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Solution of the Generalised Elastic Column Buckling Problem by the Galerkin Variational Method

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Abstract: In this paper, a system of three differential equations describing the generalized elastic column buckling problem for axial compressive load, N_x acting through the centroid of the cross-section; and such that the bimoment is zero is solved by the Galerkin variational method for columns with two pinned ends. The problem was found to reduce to an algebraic eigen-value problem for which the characteristic buckling equation was found. Two specializations of the buckling problem – doubly symmetric cross-sections and singly symmetric cross-sections were considered. For doubly symmetric cross-sections, it was found that the buckling modes are uncoupled resulting in three possibilities for the failure – Euler flexural buckling failure load in the z axis, Euler flexural buckling failure load in the y axis and the critical load in torsional buckling. The least value of the three buckling loads governs the buckling mode of failure. For columns with singly symmetric cross-sections, one of the governing equations becomes uncoupled while the other two become coupled. The characteristic buckling equation shows the column can fail by Euler flexural buckling in the yy direction and torsional – flexural buckling mode. The values of the buckling load (eigen-values) obtained using the Galerkin variational method were found to be identical with solutions for the same problem obtained in the literature using the method of undetermined parameters.

Keywords: Euler flexural buckling, singly symmetric cross-sections, eigen-value problem; characteristic buckling equation, Galerkin variational method.

I. INTRODUCTION

While buckling is a broad term, it generally refers to a situation where a structural element in compression deviates from a behaviour of elastic shortening within the original geometry and undergoes large deformations involving a change in member shape for a very small increase in load [1]. When the member has symmetry of cross-sectional geometry, section resistance and load, it may buckle in two directions; a phenomenon called bifurcation buckling. The buckling load is the load at which the deviation from the original geometry first occurs, and it is also called the critical load or bifurcation load.

If the proportional limit of linear elasticity has not been reached at any point in the member before the critical load is attained, the buckling is referred to as elastic buckling [2, 3]. Closed sections will not buckle torsionally because of their large torsional rigidity. For open thin walled sections, however, three modes of failure are considered in the analysis of instability namely – flexuaral buckling; torsional buckling and torsional flexural buckling [4, 5]. Flexural – torsional buckling mode involves simultaneous bending and twisting of the cross-section. The cross-section undergoes translation in the two axis of the cross-section and rotates an angle φ about the shear centre. The flexural – torsional buckling problem of columns has been studied by Alsayed [4] and Timoshenko and Gere [6]. It has also been studied by Allen and Bulson [7], Chajes [8] and Wang et al [9] and used in the development of the design criteria for steel design.

II. RESEARCH AIMS AND OBJECTIVES

The general objective of this work is to solve using the Galerkin variational method the generalised elastic column buckling problem defined by a system of three coupled differential equations in terms of the three unknown displacement functions, v, w and φ. The specific objectives are to solve the generalised elastic column buckling problem for pinned – pinned ends using the Galerkin variational method, and to derive the critical buckling loads for two special cases; namely – columns with doubly symmetric cross-sections and columns with singly symmetric cross-sections.

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III. METHODOLOGY

The Galerkin method belongs to the general class of variational methods, and seeks to obtain approximate solutions to differential equations with given boundary conditions by choosing the unknown functions in terms of a linear combination of coordinate (basis) shape function that satisfy the boundary conditions exactly and then proceeds to obtain the unknown parameters of the chosen basis functions by solving the Galerkin variational functional [10, 11].

The buckling problem of columns is an eigen value problem. The general eigen-value problem is given by [12]

$$Lw - \lambda Mw = 0 \qquad (1)$$

where L and M are differential operators.

The virtual work of internal forces w_i and the virtual work of external forces w_e is

$$\partial w_i + \partial w_e = \partial(w_i + w_e) = 0 \qquad (2)$$

Thus, applying the virtual work principle, the general eigen value problem can be expressed in variational form as

$$\iint (L(w) - \lambda M(w))\partial w \partial A = 0 \qquad (3)$$

Let $w = \sum c_i \, f_i(x, y)$

where $f_i(x, y)$ are functions satisfying the boundary conditions, and c_i are unknown parameters; then we obtain

$$\iint (Lw - \lambda M(w)) \sum f_i(x, y)\partial c_i dA = 0 \qquad (4)$$

$$\sum c_i \iint (L(w) - \lambda M(w)) f_i(x, y) dA = 0 \qquad (5)$$

Equation (5) will be satisfied for any small variation ∂w_i, thus the variations ∂c_i are arbitrary and Equation (5) simplifies to

$$\iint_R (Lw - \lambda Mw) f_i(x, y)\, dxdy = 0 \qquad (6)$$

where R is the domain of the problem.

Equation (6) is the Galerkin variational equation describing the eigen value problem of an n – degree of freedom system. For one dimensional domains, we have

$$\int (Lw - \lambda Mw) f_i(x)\, dx = 0, \qquad i = 1, 2, \ldots, n \qquad (7)$$

Equations (7) lead to homogeneous algebraic eigen-value – eigen-vector problems. The lowest eigen-value $\lambda_{min} = \lambda_{cr}$ determines the critical load for the case of buckling [12, 13].

IV. GOVERNING EQUATIONS OF ELASTIC COLUMN

We consider a column of length l, whose longitudinal axis is defined by the x coordinate, and the plane of the cross-section is defined by the yz coordinates. The governing differential equations that describe a generalized elastic column buckling problem for axial compressive load N_x acting through the centroid of the cross-section; the moments due to the transverse loads are zero, the applied torque vanishes; and the load is applied such that the bi-moment is zero are given by the following set of differential Equations [5].

$$EI_{zz}\frac{d^4v}{dx^4} + N_x\frac{d^2v}{dx^2} + N_x e_z\frac{d^2\varphi}{dx^2} = 0 \qquad (8)$$

$$EI_{yy}\frac{d^4w}{dx^4} + N_x\frac{d^2w}{dx^2} + N_x e_y\frac{d^2\varphi}{dx^2} = 0 \qquad (9)$$

$$EC_w\frac{d^4\varphi}{dx^4} - \left(GJ - \frac{I_E N_x}{A}\right)\frac{d^2\varphi}{dx^2} + e_z N_x\frac{d^2v}{dx^2} - e_y N_x\frac{d^2w}{dx^2} = 0 \qquad (10)$$

where $v(x)$, $w(x)$ and $\varphi(x)$ are the displacements.

E = Young's modulus of elasticity
G = modulus of rigidity or shear modulus
C_w = warping constant
I_{zz} = moment of inertia about the z axis
I_{yy} = moment of inertia about the y axis
e_z = coordinate of the shear centre
e_y = coordinate of the shear centre
N_x = load in the x – direction
I_E = polar moment of inertia about the shear centre
J = St Venant torsional stiffness of the section

$$I_E = I_{yy} + I_{zz} + (e_y^2 + e_z^2)A$$

A = area of cross-section.

V. APPLICATION OF THE GALERKIN – VARIATIONAL METHOD TO THE GENERAL COLUMN BUCKLING PROBLEM

We seek a Galerkin solution to the system of differential Equations [8], [9], and [10] for a column of length l with pinned ends at $x = 0$, $x = l$. The unknown quantities in the governing equation are the three displacements v, w and φ; and we seek a solution to the column buckling problem for the case of pinned – pinned end supports. The boundary conditions for pinned – pinned supports at $x = 0$, $x = l$ are

$$\begin{aligned} v(x=0) &= v(x=l) = 0 \\ w(x=0) &= w(x=l) = 0 \\ \varphi(x=0) &= \varphi(x=l) = 0 \\ v''(x=0) &= v''(x=l) = 0 \\ w''(x=0) &= w''(x=l) = 0 \\ \varphi''(x=0) &= \varphi''(x=l) = 0 \end{aligned} \qquad (11)$$

where the primes denote differentiation with respect to x.

Suitable displacement (buckling) functions that satisfy the pinned – pinned conditions at $x = 0$, $x = l$ are obtained as the Fourier sine series, as follows; for an infinite one parameter representation of the displacements:

$$\begin{aligned} v(x) &= \sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \\ w(x) &= \sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l} \\ \varphi(x) &= \sum_{m=1}^{\infty} \varphi_m \sin\frac{m\pi x}{l} \end{aligned} \qquad (12)$$

where v_m, w_m and φ_m are the unknown parameters of the displacement functions, and $\sin\frac{m\pi x}{l}$ are the shape (coordinate) functions of the displacements.

The Galerkin Vlasov variational integrals are

$$\int_0^l \left(EI_{zz} \frac{d^4}{dx^4}\left(\sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \right) + N_x \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} v_m \sin\frac{m\pi x}{l} \right) \right.$$

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$$+N_x e_z \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} \varphi_m \sin\frac{m\pi x}{l}\right)\Bigg)\sin\frac{m'\pi x}{l}dx = 0 \tag{13}$$

$$\int_0^l \left(EI_{yy}\frac{d^4}{dx^4}\left(\sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l}\right) + N_x \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l}\right)\right.$$

$$\left. -N_x e_z \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} \varphi_m \sin\frac{m\pi x}{l}\right)\right)\sin\frac{m'\pi x}{l}dx = 0 \tag{14}$$

$$\int_0^l \left(EC_w\frac{d^4}{dx^4}\left(\sum_{m=1}^{\infty} \varphi_m \sin\frac{\pi x}{l}\right) - \left(GJ - \frac{I_E N_x}{A}\right)\frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} \varphi_m \sin\frac{\pi x}{l}\right)\right.$$

$$\left. +N_x e_z \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} v_0 \sin\frac{m\pi x}{l}\right) - e_y N_x \frac{d^2}{dx^2}\left(\sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{l}\right)\right)\sin\frac{m'\pi x}{l}dx = 0 \tag{15}$$

Simplifying, we have form the orthogonality properties of the slope functions,

$$\sum_{m=1}^{\infty}\int_0^l \left(EI_{zz}\frac{m^4\pi^4}{l^4}v_m \sin^2\frac{m\pi x}{l} - N_x\frac{m^2\pi^2}{l^2}v_m\sin^2\frac{m\pi x}{l}\right.$$

$$\left. -N_x e_z \frac{m^2\pi^2}{l^2}\varphi_m \sin^2\frac{m\pi x}{l}\right)dx = 0 \tag{16}$$

$$\sum_{m=1}^{\infty}\int_0^l \left(EI_{yy}\frac{m^4\pi^4}{l^4}w_m \sin^2\frac{m\pi x}{l} - N_x\frac{m^2\pi^2}{l^2}w_m\sin^2\frac{m\pi x}{l}\right.$$

$$\left. +N_x e_y \frac{m^2\pi^2}{l^2}\varphi_m \sin^2\frac{m\pi x}{l}\right)dx = 0 \tag{17}$$

$$\sum_{m=1}^{\infty}\int_0^l \left(EC_w\frac{m^4\pi^4}{l^4}\varphi_m \sin^2\frac{m\pi x}{l} + \left(GJ - \frac{I_E N_x}{A}\right)\frac{m\pi^2}{l^2}\varphi_m\sin^2\frac{m\pi x}{l}\right.$$

$$\left. -e_z \frac{N_x m^2\pi^2}{l^2}v_m \sin^2\frac{m\pi x}{l} + e_y N_x\frac{m^2\pi^2}{l^2}w_m\sin^2\frac{m\pi x}{l}\right)dx = 0 \tag{18}$$

Simplifying further, we obtain:

$$\left\{\left(EI_{zz}\frac{m^4\pi^4}{l^4} - N_x\frac{m^2\pi^2}{l^2}\right)v_m - N_x e_z\frac{m^2\pi^2}{l^2}\varphi_m\right\}I_1 = 0 \tag{19}$$

$$\left\{\left(EI_{yy}\frac{m^4\pi^4}{l^4} - N_x\frac{m^2\pi^2}{l^2}\right)w_m + N_x e_y\frac{m^2\pi^2}{l^2}\varphi_m\right\}I_1 = 0 \tag{20}$$

$$\left\{\left(EC_w\frac{m^4\pi^4}{l^4} + \left(GJ - \frac{I_E N_x}{A}\right)\frac{m^2\pi^2}{l^2}\right)\varphi_m - e_z N_x\frac{m^2\pi^2}{l^2}v_m + e_y N_x\frac{m^2\pi^2}{l^2}w_m\right\}I_1 = 0 \tag{21}$$

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where
$$I_1 = \int_0^l \sin^2 \frac{m\pi x}{l} dx \tag{22}$$

Hence, Equations (19), (20) and (21) simplify further to become the following system of homogeneous equations in terms of the displacement amplitudes, v_m, w_m, φ_m

$$\left(EI_{zz}\frac{m^4\pi^4}{l^4} - N_x\frac{m^2\pi^2}{l^2}\right)v_m - N_x e_z \frac{m^2\pi^2}{l^2}\varphi_m = 0 \tag{23}$$

$$\left(EI_{yy}\frac{m^4\pi^4}{l^4} - N_x\frac{m^2\pi^2}{l^2}\right)w_m - N_x e_y \frac{m^2\pi^2}{l^2}\varphi_m = 0 \tag{24}$$

$$\left(EC_w\frac{m^4\pi^4}{l^4} + \left(GJ - \frac{I_E N_x}{A}\right)\frac{m^2\pi^2}{l^2}\right)\varphi_m - e_z\frac{m^2\pi^2}{l^2}v_m + e_y N_x \frac{m^2\pi^2}{l^2}w_m = 0 \tag{25}$$

In matrix form, we obtain the homogeneous equation

$$\begin{bmatrix} \left(EI_{zz}\frac{m^2\pi^2}{l^2} - N_x\right) & 0 & -N_x e_z \\ 0 & \left(EI_{yy}\frac{m^2\pi^2}{l^2} - N_x\right) & N_x e_y \\ -e_z N_x & e_y N_x & \left(EC_w\frac{m^2\pi^2}{l^2} + GJ - \frac{I_E N_x}{A}\right) \end{bmatrix} \begin{pmatrix} v_m \\ w_m \\ \varphi_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{26}$$

For non trivial solutions, the determinant of coefficient matrix must vanish, and we obtain the stability equation as the determinantal equation

$$\begin{vmatrix} \left(EI_{zz}\frac{m^2\pi^2}{l^2} - N_x\right) & 0 & -N_x e_z \\ 0 & \left(EI_{yy}\frac{m^2\pi^2}{l^2} - N_x\right) & N_x e_y \\ -N_x e_z & N_x e_y & \left(EC_w\frac{m^2\pi^2}{l^2} + GJ - \frac{I_E N_x}{A}\right) \end{vmatrix} = 0 \tag{27}$$

The characteristic buckling equation is obtained by expanding equation (27) and finding the roots of the resulting polynomial in N_x. We consider two special cases of this problem which can be viewed as simplifications of the general problem.

Case 1: The cross-section is doubly symmetric (such as symmetric *I* sections, crucifix + sections) and $e_y = e_z = 0$; then the stability equation simplifies to Equation [28]:

$$\begin{vmatrix} \left(EI_{zz}\frac{m^2\pi^2}{l^2} - N_x\right) & 0 & 0 \\ 0 & \left(EI_{yy}\frac{m^2\pi^2}{l^2} - N_x\right) & 0 \\ 0 & 0 & \left(EC_w\frac{m^2\pi^2}{l^2} + GJ - \frac{I_E N_x}{A}\right) \end{vmatrix} = 0 \tag{28}$$

Expanding, the buckling (characteristic) equations is found to be already in factorised form as:

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$$\left(EI_{zz}\frac{m^2\pi^2}{l^2}-N_x\right)\left(EI_{yy}\frac{m^2\pi^2}{l^2}-N_x\right)\left(EC_w\frac{m^2\pi^2}{l^2}+GJ-\frac{I_E N_x}{A}\right)=0 \tag{29}$$

The three roots of the buckling (characteristic) equation are

$$N_x = EI_{zz}\frac{m^2\pi^2}{l^2} = P_{Ezz} \tag{30}$$

$$N_x = EI_{yy}\frac{m^2\pi^2}{l^2} = P_{Eyy} \tag{31}$$

$$N_x = \frac{A}{I_E}\left(EC_w\frac{m^2\pi^2}{l^2}+GJ\right) = P_T \tag{32}$$

where P_{Ezz} is the Euler load for buckling about the zz axis, P_{Eyy} is the Euler load for buckling about the yy axis and P_T is the buckling load in torsional (twist) buckling. The stress in torsional buckling is

$$\sigma_{xx}^{T} = -\frac{1}{I_P}\left(EC_w\frac{m^2\pi^2}{l^2}+GJ\right) = \frac{P_T}{A} \tag{33}$$

where $I_P = I_E$ for doubly symmetric cross-sections.

Case 2: The cross-section is singly symmetric such as channel sections. If the zz axis is the axis of symmetry, then $e_y = 0$, $e_z \neq 0$, and the buckling equation becomes

$$\begin{vmatrix} EI_{zz}\frac{m^2\pi^2}{l^2}-N_x & 0 & -N_x e_z \\ 0 & EI_{yy}\frac{m^2\pi^2}{l^2}-N_x & 0 \\ -N_x e_z & 0 & \left(EC_w\frac{m^2\pi^2}{l^2}+GJ-\frac{I_E N_x}{A}\right) \end{vmatrix} = 0 \tag{34}$$

Expanding,

$$\left(EI_{yy}\frac{m^2\pi^2}{l^2}-N_x\right)\left\{\left(EI_{zz}\frac{m^2\pi^2}{l^2}-N_x\right)\left(EC_w\frac{m^2\pi^2}{l^2}+GJ-\frac{I_E N_x}{A}\right)-(N_x e_z)^2\right\}=0 \tag{35}$$

The roots are

$N_x = EI_{yy}\frac{m^2\pi^2}{l^2} = P_{Eyy}$ For $m = 1$, we obtain critical loads.

$$N_x = \frac{(P_T + P_{E_{zz}}) \pm \sqrt{\left[(P_T+P_{Ezz})^2 - 4\left(1-\frac{Ae_z}{I_E}\right)P_{Ezz}P_T\right]}}{2\left(1-\frac{e_z^2 A}{I_E}\right)}$$

$$N_x = \frac{1}{2\beta}\left[(P_T+P_{Ezz}) \pm \sqrt{\left((P_T+P_{Ezz})^2 - 4\beta P_{Ezz}P_T\right)}\right] \tag{36}$$

where $P_{Ezz} = EI_{zz}\frac{\pi^2}{l^2}$, $\beta = 1-\frac{Ae_z^2}{I_E}$, $P_T = \frac{A}{I_E}\left(EC_w\frac{\pi^2}{l^2}+GJ\right)$

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N_x given by Equation (36) represents the buckling load for the coupled torsional flexural buckling mode. A negative sign in equation (36) will yield smaller values, hence

$$N_x = \frac{(P_{Ezz} + P_T) - \sqrt{\left[(P_T + P_{Ezz})^2 - 4\left(1 - \frac{Ae_z^2}{I_E}\right)P_T P_{Ezz}\right]}}{2\left(1 - \frac{Ae_z^2}{I_E}\right)}$$

$$N_x = \frac{1}{2\pi}\left[(P_T + P_{Ezz}) - \sqrt{(P_T + P_{Ezz})^2 - 4\beta P_{Ezz} P_T}\right] \qquad (37)$$

The critical buckling stress for the coupled torsional flexural buckling mode is obtained as

$$\sigma_{xx\,cv}^{ET} = \frac{(\sigma_{xx\,cr}^{E} + \sigma_{xx\,cr}^{T}) - \sqrt{\left((\sigma_{xx\,cr}^{E} + \sigma_{xx}^{T})^2 - 4\left(1 - \frac{Ae_z^2}{I_E}\right)\sigma_{xx\,cr}^{E}\sigma_{xx\,cr}^{T}\right)}}{2\left(1 - \frac{Ae_z^2}{I_E}\right)}$$

$$= \frac{1}{2\beta}\left[(\sigma_{xx\,cr}^{E} + \sigma_{xx\,cr}^{T}) - \sqrt{(\sigma_{xx\,cr}^{E} + \sigma_{xx}^{T})^2 - 4\beta\sigma_{xx\,cr}^{E}\,\sigma_{xx}^{T}}\right] \qquad (38)$$

where $\sigma_{xx\,cr}^{E} = \frac{E\pi^2}{(l/r_y)^2}$, $\sigma_{xx\,cr}^{T} = \frac{E}{I_E}\left(C_w\frac{\pi^2}{l^2} + \frac{J}{2(1+\mu)}\right)$

VI. DISCUSSION AND CONCLUSION

For doubly symmetric shapes of cross-section such as (symmetric *I* sections) *I* or **+** (cruciform sections), the shear centre coincide with the centroid of the cross-section, and $e_y = e_z = 0$. The system of governing differential equations become uncoupled leading to buckling modes that are uncoupled. The characteristic buckling equation for doubly symmetric sections, given by Equation (22) has three roots, which are the Euler flexural buckling load in the *z* axis, the Euler flexural buckling load in the *y* axis, and the load in torsional (twist) buckling. The critical buckling load is the lowest value of the three buckling loads and determines how the column with doubly symmetric cross-section will fail.

For columns with singly symmetric sections like $\llcorner$, $\sqsubset$, $\top$ or $\perp$, where *zz* axis is the axis of symmetry, $e_y = 0$ and the general system of equations make Equation (9) independent of the Equations (8) and (10), with Equations (8) and (10) being coupled. The characteristic (buckling) Equation (35) reveals the column can fail by Euler flexural buckling mode in the *yy* direction and torsional – flexural buckling mode. Three values of the buckling load were determined; namely:

Euler flexural buckling load in the *yy* direction

Two (coupled) torsional flexural buckling loads given by Equation (36).

Evidently, Equation (37) gave lower values of the torsional flexural buckling load; and the mode of failure would be governed by N_x in Equation (37) or the Euler flexural buckling load in the *y* direction, whichever one is smaller. The values of buckling loads obtained using the Galerkin variational method are in excellent agreement with solutions by Det [13] and Wang et al [9].

It is seen that columns with singly symmetric sections may buckle either in Euler flexural buckling mode or in torsional – flexural buckling mode; while in columns with doubly symmetric sections the buckling differential equations are uncoupled and the buckling modes are uncoupled.

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