to the Mindlin plate problem. The specific aims and objectives are:
to present a variational formulation of the Mindlin plate problem for isotropic, homogeneous linear elastic plate materials
to obtain the Euler-Lagrange differential equations of equilibrium for the Mindlin plate problem for elastic static conditions
to obtain the governing partial differential equation of equilibrium for the Mindlin plate problem in terms of displacements, applied loads and plate material properties

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

to obtain a solution of the Mindlin plate equation so derived for clamped circular Mindlin plate under uniformly distributed load of intensity, p_0.

III. DERIVATION OF MINDLIN PLATE THEORY FROM VARIATIONAL CALCULUS

The displacement field is assumed as

$$u_r(r,z) = -z\varphi(r) \quad (1)$$

$$u_\theta(r,z) = 0 \quad (2)$$

$$u_z(r,z) = w(r, z=0) = w(r) \quad (3)$$

where u_r, u_θ, and u_z are the displacement components in the radial, circumferential and depth coordinates, respectively, r, θ and z denote the radial, circumferential and depth coordinate variables, and $w(r)$ is the transverse displacement of the plate. $\varphi(r)$ is a certain function of r, we will determine from considerations of the theory of axisymmetric elasticity.

The axisymmetric elasticity problem satisfies the kinematic equations, which for small – (finite) strain assumption yields the strain-displacement equations:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = -z\frac{\partial \varphi(r)}{\partial r} = -z\varphi'(r) \quad (4)$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = -\frac{z}{r}\varphi(r) \quad (5)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0 \quad (6)$$

$$\gamma_{rz} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0 \quad (7)$$

$$\gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} = \frac{\partial}{\partial r}w(r) + \frac{\partial}{\partial z}(-z\varphi(r)) \quad (8)$$

$$\gamma_{rz} = w'(r) - \varphi(r) \quad (9)$$

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} = 0 \quad (10)$$

The material constitutive law (Hooke's law) for axisymmetric elasticity problems can be expressed as:

$$\sigma_{rr} = \frac{E}{1-\mu^2}\left(\varepsilon_{rr} + \mu\varepsilon_{\theta\theta}\right) \quad (11)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\mu^2}\left(\varepsilon_{\theta\theta} + \mu\varepsilon_{rr}\right) \quad (12)$$

$$\tau_{rz} = G\gamma_{rz} \quad (13)$$

where E = Young's modulus of elasticity of the plate material

$$G = \text{shear modulus} = \frac{E}{2(1+\mu)} \quad (14)$$

$\mu =$ Poisson's ratio of the plate material

The total elastic strain energy functional U of the plate is given in general by the triple integral over the three dimensional plate domain of the strain energy density, thus:

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{r_1}\int_{-h/2}^{h/2}\left(\varepsilon_{rr}\sigma_{rr} + \sigma_{zz}\varepsilon_{zz} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \gamma_{r\theta}\sigma_{r\theta} + \gamma_{rz}\sigma_{rz} + \gamma_{\theta z}\sigma_{\theta z}\right)d\theta r dr\, dz \qquad (15)$$

where $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ are the normal stresses in the $r, \theta,$ and z coordinate directions and $\sigma_{r\theta}, \sigma_{rz},$ and $\sigma_{\theta z}$ are the shear stresses, and h is the thickness of the circular plate, and

$$0 \le \theta \le 2\pi; \quad 0 \le r \le r_1; \quad -\frac{h}{2} \le z \le \frac{h}{2} \qquad (16)$$

is the domain of the circular plate.
For the axisymmetric plate problems,
$\varepsilon_{zz} = 0, \ \gamma_{rz} = 0,$ and $\gamma_{z\theta} = 0,$ thus the strain energy functional U will simply become:

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{r_1}\int_{-h/2}^{h/2}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rz}\gamma_{rz}\right)d\theta r dr\, dz \qquad (17)$$

Integration with respect to θ yields the strain energy functional as the double integral over the two dimensional plate surface: $0 \le r \le r, \ -h/2 \le z \le h/2;$

$$U = \pi\int_0^{r}\int_{-h/2}^{h/2}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rz}\gamma_{rz}\right) r dr\, dz \qquad (18)$$

Substituting the material constitutive law into the strain energy functional we obtain the strain energy functional expressed in terms of strains as:

$$U = \pi\int_0^{r}\int_{-h/2}^{h/2}\left\{\frac{E}{1-\mu^2}\left[\left(\varepsilon_{rr}^2 + \mu\varepsilon_{\theta\theta}\varepsilon_{rr}\right) + \left(\varepsilon_{\theta\theta}^2 + \mu\varepsilon_{rr}\varepsilon_{\theta\theta}\right)\right] + G\gamma_{rz}^2\right\} r dr\, dz \qquad (19)$$

From variational calculus, the first variation of the strain energy functional δU is given by:

$$U = \pi\int\int\left\{\frac{E}{1-\mu^2}\left[\left(\varepsilon_{rr}^2 + \mu\varepsilon_{\theta\theta}\varepsilon_{rr}\right) + \left(\varepsilon_{\theta\theta}^2 + \mu\varepsilon_{rr}\varepsilon_{\theta\theta}\right)\right] + G\gamma_{rz}^2\right\} r dr\, dz \qquad (20)$$

Simplifying,

$$\delta U = \pi\int_0^{r}\int_{-h/2}^{h/2}\left\{\frac{2E}{1-\mu^2}\left[\left(\varepsilon_{rr} + \mu\varepsilon_{\theta\theta}\right)\delta\varepsilon_{rr} + \left(\varepsilon_{\theta\theta} + \mu\varepsilon_{rr}\right)\delta\varepsilon_{\theta\theta}\right] + 2G\gamma_{rz}\delta\gamma_{rz}\right\} r dr\, dz \qquad (21)$$

Further simplification yields:

$$\delta U = 2\pi\int_0^{r}\int_{-h/2}^{h/2}\left\{\sigma_{rr}\delta\varepsilon_{rr} + \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} + \sigma_{rz}\delta\gamma_{rz}\right\} r dr\, dz \qquad (22)$$

Using the strain-displacement relations,

$$\delta U = 2\pi\int_0^{r}\int_{-h/2}^{h/2}\left((\sigma_{rr}\delta(-z\varphi'(r)) + \sigma_{\theta\theta}\,\delta\left(-\frac{z}{r}\varphi(r)\right) + \sigma_{rz}\delta(w'(r) - \varphi(r)\right) r dr\, dz \qquad (23)$$

$$\delta U = 2\pi\int_0^{r}\int_{-h/2}^{h/2}\left[-z\sigma_{rr}\delta\varphi'(r) - \frac{z}{r}\sigma_{\theta\theta}\delta\varphi(r) + \sigma_{rz}(\delta w'(r) - \delta\varphi(r)\right] r dr\, dz \qquad (24)$$

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$$\delta U = 2\pi\int_0^r \left\{ -\int_{-h/2}^{h/2} z\sigma_{rr} dz\, \delta\varphi'(r) - \frac{1}{r}\int_{-h/2}^{h/2} z\sigma_{\theta\theta} dz \delta\varphi(r) + \int_{-h/2}^{h/2} \gamma_{rz} dz(\delta w'(r) - \delta\varphi(r)) \right\} rdr \quad (25)$$

The internal force resultants are defined as the following integrals over the plate thickness:

$$M_{rr} = \int_{-h/2}^{h/2} z\sigma_{rr}\, dz \quad (26)$$

$$M_{\theta\theta} = \int_{-h/2}^{h/2} z\sigma_{\theta\theta}\, dz \quad (27)$$

$$Q = \int_{-h/2}^{h/2} \tau_{rz}\, dz \quad (28)$$

where M_{rr}, $M_{\theta\theta}$ are bending moments distributions in the radial and circumferential directions and Q is the shear force.

Then, the first variation of the strain energy functional δU becomes:

$$\delta U = 2\pi\int_0^r \left[-M_{rr}\delta\varphi'(r) - \frac{M_{\theta\theta}\delta\varphi(r)}{r} + Q\delta(w'(r) - \varphi) \right] rdr \quad (29)$$

The potential due to external load P distributed over the plate area is

$$V = -\int_0^{2\pi}\int_0^r p\, w\, r\, dr\, d\theta \quad (30)$$

Integration with respect to θ yields:

$$V = -2\pi\int_0^r p\, w\, r\, dr \quad (31)$$

The first variation of the load potential functional becomes

$$\delta V = -2\pi\int_0^r p\, \delta w\, r\, dr \quad (32)$$

The first variation of the total potential energy function $\delta\pi$ then can be expressed, for equilibrium, following the principle of minimization of total potential energy functional as:

$$\delta\Pi = \delta(U + V) = 0 = \delta U + \delta V \quad (33)$$

By substitution of expressions for δU and δV, we have

$$\delta\Pi = 2\pi\int_0^r \left[-M_{rr}\delta\varphi'(r) - \frac{M_{\theta\theta}\partial\varphi}{r} + Q\delta(w'(r) - \varphi(r) - p\delta w \right] rdr = 0 \quad (34)$$

Alternatively, we can write $\delta\Pi$ as:

$$\int_0^r \left[-rM_{rr}\frac{d}{dr}(\delta\varphi(r)) - M_{\theta\theta}\delta\varphi(r) + rQ\frac{d}{dr}\delta(w) - rQ\delta\varphi(r) - rp\delta w \right] dr = 0 \quad (35)$$

We use integration by parts on the first and third terms of the integrand and obtain, upon simplification,

$$\int_0^r \left[(rM_{rr})' - M_{\theta\theta} - rQ\right]\delta\varphi dr + \int_0^r \left[-(rQ)' - rp\right]\delta w dr$$

$$+\left[(-rM_{rr})\delta\varphi\right]_0^r + \left[rQ\delta w\right]_0^r = 0 \qquad (36)$$

where primes are used to denote differentiation with respect to r.

The Euler-Lagrange differential equations equilibrium are then obtained as:

$$(rM_{rr})' - M_{\theta\theta} - rQ = 0 \qquad (37)$$

or $$rM'_{rr} + M_{rr} - M_{\theta\theta} - rQ = 0 \qquad (38)$$

or $$\frac{dM_{rr}}{dr} + \frac{M_{rr} - M_{\theta\theta}}{r} = Q \qquad (39)$$

and $$-(rQ)' - rp = 0 \qquad (40)$$

or $$\frac{d}{dr}(rQ) = -rp \qquad (41)$$

or $$\frac{1}{r}\frac{d}{dr}(rQ) = -p \qquad (42)$$

The boundary conditions are at $r = 0$, $r = r_1$,

$$rM_{rr} = 0 \qquad (43)$$

or φ is known

$$rQ = 0 \qquad (44)$$

or w is known/given.

The differential equations of the plate can be expressed in terms of the displacements rather than the internal force resultants. From the equation for internal force resultants,

$$M_{rr} = \int_{-h/2}^{h/2} z\sigma_{rr}\, dz = \int_{-h/2}^{h/2} \frac{Ez}{1-\mu^2}(\varepsilon_{rr} + \mu\varepsilon_{\theta\theta})dz \qquad (45)$$

$$M_{rr} = \int_{-h/2}^{h/2} \frac{Ez}{1-\mu^2}\left(-z\varphi'(r) - \mu\frac{z\varphi(r)}{r}\right)dz \qquad (46)$$

$$M_{rr} = -\int_{-h/2}^{h/2} \frac{Ez^2}{1-\mu^2}\left(\varphi'(r) - \frac{\mu}{r}\varphi(r)\right)dz \qquad (47)$$

$$M_{rr} = -\int_{-h/2}^{h/2} \frac{Ez^2}{1-\mu^2}dz\left(\varphi'(r) - \frac{\mu}{r}\varphi(r)\right) \qquad (48)$$

$$M_{rr} = -\left[\frac{Ez^3}{3(1-\mu^2)}\right]_{-h/2}^{h/2}\left(\varphi'(r) + \frac{\mu}{r}\varphi(r)\right) \qquad (49)$$

$$M_{rr} = \frac{-Eh^3}{12(1-\mu^2)}\left(\varphi'(r) + \frac{\mu}{r}\varphi(r)\right) \qquad (50)$$

$$M_{rr} = -D\left(\varphi'(r) + \frac{\mu}{r}\varphi(r)\right) = -D\left(\frac{d\varphi(r)}{dr} + \frac{\mu}{r}\varphi(r)\right) \quad (51)$$

Similarly,

$$M_{\theta\theta} = \int_{-h/2}^{h/2} \frac{Ez}{1-\mu^2}\left(-\frac{z}{r}\varphi(r) - \mu z\varphi'(r)\right)dz \quad (52)$$

$$M_{\theta\theta} = -\int_{-h/2}^{h/2} \frac{Ez^2}{1-\mu^2}dz\left(\frac{\varphi(r)}{r} + \mu\varphi'(r)\right) \quad (53)$$

$$M_{\theta\theta} = -D\left(\frac{\varphi(r)}{r} + \mu\varphi'(r)\right) = -D\left(\frac{\varphi(r)}{r} + \mu\frac{d\varphi(r)}{dr}\right) \quad (54)$$

$$Q = k\int_{-h/2}^{h/2} \tau_{rz}dz \quad (55)$$

where k = shear correction factor

$$Q = kh\tau_{rz}(z=0) \quad (56)$$

$$Q = khG\gamma_{rz}(z=0) \quad (57)$$

$$Q = khG(w'(r) - \varphi(r)) = khG\left(\frac{dw(r)}{dr} - \varphi(r)\right) \quad (58)$$

Applying the differential equation of equilibrium, we obtain

$$-\frac{Dd}{dr}\left(\varphi'(r) + \mu\frac{\varphi(r)}{r}\right) + \frac{D}{r}\left(-\varphi'(r) - \frac{\mu\varphi(r)}{r} + \mu\varphi'(r) + \frac{\varphi(r)}{r}\right) = Ghk(w'(r) - \varphi(r)) \quad (59)$$

$$\frac{1}{r}\frac{d}{dr}[Ghkr(w'(r) - \varphi(r)) = -p(r) \quad (60)$$

Simplifying,

$$D\left(\varphi''(r) + \frac{\varphi'(r)}{r} - \frac{\varphi(r)}{r^2}\right) = Gkh\left(w'(r) - \varphi(r)\right) \quad (61)$$

The second differential equation of equilibrium, yields upon integration

$$Ghkr\left(w'(r) - \varphi(r)\right) = -\int_0^r p(\bar{r})\bar{r}\,d\bar{r} + c_1 \quad (62)$$

where $\bar{r}$ is an integration variable, c_1 is a constant of integration. If there is no point (concentrated) load at the centre of the plate, c_1 = 0, and

$$w'(r) - \varphi(r) = -\frac{1}{Ghkr}\int_0^r p(\bar{r})\bar{r}\,d\bar{r} \quad (63)$$

$$\varphi(r) = w'(r) + \frac{1}{Ghkr}\int_0^r p(\bar{r})\bar{r}\,d\bar{r} \quad (64)$$

Simplifying further,

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$$\varphi(r) = w'(r) - \frac{Q}{Ghk} \quad (65)$$

Then, upon substitution of $\varphi(r)$ into Equation 65 we obtain

$$-D\left[\left(w'(r) - \frac{Q}{Ghk}\right)'' + \frac{1}{r}\left(w'(r) - \frac{Q}{Ghk}\right)' - \frac{1}{r^2}\left(w'(r) - \frac{Q}{Ghk}\right)\right] = Gkh\left(w'(r) - \varphi(r)\right) = Q \quad (66)$$

$$-D\left(w'''(r) - \frac{Q''}{Ghk} + \frac{w''(r)}{r} - \frac{Q'}{Ghkr} - \frac{w'(r)}{r^2} + \frac{Q}{Ghkr^2}\right) = Q \quad (67)$$

Simplifying further,

$$-D\left(w'''(r) + \frac{w''(r)}{r} - \frac{w'(r)}{r^2}\right) = Q - \frac{D}{Ghk}\left(Q'' + \frac{Q'}{r} - \frac{Q}{r^2}\right) \quad (68)$$

Multiplying both sides by r, we have

$$-D\left(rw'''(r) + w''(r) - \frac{w'(r)}{r}\right) = Qr - \frac{D}{Ghk}\left(rQ'' + Q' - \frac{Q}{r}\right) \quad (69)$$

We perform the differential operation $\left(\frac{1}{r}\right)\frac{d}{dr}$ on both sides of the equation to obtain:

$$D\frac{1}{r}\frac{d}{dr}\left(rw'''(r) + w''(r) - \frac{w'(r)}{r}\right) = -\frac{1}{r}\frac{d}{dr}(rQ) + \frac{D}{Ghk}\frac{1}{r}\frac{d}{dr}\left(rQ'' + Q' - \frac{Q}{r}\right) \quad (70)$$

$$D\left(\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right)w(r) = -\frac{1}{r}\frac{d}{dr}(rQ) + \frac{D}{Ghk}\frac{1}{r}\frac{d}{dr}\left(rQ'' + Q' - \frac{Q}{r}\right) \quad (71)$$

$$D\nabla^2\nabla^2 w(r) = D\nabla^4 w(r) = -\frac{1}{r}\frac{d}{dr}(rQ) + \frac{D}{Ghk}\frac{1}{r}\frac{d}{dr}\left(rQ'' + Q' - \frac{Q}{r}\right) \quad (72)$$

where ∇^2 is the Laplacian operator in polar coordinates, and ∇^4 is the biharmonic operator in polar coordinates system.

We use the differential equation of equilibrium $-\left(1/r\right) d(rQ)/dr = p(r)$, and obtain

$$\nabla^2 p = \frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) = \frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(-\frac{1}{r}\frac{d}{dr}(rQ)\right)\right] = -\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(Q' + \frac{Q}{r}\right)\right) \quad (73)$$

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

$$\nabla^2 p = -\frac{1}{r}\frac{d}{dr}\left(rQ'' + Q' - \frac{Q}{r}\right) \qquad (74)$$

The equation can be expressed as follows:

$$D\nabla^2\nabla^2 w(r) = p(r) - \frac{D}{Ghk}\nabla^2 p(r) \qquad (75)$$

or, $$D\nabla^2\nabla^2 w(r) + \frac{D}{Ghk}\nabla^2 p(r) - p(r) = 0 \qquad (76)$$

The differential equation of equilibrium is now expressed in terms of $w(r)$, the deflection and the load $p(r)$.

Solution of Mindlin plate equation for clamped circular plate under uniformly distributed load p_0

We consider a clamped circular plate of radius r_0 subjected to uniformly distributed loading of intensity p_0, over the entire plate surface. The boundary conditions are

$$w(r = r_0) = 0 \qquad (77)$$

$$\varphi(r = r_0) = 0 \qquad (78)$$

The governing partial differential equation, being a linear equation, can be solved by the superposition (linearity) principle. The solution for the deflection $w(r)$ is considered the linear superposition of the deflection $w_1(r)$ solutions for the flexural deflection when shear deformation effects are disregarded and the deflection $w_2(r)$ due to transverse shear deformations alone. The flexural deflection $w(r)$ of a circular plate clamped at the edges when shear deformation is disregarded is the solution to the partial differential equation

$$D\nabla^2\nabla^2 w(r) = p(r) \qquad (79)$$

For uniformly distributed load of intensity p_0, $p(r) = p_0$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) w(r) = \frac{p(r)}{D} \qquad (80)$$

$$\frac{1}{r}\frac{d}{dr}\left\{ r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = \frac{p(r)}{D} \qquad (81)$$

By successive integration four times with respect to r, we obtain the solution for $w(r)$ as:

$$w(r) = \int \frac{1}{r}\int r \int \frac{1}{r}\int \frac{rp(r)}{D} dr\, dr\, dr\, dr \qquad (82)$$

For uniformly distributed load of intensity $p_0(r) = p_0$, we obtain, after integration:

$$w(r) = a_1 \ln r + a_2 r^2 + a_3 r^2 \ln r + a_4 + \frac{p_0 r^4}{64D} \qquad (83)$$

where a_1, a_2, a_3 and a_4 are the four constants of integration.

For clamped edges at $r = r_0$, the boundary conditions are

$$w(r = r_0) = 0 \qquad (84)$$

$$\frac{dw}{dr}(r = r_0) = \theta(r = r_0) = 0 \qquad (85)$$

$$\frac{dw}{dr}(r = 0) = \theta(r = 0) = 0 \qquad (86)$$

$$Q_r(r = 0) = 0 \qquad (87)$$

where Q_r is the shear force for unit length acting on a plate element at any arbitrary radial distance, r from the centre.

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

$$Q_r = \frac{1}{2\pi r}\int_0^{2\pi}\int_0^r p(r)dA = \frac{1}{2\pi r}\int_0^r p_0 2\pi r dr \quad (88)$$

$$Q_r = \frac{p_0 r}{2} \quad (89)$$

For bounded deflections of the plate, $w(r=0) \neq \infty$, hence $a_1 = a_3 = 0$ Thus, the deflections become

$$w(r) = a_2 r^2 + a_4 + \frac{p_0 r^4}{64D} \quad (90)$$

Slope $$\theta(x) = \frac{dw}{dr} = 2a_2 r + \frac{p_0 r^3}{16D} \quad (91)$$

The boundary conditions at the centre ($r = 0$) are now satisfied identically since $\theta(r=0) = \frac{dw}{dr}(r=0) = 0$ and $Q_r(r=0) = 0$. The integration constants a_2 and a_4 are determined using the boundary conditions at the plate edges $r = r_0$.

$$\theta(r = r_0) = \frac{dw}{dr}(r = r_0) = 2a_2 r_0 + \frac{p_0 r_0^3}{16D} = 0 \quad (92)$$

Solving for a_2,

$$\therefore \quad a_2 = -\frac{p_0 r^2}{32D} \quad (93)$$

Similarly,

$$w(r = r_0) = a_2 r_0^2 + a_4 + \frac{p_0 r_0^4}{64D} = 0 \quad (94)$$

$$a_4 = \frac{p_0 r_0^4}{64D} \quad (95)$$

Thus

$$w(r) = \frac{pP_0 r_0^4}{64D} - \frac{p_0 r_0^2 r^2}{32D} + \frac{p_0 r^4}{64D} \quad (96)$$

$$w(r) = \frac{p_0 r_0^4}{64D}\left(1 - \left(\frac{r^2}{r_0^2}\right)\right)^2 \quad (97)$$

For a part of the circular plate, at an arbitrary radial distance r from the centre equilibrium yields
But the shear force resultant is

$$Q = \frac{1}{2\pi r}\int\int p(r)dA = \frac{1}{2\pi r}\int_0^{2\pi}\int_0^r p(r) r dr\, d\theta = \frac{p_0 r}{2} \quad (98)$$

$$Q = \int_{-h/2}^{h/2} \tau_{rz} dz = k\tau_{rz}(z=0)h \quad (99)$$

where $\tau_{rz}(z=0)$ is the shear stress at the middle surface

$$\therefore \quad \tau_{rz}(z=0) = \frac{Q}{kh} = \frac{p_0 r}{2kh} \quad (100)$$

The shear strain γ_{rz} is

$$\gamma_{rz} = -\alpha = -\frac{dw_2(r)}{dr} \quad (101)$$

where $w_2(r)$ is the displacement due to shear deformation alone.

$$\tau_{rz}(z=0) = -G\gamma_{rz} = -G\frac{dw_2(r)}{dr} \quad (102)$$

Then, by substitution,

$$\frac{dw_2(r)}{dr} = -\frac{p_0 r}{2khG} \quad (103)$$

$$dw_2(r) = -\frac{p_0 r}{2Ghk}dr \quad (104)$$

Integration yields,

$$w_2(r) = \int \frac{-p_0 r}{2Ghk}dr \quad (105)$$

$$w_2(r) = \frac{p_0}{2Ghk}\left(-\frac{r^2}{2} + c_1\right) \quad (106)$$

$$w_2(r) = \frac{p_0}{4Ghk}\left(-r^2 + \overline{c}_1\right) \quad (107)$$

where $\overline{c}_1 = 2c_1$

$$w_2(r) = \frac{P_0 r_0^2}{4Ghk}\left(c_2 - \frac{r^2}{r_0^2}\right) \quad (108)$$

where $c_2 = \frac{\overline{c}_1}{r_0^2}$

It can be observed that $w_2(r)$ satisfies the homogeneous part of the governing equation given as

$$\nabla^2\nabla^2 w_2(r) = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w(r) = 0 \quad (109)$$

The general solution then becomes:

$$w(r) = w_1(r) + w_2(r) = \frac{p_0 r_0^4}{64D}\left(1 - \left(\frac{r}{r_0}\right)^2\right)^2 + \frac{p_0 r_0^2}{2Ghk}\left(c_1 - \frac{r^2}{r_0^2}\right) \quad (110)$$

We apply the boundary conditions to obtain the integration constant c_1. The boundary condition is

$$w_2(r = r_0) = 0 \quad (111)$$

Thus,

$$w_2(r = r_0) = \frac{p_0 r_0^2}{4Ghk}\left(c_1 - \frac{r_0^2}{r_0^2}\right) = 0 \quad (112)$$

Solving, $c_1 = 1$ (113)

Also, the boundary condition is

International Journal for Research in Applied Science & Engineering Technology (IJRASET)

$$\varphi(r = r_0) = w'(r = r_0) + \frac{1}{Ghkr_0}\int_0^{r_0} p_0 \bar{r}\, d\bar{r} = 0 \quad (114)$$

Thus,

$$w(r) = \frac{p_0 r_0}{64D}\left(1 - \frac{r^2}{r_0^2}\right)^2 + \frac{p_0 r_0^2}{4Ghk}\left(1 - \frac{r^2}{r_0^2}\right) \quad (115)$$

where, $G = \dfrac{E}{2(1+\mu)}$ $\quad D = \dfrac{Eh^3}{12(1-\mu^2)}$

$$E = \frac{12D(1-\mu^2)}{h^3} \quad (116)$$

$$G = \frac{12D(1-\mu^2)}{2h^3(1+\mu)} = \frac{6D(1-\mu)}{h^3} \quad (117)$$

$$w(r) = \frac{P_0 r_0^4}{64D}\left(1 - \frac{r^2}{r_0^2}\right)^2 + \frac{P_0 r_0^2 h^3}{24D(1-\mu)hk}\left(1 - \frac{r^2}{r_0^2}\right) \quad (118)$$

$$w(r) = \frac{P_0 r_0^4}{64D}\left(1 - \frac{r^2}{r_0^2}\right)^2 + \frac{P_0 r_0^2 h^2}{24D(1-\mu)k}\left(1 - \frac{r^2}{r_0^2}\right) \quad (119)$$

When $k = 4/3$ which corresponds to the parabolic variation of the shear stress distribution over the plate thickness, $w(r) = \dfrac{p_0 r_0^4}{64D}\left(1 - \dfrac{r^2}{r_0^2}\right)^2 + \dfrac{p_0 r_0^2 h^2}{32D(1-\mu)}\left(1 - \dfrac{r^2}{r_0^2}\right)$ (120)

This is now identical with the result obtained using rigorous methods of the theory of elasticity and presented in Love (1944). The deflection is in two parts, deflection due to pure bending behaviour and deflection due to shear deformation. The deflection due to shear deformation is seen form Equation (105) to be

$$w_s = \frac{p_0 r_0^2 h^2}{32D(1-\mu)}\left(1 - \frac{r^2}{r_0^2}\right) \quad (121)$$

The maximum deflection occurs at the centre of the plate when $r = 0$, and is obtained as

$$w_{\max} = w(r=0) = \frac{p_0 r_0^4}{64D} + \frac{p_0 r_0^2 h^2}{32D(1-\mu)} \quad (122)$$

For $\mu = 0.30$,

$$w_{\max} = w(r=0) = \frac{p_0 r_0^4}{64D} + \frac{p_0 r_0^2 h^2}{22.4D} \quad (123)$$

The expression for the additional deflection at the centre due to the shear deformation is

$$w_s = \frac{p_0 r_0^4 h^2}{22.4D} \quad (124)$$

IV. DISCUSSION AND CONCLUSIONS

In this research work, a variational formulation of the Mindlin plate theory has been presented; for the case of circular plates. The circular plate problem was presented as an axisymmetric elasto-static problem and shear deformation was considered, thus rendering the formulation suitable for thick plates; where shear deformations play a significant role in their flexural behaviour. The strain energy functional and the work potential functional were derived and added to obtain the total potential energy by functional for the plate. First variation of the total potential energy functional was performed and equated to zero, following the principle of minimum total potential energy to obtain the Euler-Lagrange differential equations of equilibrium (Equations (39), (42)) and the boundary conditions (Equations (43), (44)) of the Mindlin plate problem. The governing partial differential equation (PDE) of equilibrium was also obtained as Equation (75). The governing PDE was solved for clamped circular plate subject to uniformly distributed load p_0 over the entire plate surface. The solution obtained for the deflection is shown in Equation (104). It is observed that the deflection is the sum of the deflection of clamped thin plates and another deflection function accounting for the effect of shear deformation. The deflection due to shear deformation is found as Equation (121). Maximum deflection occurs at the centre of the plate where $r = 0$, and the expression for the maximum deflection was obtained as Equation (122) and Equation (123) for plates with Poisson's ratio of 0.30. The expression obtained for the additional deformation at the plate centre due to the shear deformations was obtained as Equation (124). In a bid to investigate the effect of the plate thickness on the maximum deformation at the centre, the values of the maximum deflection at the centre were computed for varying ratios of h/r_0 ranging from 0.001 to 1.00 for a Poisson's ratio μ of 0.30 and presented in Table 1. It was observed from Table 1 that for h/r_0 less than 0.05, corresponding to thin circular plates, shear deformation has insignificant contribution to the overall deformation, while when h/r_0 is greater than 0.05, shear deformation begins to have a significant effect on the deformation of the plate in agreement with technical literature.

Table 1: Coefficients for maximum deflection of Mindlin plate and Kirchoff-Love plate subjected to uniformly distributed load p_0 over the entire plate region for different $\left(h/r_0\right)$ ratios, and $\mu = 0.30$

$\frac{h}{r_0}$	$w_b\left(\times\frac{p_0 r_0^4}{D}\right)$ (Kirchoff-Love plate)	$w_{\text{Mindlin}}\left(\times\frac{p_0 r_0^4}{D}\right)$ (Mindlin plate)	$w_s\left(\times\frac{p_0 r_0^4}{D}\right)$
0.001	0.015625	0.015625	4.4643×10^{-8}
0.01	0.015625	0.0156295	4.4642×10^{-6}
0.05	0.015625	0.01573661	1.11607×10^{-4}
0.10	0.015625	0.01607143	4.46429×10^{-4}
0.15	0.015625	0.0166295	1.004464×10^{-3}
0.20	0.015625	0.0174107	1.785714×10^{-3}
0.50	0.015625	0.0267857	0.0111607
0.70	0.015625	0.0375	0.021875
1.00	0.015625	0.0602679	0.044643

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