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Topical Advancements in Various Spline Techniques for Boundary Value Problems

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***Abstract**: the present paper surveys and reviews papers on numerical solution of different types of boundary value problems by Spline based Technique. Among a number of numerical methods used to solve boundary value problems such as Variation Iteration Method, Adomian Decomposition Method, RKM etc. spline methods provide an efficient tool and more accurate results. Methodology used in this paper include cubic splines, quartic spline, quantic spline, non-polynomial splines, parametric spline, exponential etc. which has been developed by many Mathematicians & Researchers over the years*.
***Keywords:** Boundary value Problems, B-spline methods,Cubic spline, quartic spline, quintic spline.*

I. INTRODUCTION

Boundary value problems (ordinary and partial) arises in various field of sciences and engineering such as heat transfer, problems of the deflection of plates, fluid mechanics, viscoelasticity, chemistry, physics, finance, bio-sciences, physics and many other field. So it becomes necessary to find the solution of boundary value problem. Sometimes it is not possible to find the solution in closed form or it is too complicated. So we required numerical methods to solve such BVPs. Many Authors have used different methods to solve different type of BVP such as Variational Iteration Method, Reproducing Kernel Meethod, Finite Difference Method, Adomian Decomposition Metthod etc. but spline based method has been used more in the literature due to effectiveness and accuracy of the method. In this paper we summarized the papers since 2013 related to numerical solution of BVP by Spline based methods which includes the cubic B-spline method, quartic, quintic ,sextic ,septic spline method , Non-polynomial spline method, exponential spline method etc. The present paper is organized as follows,

Section II: In this section different research papers has been summarized according to the order and nature of spline method such as cubic spline, quartic spline , quintic spline, nonpolynomial spline , exponential spline etc.

Section III: the conclusion has been given in this section.

II. SPLINE SOLUTION OF BOUNDARY VALUE PROBLEMS: RECENT DEVELOPMENTS

In this section we surveyed research articles published since 2013 related to numerical solution of boundary value problems by spline methods in increasing order of the degree of the spline start with third degree spline (cubic spline)

A. *Cubic Splines*

In the year 2013 Bialecki and Karageorghis [1] formulate a fourth order modified nodal cubic spline collocation scheme for variable coefficient second order partial differential equations in the unit cube subject to nonzero Dirichlet boundary conditions. They consider the general variable coefficient nonzero Dirichlet boundary value problem (BVP)

$$Lu = f(x,y,z), \quad (x,y,z) \in \Omega \qquad u((x,y,z) = g(x,y,z), \quad (x,y,z) \in \partial\Omega \tag{1}$$

Where

$$Lu = p_1(x,y,z)D_x^2 u + p_2(x,y,z)D_y^2 u + p_3(x,y,z)D_z^2 u + p_{12}(x,y,z)D_x D_y u + p_{13}(x,y,z)D_x D_z u$$
$$+ p_{23}(x,y,z)D_y D_z u + r_1(x,y,z)D_x u + r_2(x,y,z)D_y u + r_3(x,y,z)D_z u + q(x,y,z)u \tag{2}$$

To describe the method, the authors have taken L=Δ, g=0 in equation (1) that becomes a Poisson’s equation with zero Dirichlet boundary condition as follows

$$\Delta u = f(x,y,z), \quad (x,y,z) \in \Omega \qquad u(x,y,z) = 0, \quad (x,y,z) \in \partial\Omega \tag{3}$$

The authors develop a very efficient matrix decomposition algorithm (MDA) for the solution of the resulting linear system. In general, an MDA is a direct method which reduces the solution of a linear system to solving a collection of independent linear systems corresponding to one-dimensional problems. Convergence of the method has been shown with the help of examples.

1) Remarks: The proposed method has applied only on the problem of non-zero Dirichlet boundary conditions. In future,

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Researchers can apply the (MNCS) for boundary conditions other than Dirichlet.

In the same year Mittal and Jain [2] considered the nonlinear Fisher's reaction–diffusion equations as

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta u(1-u) \qquad a \le x \le b \quad t \ge 0 \tag{4}$$

$$\begin{aligned} &u(x,0) = u_0(x) && a \le x \ge b \\ &u(a,t) = g_0(t), \quad u(b,t) = g_1(t) && t \in [0,T] \end{aligned} \tag{5}$$

To solve this problem first the author has described the cubic B-spline function then they modified cubic B-spline basis functions to obtain a diagonally dominant system of differential equations for handling Dirichlet boundary conditions. The procedure for modifying the basis functions is given as follows:

$$\begin{aligned} &\tilde{B}_0(x) = B_0(x) + 2B_{-1}(x) \\ &\tilde{B}_1(x) = B_1(x) - B_{-1}(x) \\ &\tilde{B}_j(x) = B_j(x), \qquad j=2....N-2 \\ &\tilde{B}_{N-1}(x) = B_{N-1}(x) - B_{N+1}(x) \\ &\tilde{B}_N(x) = B_N(x) + 2B_{N+1}(x) \end{aligned} \tag{6}$$

the approximate solution of eq. (4) using the modified cubic B-spline basis functions is given in the form

$$\begin{aligned} &U^N(x_0,t) = g_0(t), && \text{for } j=0 \\ &U^N(x_j,t) = \sum_{j=0}^{N} \alpha_j \tilde{B}_j(x), && \text{for } j=0,1.......N-1 \\ &U^N(x_N,t) = g_1(t), && \text{for } j=N \end{aligned} \tag{7}$$

Using approximate solution (7) and modified cubic B-spline function (6), the approximate values of $U_t^N(x)$ at the nodes are determined in terms of the time parameters α_j as follows:

$$\begin{aligned} &(U_t)_0 = \dot{g}_0(t), && \text{for } j=0 \\ &(U_t)_j = \sum_{j=0}^{N} \dot{\alpha}_j \tilde{B}_j(x), && \text{for } j=0,1.......N-1 \\ &(U_t)_N = \dot{g}_1(t), && \text{for } j=N \end{aligned} \tag{8}$$

To solve the resulting system of O.D.E, SSP-RK54 method has been used. The method has been tested on three examples and results show the effectiveness of the method.

2) Remark: The main advantage of the proposed method is that it needs less storage which causes less numerical errors.

In 2014 Lang & Xu [3] used cubic B-spline on uniform partitions to solve the following problem

$$\begin{aligned} &u'' = F(x,u,u',v,v'), \quad x \in [a,b] \\ &v'' = G(x,u,u'v,v'), \quad x \in [a,b] \\ &u(a) = u_0, \quad u(b) = u_1, \\ &v(a) = v_0, \quad v(b) = v_1, \end{aligned} \tag{9}$$

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where F and G are two nonlinear functions of x, u, u_0, v, and v_0. On discretizing equation (9) at inner knots

$$u'(x_j) = F(x_j, u(x_j), u'(x_j), v(x_j), v'(x_j)),$$
$$v'(x_j) = G(x_j, u(x_j), u'(x_j), v(x_j), v'(x_j)). \tag{10}$$

After algebraic calculations 2n+6 linear equations have been obtained and gives the solution

$$u(x) = \sum_{i=-1}^{n+1} c_i B_i(x) \; and \; \mathrm{v}(x) = \sum_{i=-1}^{n+1} d_i B_i(x) \tag{11}$$

3) Remark: The advantage of the method is that the 4th order & 6th order two point BVP can also be solved by the proposed method. The presented method has been proved of 4th order convergent. Four examples have been solved which shows the effectiveness of the method.

In the same year Chen & Wong [4] considered a system of second order boundary value problems of the type

$$y''(x) = \begin{cases} f(x) & a \le x \le c \\ g(x)y(x)+f(x)+r & c \le x \le d \\ f(x) & d \le x \le b \end{cases} \tag{12}$$

$$y(a) = \alpha \qquad y(b) = \beta$$

with continuity conditions of *y* and y' at *c* and *d*. Here, *f* and *g* are continuous functions on [*a*, *b*] and [*c*, *d*] respectively, *r, α and β* are real finite constants. Such type of systems has important applications in many fields of science and engineering. First the author defines the deficient discrete cubic spline $S_i(x)$ as a polynomial of degree 3 or less which will be equivalent to

$$D_h^{\{j\}} S_i(x_i) = D_h^{\{j\}} S_{i+1}(x_i) \qquad j=0,1 \quad 1 \le i \le n-1$$
$$D_h^{\{2\}} S_i(x_i) = D_h^{\{2\}} S_{i+1}(x_i) \tag{13}$$

Then solution of eq. (11) approximated by the deficient discrete cubic spline

$$y''(x) = \begin{cases} f(x), & x \in [a,c) \cup (d,b] \\ g(x)S(x;h)+f(x)+r, & x \in (c,d) \end{cases} \tag{14}$$

$$y(x) \cong S(x;h), \quad y'(x) \cong D_h^{(1)} S(x;h), \quad x \in [a,b] \tag{15}$$

and the values of unknowns has been determined by solving the system of linear equations. Convergence of the method has been discussed and shown that method has of 2nd order convergence.

4) Remark: The main feature of the paper is that the application of equation (12) to obstacle BVP has been discussed.

In the paper [5] authors has discussed the cubic trigonometric B-spline collocation method. They have considered a vibrating elastic string of length L which is located on the x-axis of the interval [0, L]. Then, the vertical displacement u (x, t) of the elastic string at point x units from the origin after a time t elapsed is given by the one-dimensional wave equation

$$\frac{\delta^2 u}{\delta t^2}(x,t) - \alpha^2 \frac{\delta^2 u}{\delta x^2}(x,t) = q(x,t), \qquad 0 \le x \le L, \quad 0 \le t \le T \tag{16}$$

subject to

Initial displacement: $u(x, t=0) = e_1(x) \quad 0 \le x \le L,$

Initial velocity: $u_t(x, t=0) = e_2(x) \quad 0 \le t \le T$ (17)

boundary conditions

$$\beta u(x=0,t)+\gamma u_x(x=0,t)=f_1(t) \tag{18}$$

and nonlocal conservation boundary conditions

$$\int_0^L u(x,t)dx = f_2(t) \qquad 0 \le t \le T \tag{19}$$

where q, g_1, g_2, f_1 and f_2 are the known functions and b, c are constants This type of problems arises in non-local reactive transport in underground water flows in porous media, semi -conductor modeling, non-Newtonian fluid flows and radioactive nuclear decay in fluid flows. To solve equation (16) the authors used finite difference method to discretize the time derivative and for the space dimension they used a new Cubic trigonometric B-spline method.

The solution domain a≤ x ≤b is equally divided by knots x_i into n subintervals $[x_i, x_{i+1}]$, i= 0,1,2…n-1 where $a=x_0< x_1< \ldots\ldots <x_n = b$. the approximate solution of eq.(16) is as

$$U_i(x,t)=\sum_{i=-3}^{n-1} C_i(t)TB_i^{(4)}(x) \tag{20}$$

where $C_i(t)$ are to be determined for the approximated solutions $U_i(x, t)$ to the exact solutions u(x, t) at the point (x_i, t_j) and $TB^4_i(x)$ are cubic trigonometric B-spline basis functions defined as

$$TB_i^{(4)}(x)=\frac{1}{\omega}\begin{cases} p^{(3)}(x_i) & x\in[x_i,\ x_{i+1}] \\ p(x_i)p(x_i)q(x_{i+2})+q(x_{i+3})p(x_{i+1})+q(x_{i+4})p(x_{i+2}) & x\in[x_{i+1},\ x_{i+2}] \\ q(x_{i+4})p(x_{i+1})\ q(x_{i+3})+q(x_{i+4})p(x_{i+2})+p(x_i)q^{(2)}(x_{i+3}) & x\in[x_{i+2},\ x_{i+3}] \\ q^{(3)}(x_{i+4}) & x\in[x_{i+3},\ x_{i+4}] \end{cases} \tag{21}$$

Where

$$p(x_i)=\sin(\frac{x-x_i}{2}),\qquad q(x_i)=\sin(\frac{x_i-x}{2}),\qquad \omega=\sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2}) \tag{22}$$

By approximating the solution and using the conditions of spline the system is converted into a tridigonal system which can be solved easily. Stability of the method has been proved unconditionally stable by von Neumann approach and some examples are solved to show the feasibility of the proposed method.

5) Remarks: In this paper a new cubic trigonometric B-spline collocation method is applied on the one-dimensional wave equation not only with classical boundary conditions but for given initial condition and non-local conservation condition. The proposed method less storage and CPU time as compare to CuBSM .

In the year 2015 Mittal, and Dahiya [6] considered the two dimensional hyperbolic equations, which include diffusion and constant convection in rectangular domain $\Omega \in R^2$

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \qquad (x,y)\in\Omega \qquad t\in(0,T] \tag{23}$$

where P and Q are constants which are less than one in absolute value i.e. both P and Q are less than the diffusion coefficients and Initial conditions are given by

$$\begin{aligned} &u(x,y,0)=u_0(x,y) && (x,y)\in\Omega \\ &\frac{\partial u}{\partial t}(x,y,0)=u_1(x,y) && (x,y)\in\Omega \end{aligned} \tag{24}$$

and Dirichlet boundary conditions

$$u(x,y,t)=f(x,y,t) \qquad (x,y)\in\Omega \quad t\in(0,T] \tag{25}$$

Introducing an auxiliary function ω, Eq. (23) reduces to a system of partial differential equations given as follows

$$\frac{\partial \omega}{\partial t} + P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (x,y) \in \Omega \qquad t \in (0,T]$$

$$\omega = \frac{\partial u}{\partial t} + u \qquad (x,y) \in \Omega \qquad t \in (0,T] \tag{26}$$

The initial value and boundary value conditions on ω are given by system in Eq. (26) as

$$\omega(x,y,0) = \omega_0(x,y) \qquad (x,y) \in \Omega$$

$$\omega(x,y,t) = g(x,y,t) \qquad (x,y) \in \partial\Omega \quad t \in (0,T] \tag{27}$$

In this method cubic B-spline method has been used to compute the weighting coefficients in place of using method of Lagrange interpolation. By using this method hyperbolic problem reduces into a system of nonlinear ordinary differential equations which has been solved by Runge–Kutta (SSP-RK43) method.

6) Remarks: The unconditional stability of the proposed method is the main advantage. In this method less no. of grid points required that causes less calculation and less errors.

In paper [7] the author, takes the time fractional Burgers equation into consideration which is the simplest nonlinear model equation for diffusive waves in fluid dynamics and has following form

$$\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} + U(x,t\frac{\partial U(x,t)}{\partial x} - \upsilon\frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t) \tag{28}$$

where v is a viscosity parameter and

$$\frac{\partial^\gamma U(x,t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-\tau)^\gamma \frac{\partial U(x,\tau)}{\partial \tau} d\tau \tag{29}$$

is the fractional derivative given in the Caputo's sense.

The boundary conditions of the model problem (34) given in the interval a$\leq x \leq b$ as

$$U(a,t) = h_1(t), \qquad U(b,t) = h_2(t), \qquad t \geq 0 \tag{30}$$

and the initial condition as

$$U(x,0) = g(x) \qquad a \leq x \leq b \tag{31}$$

To solve the above problem, the author first defines cubic B-spline functions and then writes an approximate solution $U_N(x, t)$ in terms of the cubic B-splines functions as follows:

$$U_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\Phi_m(x) \tag{32}$$

where $\delta_m(t)$'s are unknown, time-dependent quantities to be determined from the initial, boundary and cubic B-spline collocation conditions. Three examples have been solved with this method and results are compared with the analytic results confirmed the effectiveness of the method. The accuracy of the method has been measured by the error norm L_2 given as

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \approx \sqrt{h\sum_{j=0}^{N}\left|U_j^{exact} - (U_N)_j\right|^2}$$

7) Remarks: In this paper the time fractional derivative has considered of the Caputo form.

In paper [8] the Authors have proposed a scheme based on cubic spline in compression and they have considered the following boundary value problem (BVP) for the delay differential equation (DDE):

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x-1) = f(x) \qquad 0 \leq x \leq 2 \tag{33}$$

subject to the interval and boundary conditions

$$y(x) = \phi(x), \qquad x \in [-1,0]$$

$$y(2) = \beta \tag{34}$$

where $0<\varepsilon< 1$ and $a(x) \geq \alpha > 0$, $a(x)$, $b(x)$, $f(x)$ are given sufficiently smooth functions on [0,2], $\varphi(x)$ is a smooth function on [-1,0]

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and β is a given constant which is independent of ε. A function $s(x, \tau) = s(x)$ satisfying the differential equations in $[x_i, x_{i+1}]$ is

$$s''(x)+\tau s(x)=[s''(x_i)+\tau s(x_i)]\frac{(x_{i+1}-x)}{h}+[s''(x_{i+1})+\tau s(x_{i+1})]\frac{(x-x_i)}{h} \quad (35)$$

where $s(x_i) = y_i$ and $\tau > 0$ is termed cubic spline in compression. On Solving (41) as a linear second order differential equation, we get

$$s(x_i)=A\cos\frac{\lambda}{h}x_i+B\sin\frac{\lambda}{h}x_i+\frac{(M_i+\tau y_i)}{\tau}\frac{(x_{i+1}-x)}{h}\frac{(M_{i+1}+\tau y_{i+1})}{\tau}\frac{(x-x_i)}{h} \quad (36)$$

After using the continuity of derivative and approximation of derivatives a tridiagonal system has been obtained which give the solution.

8) Remarks: The proposed method is helpful to solve those singular perturbation problems which are unstable and fail to give accurate results when the perturbation parameter ε is small because the accuracy of the method accuracy does not depend on the parameter value ε. The proposed method is uniform convergent.

Wang, Zhou, and Cui, in their paper [9] have considered following class of boundary value problem

$$u''(x)=\mathrm{f}(\mathrm{x},\mathrm{u}(\mathrm{x})),\qquad \mathrm{x}\in I=(\mathrm{a},\mathrm{b})$$
$$u(a)=\mathrm{a}_0,\quad \mathrm{u(b)}=\mathrm{a}_1 \quad (37)$$

Where f(x, u) is an unknown continuous function the author has used parametric cubic spline function with unequally spaced nodes on Legendre-gauss-Labatto nodes. Divide the interval [a,b] into N small intervals $[x_i, x_{i+1}]$ by $x_i=x(t_i)$, i=0,1 ….N-1. The spline function $S(x,\tau)=S(x)$ satisfies the given problem on the interval $[x_i, x_{i+1}]$ and given as

$$S''(x)+\tau S(x)=(S''(x_i)+\tau S(x_i))\frac{x_{i+1}-x}{h_{i+1}}+(S''(x_{i+1})+\tau S(x_{i+1}))\frac{x-x_i}{h_{i+1}} \quad (38)$$

After writing the general solution of equation (44) by using interpolating conditions and using the continuity of derivative at the nodes following relation has been obtained

$$\alpha_i M_{i+1}+(\beta_{i+1}+\beta_i)M_i+\alpha_{i+1}M_{i+1}=\frac{u_{i+1}}{h_{i+1}}-(\frac{1}{h_{i+1}}+\frac{1}{h_i})u_i+\frac{u_{i-1}}{h_i} \quad (39)$$

These equations form a tridigonal system about the numerical solution $u_1\, u_2, \ldots\ldots u_{N-1}$ which approximate the solution u(x).

9) Remarks: The author has shown that the results obtained by using the proposed method have given better results than other method discussed in literature. The main feature of the method is the adjustable parameter τ involved in the cubic spline function. Akram and Tariq in [10] have used cubic polynomial spline functions for the approximate solutions of the following fractional boundary value problems (FBVPs)

$$D^{\alpha}\,y(x)+y(x)=f(x),\quad x\in[a,\,b],\qquad 1\le\alpha<2, \quad (40)$$

subject to

$$\mathrm{y\,(a) = A,\ y(b) = B.} \quad (41)$$

where A and B are real constants. Also *f (x)* is continuous function on the interval [*a*, *b*] and D^{α} denotes fractional derivative in Caputo's sense. Caputo's fractional derivative is widely applied in modelling of the material's mechanical properties, modelling of the viscoelastic behavior, signal processing, diffusion problems, bioengineering and mathematical finance models etc. The right and left sided Caputo's fractional derivative of order α has defined as

$$D^{\alpha}_{-b}y(x)=\begin{cases} I^{m-\alpha}_{-b}D^{m}y(x), & \mathrm{m}-1\le\alpha\le\mathrm{m},\ \mathrm{m,n}\in\mathrm{N} \\ \dfrac{D^{m}y(x)}{Dx^{m}}, & \alpha=m \end{cases} \quad (42)$$

And

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$$D_{a+}^{\alpha} y(x) = \begin{cases} I_{a+}^{m-\alpha} D^m y(x) \;, & m-1 \le \alpha \le m \;,\; m,n \in N \\ \dfrac{D^m y(x)}{Dx^m} \;, & \alpha = m \end{cases} \tag{43}$$

and then numerical solution of given FBVP has discussed with left differential operator (first case) and with right differential operator (second case).

In first case the FBVP (1) becomes

$$D_{\alpha+}^{\alpha} y(x) + y(x) = f(x) \qquad 1 \le \alpha \le 2 \tag{44}$$

The cubic spline solution of above problem is given as

$$\psi_i(x) = \hat{a}_i (x - x_{i-1})^3 + \hat{b}_i (x - x_{i-1})^2 + \hat{c}_i (x - x_{i-1}) + \hat{d}_i \qquad i=1, 2 \ldots n \tag{45}$$

where $\hat{a}_i, b_i, \hat{c}_i, \hat{d}_i$, are undetermined coefficients and determined by using the continuity of derivative and boundary conditions.

In second case, the cubic spline solution of the FBVP (1) is given as

$$\psi_i(x) = a_i (x_{i+1} - x)^3 + b_i (x_{i+1} - x)^3 + c_i (x_{i+1} - x)^3 + d_i \quad , i=1, 2 \ldots n. \tag{46}$$

where a_i, b_i, c_i, d_i, are undetermined coefficients and determined by same procedure as given above.

10) Remarks:Convergence of method has been discussed and proved that the method is second order convergent. Two numerical examples have given for accuracy of the method the results of same examples are compared with the other method and found that results of suggested method are more accurate. The suggested method also utilizes the properties of fractional derivatives in order to solve this problem.

B. *Quartic and Quintic splines:*

In paper [11] Li, Chen and Ma have considered the linear sixth order boundary value problems of the form

$$y^{(6)} + f(x) y = g(x), \qquad x \in [a,b] \tag{47}$$

Subject to the conditions

$$\begin{aligned} y(a) &= A_0, \quad y'(a) = A_1, \quad y''(a) = A_2, \\ y(b) &= B_0, \quad y'(b) = B_1, \quad y''(b) = B_2, \end{aligned} \tag{48}$$

Where (x) and (x) are continuous functions on $[a, b]$, $A_i\,(i = 0, 1, 2)$ and $B_i\,(i = 0, 1, 2)$ are given finite real constants. Such sixth boundary value problem arises in astrophysics. For an interval $[a, b]$, introduce a set of equally spaced knots of partition $\Omega = \{x_0, x_1 \ldots x_n\}$, and assume that $n \ge 5$, $x_i = a + ih$, $(i = 0, 1. . . n)$, $x_0 = a$, $x_n = b$. Let $S_4[\pi]$ be the space of continuously differentiable, piecewise, quartic-degree polynomials on π.

The zero-degree B-spline is defined as

$$N_{i,0}(x) = \begin{cases} 1 & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

And for positive constant p, it is defined in the following recursive form:

$$N_{i,p}(x) = \frac{x - x_{i-1}}{x_{i+p} - x_i} N_{i,p-1}(x) + \frac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x) \qquad p \ge 2 \tag{50}$$

The quartic B-spline in $S_4[\pi]$ is

$$N_{i,4}(x)=\frac{1}{24h^4}\begin{cases}(x-x_{i-2})^4 , & x\in[x_{i-2},x_{i-1}]\\ (x-x_{i-2})^4-5(x-x_{i+1})^4 & x\in[x_{i-1},x_i]\\ (x-x_{i-2})^4-5(x-x_{i-1})^4+10(x-x_i)^4 & x\in[x_i,x_{i+1}]\\ (x-x_{i+3})^4-5(x-x_{i+2})^4 & x\in[x_{i+1},x_{i+2}]\\ (x-x_{i+3})^4 & x\in[x_{i+2},x_{i+3}]\\ 0 & \text{otherwise}\end{cases} \tag{51}$$

Let s(x) = $\sum_{i=-2}^{n+1} c_i\, N_{i,4}(x)$ be the approximate solution of eq. (47). After discretize (47) at the knots x_i (i = 2, 3 . . . n − 2), we get

$$\frac{1}{2h^6}\left(c_{i-4}-5c_{i-3}+9c_{i-2}-5c_{i-1}-5c_i+9c_{i+1}-5c_{i+2}c_{i+3}\right)+\frac{f(x_i)}{24}(c_{i-2}+11c_{i-1}+11c_i+c_{i+1})=g(x_i)+o(h^2) \tag{52}$$

On simplification,

$$12\left(c_{i-4}-5c_{i-3}+9c_{i-2}-5c_{i-1}-5c_i+9c_{i+1}-5c_{i+2}+c_{i+3}\right)+f(x_i)(c_{i-2}+11c_{i-1}+11c_i+c_{i+1})h^6=24h^6g(x_i)+o(h^8) \tag{53}$$

Dropping the term o (h^2) from (52), they get a linear system with $n-3$ linear equations in $n+4$ unknown's c_i (i =-2, -1…. n+1)

$$12\left(c_{i-4}-5c_{i-3}+9c_{i-2}-5c_{i-1}-5c_i+9c_{i+1}-5c_{i+2}c_{i+3}\right)+f(x_i)(c_{i-2}+11c_{i-1}+11c_i+c_{i+1})h^6=24h^6g(x_i) \tag{54}$$

For solution of (47) seven more equations are needed. These equations are obtained by boundary conditions at x=a. By boundary condition at x=a

$$\begin{aligned}&c_{-2}+11c_{-1}+11c_0+c_1=24A_0\\ &-c_{-2}-3c_{-1}+3c_0+c_1=6hA_1\\ &c_{-2}-c_{-1}-c_0+c_1=2h^2A_2\end{aligned} \tag{55}$$

By boundary condition at x= b we get

$$\begin{aligned}&c_{n-2}+11c_{n-1}+11c_n+c_{n+1}=24B_0\\ &-c_{n-2}-3c_{n-1}+3c_n+c_{n+1}=6hB_1\\ &c_{n-2}-c_{n-1}-c_n+c_{n+1}=2h^2B_2\end{aligned} \tag{56}$$

And finally after algebraic manipulation we get

$$5c_{-2}-33c_{-1}+93c_0-145c_1+135c_2-75c_3+23c_4-3c_5=24h^6y^6(a) \tag{57}$$

Equations (54),(55),(56) & (57) form a system of (n+4) linear equations with c_i (i= -2,-1….n+1) as unknowns. The solution of equation (47) has been found by solving this system by matrix method. The proposed method has convergence of 2nd order.

1) Remarks: In the paper, the numerical example has been given and the results are compared with Adomian Decomposition method, Variational Iteration Method which shows that the proposed method gives less error.

In paper [12] Viswanadham and Ballem have used galerkin method with Quintic B- splines. they have considered a general eighth order linear boundary value problem given by

$$a_0(x)y^{(8)}(x)+a_1(x)y^{(7)}(x)+a_2(x)y^{(6)}(x)+a_3(x)y^{(5)}(x)+a_4(x)y^{(4)}(x)+a_5(x)y'''(x)+a_6(x)y''(x)+a_7(x)y'(x)+a_8(x)y(x)=b(x) \tag{58}$$

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subject to the boundary conditions

$$y(c)=A_0\ ,y(d)=C_0\ ,y'(c)=A_1\ ,y'(d)=C_1\ ,y''(c)=A_2\ ,y''(d)=C_2,y'''(c)=A_3\ ,y'''(d)=C_3 \tag{59}$$

where A_0, C_0, A_1, C_1, A_2, C_2, A_3, $C_3$are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$,$a_6(x)$, $a_7(x)$, $a_8(x)$, b(x) are all continuous functions defined on the interval [c, d].The authors have used Quintic B-spline as basis function.

2) Remarks: In this paper, the authors set new quintic B-spline basis which has been set zero on the boundary. Numerical examples have been solved and results are compared with the exact solution. For solving the nonlinear problems method of quasilinearization has been used.

In paper [14] Viswanadham and Reddy solved ninth order boundary value problem by using patrov-galerkin method and they have used Quintic B-spline as basis function and septic-spline as weighted function.

In paper [13] Saini and Mishra considered the self-adjoint third-order singularly perturbed boundary value problem of the form:

$$\begin{aligned} &Ly(x)=-\varepsilon y(x)'''(x)+u(x)y(x)=f(x) \qquad u(x)\geq 0\\ &y(0)=\alpha\ ,\quad y(1)=\beta \quad y'(0)=\gamma\ , \end{aligned} \tag{60}$$

Where α, β, γ are constants and ε is a small positive parameter (0< ε≤1) are sufficiently smooth functions. To solve this problem first quartic b-spline function has been defined then solution is approximated by spline function as

$$y(x)=r(x)=\sum_{i=-4}^{n-1} c_i B_i(x) \tag{61}$$

The eq. (60) becomes

$$\begin{aligned} &-\varepsilon\sum_{j=-4}^{n-1} c_j B''_j(x_i)+u(x_i)\sum_{j=-4}^{n-1} c_j B_j(x_i)=f(x_i), \qquad i=0,........n\\ &\sum_{j=-4}^{n-1} c_j B''_j(x_0)=\alpha\\ &\sum_{j=-4}^{n-1} c_j B_j(x_n)=\beta\\ &\sum_{j=-4}^{n-1} c_j B'_j(x_0)=\gamma \end{aligned} \tag{62}$$

by using the continuity of derivative and boundary conditions that converted in a system of (N+4)x(N+4) equation in (N+4) unknown whose solution has been obtained by matrix method.

3) Remarks: This method is very efficient and implementation of the method is very easy and accurate. The comparison between results obtained by different methods is shown with the help of graphs. Two examples have been solved to show the effectiveness of the method.

In 2016 Akram and Tariq [15] used Quintic spline collocation method for solving fractional boundary value problems. They considered the following fourth order linear fractional differential equations

$$y^{(4)}(x)+D^{\alpha}\ p(x)y(x)=g(x), \qquad x\in[a,\ b], \tag{63}$$

subject to

$$y(a)-A_1=y(b)-A_2=y''(a)-B_1=y''(b)-B_2 \tag{64}$$

where A_i, B_i ,i= 1, 2 are real constants. The functions p(x) and g(x) are continuous on the interval [a, b] and D^{α} denotes fractional derivative in Caputo's sense. The analytic solution of Eq. (63) and (64) cannot be obtained for arbitrary choices of p(x) and g(x). When α=0 eq. (63) has reduced to the classical fourth order boundary value problem. Caputo's fractional derivative of order α has been defined by the author. First divide the interval [a,b] into subintervals $[x_{i-1},x_i]$.Let y(x) be the exact solution of eq. (63) and S_i be an approximation to y_i= $y(x_i)$ obtained by the spline function $T_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Consider that

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each quintic polynomial spline segment $T_i(x)$ has the following form:

$$T_i(x) = a_i(x - x_{i-1})^5 + b_i(x - x_{i-1})^4 + c_i(x - x_{i-1})^3 + d_i(x - x_{i-1})^2 + e_i(x - x_{i-1}) + f_i \tag{65}$$

The solution of eq. (63) has been found by using the continuity of the derivative and end term conditions.

4) Remark: The strong advantage of this scheme is to provide smooth continuous approximations to exact solutions at every point of the range of integration. The suggested method also utilizes the properties of fractional derivatives in order to solve this problem. Convergence of the method has been discussed and found that the method is 2nd order convergent. Two numerical examples have been solved by the method that shows the effectiveness of the method.

In the same year Korkmaza and Dag [16] solved numerically the one-dimensional advection-diffusion equation (ADE) which is given in the following form

$$\frac{\partial U}{\partial t} + \alpha \frac{\partial U}{\partial x} - \beta^2 \frac{\partial^2 U}{\partial x^2} = 0, \qquad 0 \le x \le L \tag{66}$$

with the initial condition

$$U(x,0) = U_0(x), \qquad 0 \le x \le L \tag{67}$$

and the boundary conditions

$$U(0,t) = f(t), \qquad U(L,t) = g(t) \tag{68}$$

In a finite domain [0, L] where α and β are parameter. Consider the uniform grid distribution $a=x_1<x_2<\cdots<x_N=b$ with N *grid* points of the interval $[a,b]$ and the functional values of $U(x,t)$ at each grid and time t are defined as $U(x_i,t), i=1,\ldots,N$. The approximation to derivatives of $U(x,t)$ with respect to the variable x is

$$\frac{\partial U^{(r)}(x_i,t)}{\partial x^{(r)}} = \sum_{j=1}^{N} w_{ij}^{(r)} U(x_j,t) \tag{69}$$

Where $w^{(r)}_{ij}$ stands for the weighting coefficients of the approximation and r is the order of the derivative. According to the method, before approximate to derivative terms, the weighting coefficients $w^{(r)}ij$ should be computed by substituting the basis functions in Eq. (69) instead of $U(x_i,t)$.In this study both quartic and quantic B-spline function are considered to determine the weighting coefficients. First substitution of each quartic B-spline function into the DQM equation (69) for a fixed x_i and r give

$$\frac{d^{(r)}\varphi_m(x_i)}{dx^r} = \sum_{j=m-1}^{m+2} w_{ij}^{(r)} \varphi_m(x_j), \quad \text{m=-1,0,.......N+1} \tag{70}$$

In order to obtain all weighting coefficients $w^{(r)}_{ij}$, the system is solved for each grid x_i covering the whole problem domain. Same procedure is applied for quantic B-spline method.

Stability of quartic and quintic method has been discussed by matrix stability. The numerical examples have been solved and numerical approximation error is computed by use of the discretized maximum norm and L_2 norm defined by

$$L_\infty = \max_i \left| U_i^{exact} - U_i^{numeric} \right|$$
$$L_2 = (h \sum_{i=1}^{N} \left\| U_i^{exact} - U_i^{numeric} \right|)^{1/2} \tag{71}$$

5) Remarks: When the results are compared with least square elements and with analytic solution the result by both QRTDQ and QNTDQ seem more accurate. When QRTDQ and QNTDQ are compared QNTDQ is more accurate.

In paper [17] Lodhi and Mishra consider the following form of the problem:

$$Ly(x) = \varepsilon y^{(4)}(x) + \frac{p}{x} y''(x) + \frac{p}{x} y''(x) + \frac{r}{x} y'(x) + \frac{w}{x} y(x) = f(x) \qquad x \in (0,1) \tag{72}$$

$$y(0) = \alpha_0, \quad y''(0) = \alpha_1, \quad y(1) = \beta_0, \quad y''(1) = \beta_1, \quad \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathrm{R} \tag{73}$$

Eq. (72) has regular singularity at x = 0. Therefore, at x = 0, Eq. (72) has been simplify as

$$(\varepsilon + p) y^{(4)}(x) + q y'''(x) + r y''(x) + w y'(x) = f(x) \tag{74}$$

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Let S(x) be the B-spline interpolating the function y(x) at the nodal points and given as

$$S(x) = \sum_{i=-2}^{N+2} c_i B_i(x) \qquad (75)$$

where c_i are unknown coefficient and $B_i(x)$'s fifth degree B-spline functions. To solve fourth order boundary value problem, the spline function is evaluated at the nodal points $x = x_i$ by using the conditions of spline functions and continuity of derivative (N + 5) × (N + 5) system with (N + 5) unknowns is obtained which is solved by converting the system into matrix form.

The convergence analysis has been given and the method is shown to have uniform convergence of the second order. Two examples have been solved by the method.

6) Remark: In the proposed method there is no need to reduce the order of 4h order boundary value problem.

In paper [18] Pandey, has considered following third order boundary value problem

$$u'''(x) = f(x,u), \qquad a \le x \le b \qquad (76)$$

Subject to the boundary conditions

$$u(a) = \alpha, \quad u'(a)=\beta, \quad \text{and} \quad u'(b)=\gamma. \qquad (77)$$

Where α, β, γ are real constants. First divide the given interval into n equal sub intervals, then

The equation (76) at the node $x=x_i$ has been written as

$$u_i''' = f_i, \qquad a \le x \le b, \qquad (78)$$

And the boundary conditions become

$$u_0 = \alpha, \quad u_0'=\beta, \quad u_0'=\gamma. \qquad (79)$$

Let $S_i(x)$ be the approximate solution of equation (76) passing through $(x_{i-1/2}, S_{i-1/2})$ and $(x_{i+1/2}, S_{i+1/2})$ and given as

$$S_i(x) = c_{i0}(x - x_{i-1/2})^4 + c_{i1}(x - x_{i-1/2})^3 + c_{i2}(x - x_{i-1/2})^2 + c_{i3}(x - x_{i-1/2}) + c_{i4} \qquad (80)$$

where $c_{i0}, c_{i1}, c_{i2}, c_{i3}, c_{i4}$ are real finite constants.

After discretizing the problem (78) at the nodes a system of linear or nonlinear equations has been obtained and the gauss-seidel or Newton- Raphson method has been used for solving the system.

7) Remarks: The rate of the convergence of the method is quadratic. Numerical examples have shown the accuracy and efficiency of the method. The presented method can also be used for numerical solution of higher order BVP.

C. *Higher degree Splines:*

In this part the papers in which higher degree spline used to solve BVP's summarized.

In paper [19] Khandelwal, and Sultana, have obtained the numerical solution for the linear sixth-order two-point boundary value problems of the form

$$y^{(6)} + f(x)y = g(x), \qquad x \in [a,b] \qquad (81)$$

Subject to the boundary conditions:

$$\begin{aligned} y(a) &= \alpha_0, \quad y(b) = \alpha_1 \\ y(a) &= \beta_0, \quad y'(b) = \beta_1 \\ y''(a) &= \gamma_0, \quad y''(b) = \gamma_1 \end{aligned} \qquad (82)$$

Where $\alpha_j, \beta_j, \gamma_j$ are finite real arbitrary constants while f(x) and g(x) are continuous functions defined on the interval [a, b]. To develop the spline approximation to the problem (81) with the boundary conditions (82), first introduce a finite set of grid point $x_j=a+jh$, j= 0, 1, 2….n by dividing the interval [a, b] into n equal parts where x_0=a, x_n=b, $h=\frac{b-a}{n}$.

A function $S_\Delta(x,\tau)$ of class $C^6[a,b]$ which interpolates y(x) at the mesh points x_j depends on a parameter τ, reduces to ordinaryseptic spline $S_\Delta(x)$ in [a, b] as $\tau \to 0$ is termed as parametric septic spline function. Since the parameter τ can occur in $S_\Delta(x,\tau)$ in many ways such a spline is not unique. If $S_\Delta(x,\tau)$ is a parametric septic spline satisfying the following differential equation in the subinterval $[x_{j-1}, x_j]$

$$S_{\Delta}^{(6)}[x,\tau]+\tau^2 S_{\Delta}^{(4)}[x,\tau]=(Q_j+\tau^2F_j)\frac{x-x_{j-1}}{h}+(Q_{j-1}+\tau^2F_{j-1})\frac{x_j-x}{h}=V_jz+V_{j-1}\bar{z} \quad (83)$$

Where

$$z=\frac{x-x_{j-1}}{h},\quad \bar{z}=\frac{x_j-x}{h},\quad V_j=Q_j+\tau^2F_j,\quad S_{\Delta}^{(4)}[x_j,\tau]=F_j \quad (84)$$

$$S_{\Delta}^{(6)}[x_j,\tau]=Q_j$$

After solving eq. (83) and using derivative continuities following consistency relations are obtained

$$M_{j+1}+4M_j+M_{j-1}=\frac{6}{h^2}(y_{j+1}-2y_j+y_{j-1})+\frac{h^2}{60}(7F_{j+1}+16F_j+7F_{j-1})+6h^4(\alpha_2Q_{j+1}+2\beta_2Q_j+Q_{j-3})$$

$$M_{j+1}-2M_j+M_{j-1}=\frac{h^2}{6}(F_{j+1}+4F_j+F_{j-1})+h^4(\alpha_1Q_{j+1}+2\beta_1Q_j+\alpha_1Q_{j-1})$$

$$F_{j+1}-2F_j+F_{j-1}=h^2(\alpha Q_{j+1}+2\beta_1Q_j+\alpha Q_{j-1}) \quad (85)$$

Where

$$\alpha_2=\frac{7}{360\omega^2}+\frac{1}{\omega^2}\{\frac{1}{6\omega^2}-\frac{1}{\omega^4}(\omega\cos ec\omega-1)$$

$$\beta_2=\frac{8}{360\omega^2}+\frac{1}{\omega^2}\{\frac{1}{3\omega^2}-\frac{1}{\omega^4}(1-\omega\cot\omega)\} \quad (86)$$

$$\alpha=\frac{1}{\omega^2}(\omega\cos ec\omega),\qquad \beta=\frac{1}{\omega^2}(1-\omega\cot\omega)$$

$$\alpha_1=\frac{1}{\omega^2}(\frac{1}{6}-\alpha),\qquad \beta_1=\frac{1}{\omega^2}(\frac{1}{3}-\beta)$$

By using these consistency relations following relation is obtained in terms of sixth derivative of spline Q_j and y_j.

$$y_{j-3}-6y_{j-2}+15y_{j-1}-20y_j+15y_{j+1}-6y_{j+2}+y_{j+3}=h^{(6)}(pQ_{j-3}+qQ_{j-2}+rQ_{j-1}+sQ_j+rQ_{j+1}+qQ_{j+2}+pQ_{j+3}) \quad (87)$$

Where $Q_j=-f_iy_j+g_j$, with $f_j=f(x_j)$, $g_j=g(x_j)$, j=3,4…..n-3 and

$$p=\left(\frac{\omega-\sin\omega}{\omega^6\sin\omega}-\frac{1}{6\omega^3\sin\omega}+\frac{1}{120\omega\sin\omega}\right)$$

$$q=\left(\frac{6}{\omega^6}-\frac{4+2\cos\omega}{\omega^5\sin\omega}-\frac{1-\cos\omega}{3\omega^3\sin\omega}+\frac{13-\cos\omega}{60\omega\sin\omega}\right) \quad (88)$$

$$r=\left(\frac{-15}{\omega^6}-\frac{7+8\cos\omega}{\omega^5\sin\omega}+\frac{5+4\cos\omega}{6\omega^3\sin\omega}+\frac{67-52\cos\omega}{120\omega\sin\omega}\right)$$

$$s=\left(\frac{20}{\omega^6}-\frac{8+12\cos\omega}{\omega^5\sin\omega}-\frac{2+6\cos\omega}{3\omega^3\sin\omega}+\frac{13-33\cos\omega}{30\omega\sin\omega}\right)$$

As $\omega\to 0$, then $(\alpha,\beta,\alpha_1,\beta_1,\alpha_2,\beta_2)\to(\frac{1}{6},\frac{1}{3},\frac{-7}{360},\frac{-8}{360},\frac{-31}{15120},\frac{-7}{945})$ and (p, q, r, s) $\rightarrow$ $\left(\frac{1}{5040},\frac{120}{5040},\frac{1191}{5043},\frac{2416}{3040}\right)$ then, the spline defined by (88) reduces to an ordinary septic spline and the spline relations reduce to corresponding ordinary septic spline relations.

The relation (87) gives (n-5) linear algebraic equations in (n-1) unknowns so four more relations has been obtained two at each end of range of integration. That will complete the solution. Convergence of the method has been discussed by standard procedure and shown that the method is sixth order convergent.

1) Remarks: The_numerical results are compared with the decomposition method and other spline methods which shows superiority of the presented method over the these methods

In paper [20] Akram has used sextic spline function to develop a technique for the solution of the following system

$$y^{(5)}(x)=\begin{cases} f(x) & a\le x\le c \\ f(x)+y(x)\,g(x)+r & c\le x\le d \\ f(x) & d\le x\le b \end{cases} \tag{89}$$

Along with the boundary conditions

$$\begin{aligned} &y(a)=y(b)=\alpha_0 \quad , \quad y^{(1)}(a)=y^{(1)}(b)=\alpha_1 \\ &y(c)=y(d)=\alpha_2 \quad , \quad y^{(1)}(c)=y^{(1)}(d)=\alpha_3 \\ &y^{(2)}(a)=\alpha_4 \qquad , \quad y^{(2)}(c)=y^{(2)}(c)=\alpha_5 \end{aligned} \tag{90}$$

Where r and a_i, i=0, 1, . . .5 are finite real constants and the functions f(x) and g(x) are continuous on [a, b] and [c, d] respectively. Such type of systems arises in connection with contact, obstacle and unilateral problems. To develop the sextic spline approximation S to the problem (89) the interval [a, b] is divided into k equal sub intervals (such that k is divisible by 4), using the grid points x_i= a +ih, i=0, 1. . .. k, where h=b-a/n. The restriction S_i of S to each subinterval [x_i, x_{i+1}] i=0,1…k-1 has been defined as

$$S_i(x)=a_i(x-x_i)^6+b_i(x-x_i)^5+c_i(x-x_i)^4+d_i(x-x_i)^3+e_i(x-x_i)^2+f_i(x-x_i)+g_i \tag{91}$$

For

$$\begin{aligned} &S_i(x_i)=y_i, \quad S_i^{(1)}(x_i)=m_i \\ &S_i^{(5)}(x_i)=t_i, \quad S_i^{(3)}(x_i)=n_i \end{aligned} \tag{92}$$

And assuming y(x) to be the exact solution of the system (89) and y_i be an approximation to y(x_i) obtained by the spline S(x_i). Now applying the second, third and fourth derivative continuities at the knots following consistency relations are obtained

$$t_{i-3}+57t_{i-2}+302t_{i-1}+302t_i+57t_{i+1}+t_{i+2}=\frac{-720}{h^5}\left(y_{i-3}-5y_{i-2}+10y_{i-1}-10y_i+5y_{i+1}-y_{i+2}\right) \tag{93}$$

For i=3, 4…k-2. The end conditions corresponding to the system (91) with (92) are determined as

$$\begin{aligned} &\text{(i)} \sum_{k=i-1}^{i+3} b_k y_k + c_0 h y_{i-1}^{(1)} + d_0 h^5 y_{i-1}^{(5)} + h^5 \sum_{k=i-1}^{i+3} d_k t_k = 0, \quad \text{i=1,n+1, 3n+1} \\ &\text{(ii)} \sum_{k=i-1}^{i+3} e_k y_k + c_1 h y_{i-2}^{(1)} + d_1 h^2 y_{i-2}^{(2)} + h^5 \sum_{k=i-1}^{i+3} I_k t_k = 0, \quad \text{i=2,n+2, 3n+2} \\ &\text{(iii)} \sum_{k=i-1}^{i+3} m_k y_k + c_2 h y_{i+1}^{(1)} + d_2 h^5 y_{i+1}^{(5)} + h^5 \sum_{k=i-1}^{i+3} n_k t_k = 0, \quad \text{i=n-1,3n-1, 4n-1} \end{aligned} \tag{94}$$

where b_k, d_k,e_k,y_k,t_k,I_k,n_k i=0,1,2 are arbitrary parameters to be determined.

Numerical examples also illustrate the accuracy of the method and results shows that the method is second order convergent. The application of the method has been described.

2) Remarks: the presented sextic spline method is better than quartic spline method and a powerful mathematical tool for the solution of system of fifth order BVPs.

In paper [21] Kalyani,and Lemma considered the linear seventh order differential equation

$$y^{(7)}+f(x)y(x)=r(x), \qquad x\in[a,b] \tag{95}$$

with the boundary conditions

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$$y(x_0)=\alpha \quad, y(x_n)=\beta \quad, y'(x_0)=\alpha' \,, \quad y'(x_n)=\beta' \tag{96}$$

$$y''(x_0)=\alpha'' \quad, y''(x_n)=\beta'' \quad, y'''(x_0)=\alpha''' \,, \quad y'''(x_n)=\beta''' \tag{97}$$

First the authors have defined a ninth degree spline $S(x)$, which is an approximate solution of $y(x)$ and is represented in the form

$$S(x)=a+b(x-x_0)+c(x-x_0)^2+d(x-x_0)^3+e(x-x_0)^4+g(x-x_0)^5+h(x-x_0)^6+ \\ j(x-x_0)^7+k(x-x_0)^8+\sum_{i=0}^{n-1} l_i(x-x_0)^9 \tag{98}$$

$S(x)$ and its first six derivatives are continuous across nodes. now taking spline approximation in (1) at $x_i= x$, for $i= 0,1, 2,3, 4\ldots. n$ $(n + 8)$ equations has been obtained in $(n + 9)$ unknowns a,$b,c,d,e, g,h, j, k,l_0,l_1\ldots\ldots l_{n-1}$. To find the values of all unknowns one more equation is required for this the eq. will be $l_{n-1}=l_{n-2}$, that will convert the system into solvable form.

3) Remark: When the results obtained by proposed method are compared with the results obtained by eighth degree spline functions with different step length it was observed that the ninth degree spline solutions are more accurate and the accuracy increase when small step length has been used.

In paper [22] Zahra and Ashraf have considered the two-parameter singularly perturbed semi-linear boundary value problem of the following form:

$$Ly(x)=-\varepsilon_d y''+\varepsilon_c y'+f(x,y)=0, \qquad x\in(0,1) \tag{99}$$

where the boundary conditions are

$$y(0)=\mu_1 \quad, y(1)=\mu_2. \tag{100}$$

with two small parameters $0<\varepsilon_c<<1$, $0<\varepsilon_d<< 1$, where p(x) and f(x,y) are sufficiently smooth functions. In the presented method piecewise uniform Shishkin mesh has been used.The exponential function has been defined as

$$Q_i(x)=a_i e^{k(x-x_i)}+b_i e^{-k(x-x_i)}+c_i(x-x_i)+d_i, \qquad i=0,1,....n \tag{101}$$

After that consistency relation has been obtained that gives n-1equations in n-1 unknowns. The proposed singularly perturbed problem has been discretized at $x=x_i$ as

$$-\varepsilon_d(S_{i-1}-2S_i+S_{i+1})+\frac{\varepsilon_c h}{2}(D_i S_{i-1}+E_i S_i+A_i S_{i+1})=-h^2(\alpha f_{i-1}+\beta f_i+\alpha f_{i+1}) \tag{102}$$

4) Remarks: The convergence of the method is the main highlight of the given method because the convergence does not depend upon the perturbation parameters.

In continuitation Exponential B-Spline has been used in paper [23] by Mohammadi to find the solution of Convection-Diffusion Equation which is given as

$$\frac{\partial u}{\partial t}+\varepsilon\frac{\partial u}{\partial x}=\gamma\frac{\partial^2 u}{\partial x^2} \quad, 0\le x\le L\,, 0\le t\le T \tag{103}$$

subject to the initial conditions

$$u(x,0)=\phi(x) \quad, \qquad 0\le x\le L \tag{104}$$

and also with appropriate Drichlet boundary conditions

$$u(0,t)=g_0(t), \qquad u(L,t)=g_1(t) \quad, \quad 0\le t\le T. \tag{105}$$

where the parameter γ is the viscosity coefficient and ε is the phase speed and both are assumed to be positive. $\phi(x)$, $g_0(t)$, $g_1(t)$are known functions with sufficient smoothness. The region [0, L] is partitioned into n finite elements of equal length h by knots x_i such that $0=x_0< x_1<\ldots\ldots x_n=L$. let B_i, i=-1,0.... .n+1 be the exponential spline with both knots x_i and two additional knots outside the region. An approximate solution $u_n(x,t)$ to the analytical solution $u(x, t)$ will be sought in form of an expansion of B-splines:

$$u_n(x,t)=\sum_{i=-1}^{n+1}\delta_i(t)B_i(x) \tag{106}$$

where δ_i are time dependent parameters to be determined from the exponential B-spline collocation form. The nodal value u and its first and second derivative at the knots x_i are obtained. The author has replaced the time derivative by finite difference representation

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and the first-order space and second-order space derivatives by exponential B-spline. The author showed that the proposed method is unconditionally stable by using the Von Neumann method

5) Remarks: In the proposed method Crank-Nicolson formulation has been used for time integration and exponential B-spline functions for space integration. The method has improved the accuracy of B-spline method by involving some parameters, which enable us to obtain the classes of methods. Application of the proposed method is simple in comparison with the other well-known methods.

In 2015Akram [24] [has used exponential spline technique for solving fractional boundary value problem of 2nd order of the form

$$y''(x) + D^{\alpha} y(x) = f(x) \qquad x \in [0,1] \qquad m-1<\alpha<m \tag{107}$$

subject to

$$y(0) = 0,\ y(1) = 0, \tag{108}$$

where $f(x)$ is continuous function on the interval [0, 1] and D^{α} denotes fractional derivative in Caputo's sense. Let $x_i = a + ih$ (i= 0, 1, . . . n, h = b–a/n be grid points of the uniform partition of [a, b] into the subintervals [x_{i-1}, x_i]. Let $y(x)$ be the exact solution of Eq. (107) and S_i be an approximation to $y_i = y(x_i)$ obtained by the exponential spline function ψ_i passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}).the author has defined in each subinterval the exponential spline segment in each subinterval as:

$$\psi_i = a_i e^{k(x_{i+1}-x)} + b_i e^{-k(x_{i+1}-x)} + c_i (x_{i+1} - x) + d_i \qquad i=0,1...n-1 \tag{109}$$

where a_i, b_i, c_i and d_i are undetermined coefficients. These coefficients are expressed in terms of S_i and M_i, as

$$\begin{aligned} &\psi_i(x_i) = S_i \ , \qquad \psi_i(x_{i+1}) = S_{i+1} \ , \\ &\psi_i''(x_i) = M_i \ , \qquad \psi_i''(x_{i+1}) = M_{i+1} \ , \end{aligned} \tag{110}$$

After find the values of unknowns and by using the continuity of derivative following relations are obtained

$$S_{i+1} - 2S_i + S_{i-1} = h^2 [M_{i+1}(\frac{1}{\theta^2} - \frac{1}{\theta \sinh \theta}) + M_i(-\frac{2}{\theta^2} + \frac{2\coth\theta}{\theta}) + M_{i-1}(\frac{1}{\theta^2} - \frac{1}{\theta \sinh \theta})] \tag{111}$$

The eq. (111) gives (n-1) linear algebraic equations in n-1 unknowns which can be solved easily.

6) Remarks: The convergence of the method is discussed and the order of convergence is $O(h^{2-\alpha})$. The application of the problem also shown with the help of some examples.

In the same year exponential B-splines has been used with Galerkin Method by Gorgulu,Dag and Irk [25] to solve the regularized long wave (RLW) equation of the form

$$u_t + u_x + \varepsilon u u_x - \mu u_{xxt} = 0 \tag{112}$$

where x is space coordinate, t is time, u is the wave amplitude and ε and μ are positive parameters. Boundary and initial conditions of the equation (112) are

$$\begin{aligned} &u(a,t) = \beta_1, \quad u(b,t) = \beta_2, \\ &u_x(a,t) = 0, \quad u(b,t) = 0, \ t \in (0,T] \\ &u(x,0) = f(x), \quad x \in [a,b]\ . \end{aligned} \tag{113}$$

Applying the Galerkin method to the RLW equation with the exponential B-splines as weight function over the element [a, b] gives

$$\int_a^b B_i(x)(u_t + u_x + \varepsilon u u_x - \mu u_{xxt}) = 0 \tag{114}$$

The approximate solution U_N over the element [x_m , x_{m+1}] can be written as

$$U_N^e = B_{m-1}(x)\delta_{m-1}(t) + B_m(x)\delta_m(t) + B_{m+1}(x)\delta_{m+1}(t) + B_{m+2}(x)\delta_{m+2}(t) \tag{115}$$

where quantities $\delta_j(t)$, j=m-1….. m+2 are element parameters and $B_j(x)$,j = m-1…..m + 2 are known as the element shape functions

7) Remarks: The author has tested the method on the propagation of the single solitary wave, the interaction of two solitary waves and wave generation. The numerical examples have been solved by this method and the accuracy of the results has been checked by L_∞ error norm.

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In paper [26] Jha, Mohanty and Chauhan considered the fourth order boundary value problem that is expressed in the coupled form as

$$-u^{2}(x)+v(x)=0$$
$$-v^{2}(x)+\Upsilon(x,u(x),u'(x),v(x),v'(x))=0 \qquad (116)$$
$$u(\alpha)=\alpha_{**}\ ,\quad \mathrm{v}(\alpha)=\alpha_{**}\ ,u(\beta)=\beta_{**}\ ,\quad \mathrm{v}(\beta)=\beta_{**}\ ,$$

The author has been derived non-polynomial spline basis and finite difference discretization's to solve above problem. The method has convergence of third order.

Remarks:

The main feature of the presented method is that it can be apply for the solution for the linear, nonlinear, singular, non-singular problems of higher even order also.

In paper [27] Pedas and Tamme have studied the convergence behavior of a high order numerical method for the solution of nonlinear boundary value problems of the form

$$(D_{*}^{\alpha}y)(t)=f(t,y(t))\ ,\qquad \mathrm{o}\le \mathrm{t}\le \mathrm{b}, \qquad (117)$$

$$\sum_{j=0}^{n_0}\alpha_{ij}y^{(j)}(0)+\sum_{j=0}^{n_1}\beta_{ij}y^{(j)}(b_1)=\gamma_i,\quad 0\le \mathrm{b}_1\le \mathrm{b},\ \ \mathrm{i=0,......n\text{-}1}. \qquad (118)$$

8) Remarks: The authors have used integral equation reformulation of the problem and piecewise polynomial approximations on special non-uniform grids reflecting the possible singular behaviour of the exact solution. The convergence of the method has been analyzed and numerical examples have been solved by the proposed method.

In paper [28] Aminikhah and Aavi used B-spline collocation and quasi-interpolation methods for solving boundary layer flow and convection heat transfer over a flat plate which in mathematical form can be written as

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0$$
$$\theta''(\eta)+\frac{\mathrm{Pr}}{2}f(\eta)\theta'(\eta)=0 \qquad (119)$$

And the boundary conditions are

$$f(0)=f'(0)=\theta(0)=0,\quad \mathrm{f}''(0)=\theta'(0)=\sigma \qquad (120)$$

First the author has solved the problem by quartic b-spline method as usual which has convergence of order four. In the B-spline quasi interpolation method, first the jth B-spline of degree d for knot sequence $X_N=\{x_j\}_{j=-d}^{+d}$ has been defined as

$$BX_{N,j,d}(r)=\omega X_{N,j,d}BX_{N,j,d-1}(r)+(1-\omega X_{N,j+1,d})BX_{N,j,d-1}(r) \qquad (121)$$

Where

$$\omega X_{N,j,d}(r)=\frac{r-x_j}{x_{j+d-1}-x_j},\qquad \mathrm{BX}_{N,j,0}(r)=\begin{cases}1 & \mathrm{x}_j\le \mathrm{r}\le x_{j+1}\\ 0 & \text{otherwise}\end{cases} \qquad (122)$$

The univariate B-spline quasi-interpolation has been defined as operators of the form

$$Q_{d,f}=\sum_{j=1}^{n+d}\mu_j(f)\,\mathrm{B}_j \qquad (123)$$

The equations (121) and (123) have been applied on first order ordinary differential equation a nonlinear system of n nonlinear equations has been formed whose solution has been obtained by trust-region-dogleg method. This method has been proved of fifth order convergent.

9) Remarks: The presented method has always leaded to a system of nonlinear equations which can be solved easily. The advantage of the method is that when the step size has been reduced more accurate solution has been obtained.

In paper [29] Hamasalh and Muhammad considered a new fractional spline of non-polynomial form to solve

$$(D^{2\alpha}+\eta D^{\alpha}+\mu)y(x)=f(x),\quad \alpha\mathrm{=1.5}\quad \mathrm{x}\in\mathrm{[a,b]} \qquad (124)$$

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Subject to boundary conditions

$$y(a) = y(b) = 0 \qquad (125)$$

where η, μ are all real constants. The function f(x) is continuous on the interval [a, b] and the operator D^{α} represents the Caputo fractional derivative. When α = 1, then equation (125) has been reduced to the classical second order boundary value problem. the author has derived a new fractional spline method by using non- polynomial spline which is defined as

$$P_i(x) = a_i + b_i(x - x_i)^{\alpha} + c_i \sin_{\alpha} k(x - x_i)^{\alpha} + d_i \cos_{\alpha} k(x - x_i)^{3/2} \qquad (126)$$

Where a_i, b_i, c_i, d_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method. The values of unknowns have been obtained and by using the continuity condition $D^{2\alpha} P_{i-1}(x_i) = D^{2\alpha} P_i(x_i)$ following consistency relation has been obtained

$$\frac{1}{h^{2\alpha}}(y_{i+1} - 2y_i + y_{i-1}) = \lambda M_{i+1} + 2\beta M_i + \lambda M_{i-1}\,. \qquad (127)$$

10) Remarks: Convergence of the proposed method is discussed and numerical examples have been solved by the method.

In paper [30] authors have presented a new class of Linear Multistep Method (LMM) based on B-spline method for this they considered following boundary value problem

$$\Im(y)(t) = f(t, y), \qquad t \in [a,b] \;\; y(a)=y_a, \;\; y(b) = y_b. \qquad (128)$$

The coefficients characterizing the main k-step BS2 method for the second order BVP in (128) are given by

$$\alpha_i^{(2)} := B''(k+1-i), \;\; \alpha_i^{(1)} := B'(k+1-i), \;\; \beta_i := B(k+1-i), \;\; i=0,....k \qquad (129)$$

where B is the cardinal B-spline of degree k+1 (k≥2) with integer knots 0,….k+2. After defining the spline of degree k+1 and state that any spline s_h of degree k+1 and knots at the mesh points satisfies the following relation

$$\sum_{i=-k_1}^{k_2} \alpha_{i+k_1} s_h(t_{j+i}) = h^2 \sum_{i=-k_1}^{k_2} \beta_{i+k_1} \Im(s_h)(t_{j+i}), \qquad j=k_1.....n-k_2. \qquad (130)$$

11) Remarks: The authors have analyzed the 2-step BS2 method. The method has absolute stability for k=2 and k=4. Two examples have been solved.

In paper [31] the authors have considered the 1D quasilinear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = A(x,t,u)\frac{\partial^2 u}{\partial x^2} + f(x,t,u,u_x,u_t)\,,\; 0 < x < 1,\; t > 0\;, \qquad (131`)$$

The initial conditions are given by

$$u(x,0) = \phi(x), u_t(x,0) = \psi(x), \;\; 0 \le x \le 1\;, \qquad (132)$$

and the boundary conditions are given by

$$u(0,t) = a_0(t),\; u(1,t) = a_1(t), t \ge 0\,. \qquad (133)$$

We assume that the functions f (x, t, u, u_x, u_t), ϕ(x), ψ(x), a_0(t) and a_1(t) are sufficiently smooth and their required higher order derivatives exist. The wave equations are important second order hyperbolic partial differential equations for the description of waves as they occur in most scientific and engineering disciplines such as sound waves, light waves, water waves, acoustics waves, electromagnetic waves, and fluid dynamics, optics, electromagnetism, solid mechanics, structural mechanics, quantum mechanics, etc. first the author has discussed spline in compression and its properties and then described the new three level implicit method based on spline in compression approximation. To solve equation (131) the author has used spline in a tension function by approximating of its first order space derivative in *x*-direction and central difference approximations of time derivative in *t*-direction.

12) Remarks:The proposed method is of $o(k^2+h_l k^2+h_l^3)$ which is better than the available numerical methods based on spline in compression approximations for the numerical solution of second order quasilinear hyperbolic equations on a variable mesh are which are of $o(k^2+h_l)$ accuracy only. The author has also discussed the application of the proposed method to a wave equation with singular coefficients. Stability analysis of a linear scheme and convergence analysis of a general nonlinear scheme has also been discussed in this paper.

In paper [32] the fifth-order boundary value problem (BVP) of the following form has been considered

$$\begin{aligned} &y^{(5)}(x)+f(x)y(x)=g(x)\ ,\qquad x\in[a,b]\\ &y(a)=\alpha_0,\quad y(b)=\beta_0\\ &y'(a)=\alpha_1,\quad y'(b)=\beta_1\\ &y''(a)=\alpha_2. \end{aligned} \tag{134}$$

where a_0, a_1, a_2, b_0 and b_1 are finite real constants, also f(x)and g(x) are continuous on [a,b]. Consider the following restriction S_i of S to each subinterval $[x_i, x_{i+1}]$ i=0, …. n-1.

$$S_i(x)=a_i\left\{\cos k(x-x_i)+e^{(x-x_i)}\right\}+b_i(x-x_i)^5+c_i(x-x_i)^4+d_i(x-x_i)^3+e_i(x-x_i)^2+f_i(x-x_i)+g_i \tag{135}$$

From (135) the values of unknowns have been determined and by using the end term conditions solution has been obtained. To check the accuracy of the proposed method, three boundary value problems are considered in this section and maximum absolute error has been calculated for each examples. The order of truncation error has been discussed with the given boundary conditions and also with the improved boundary conditions.

13) Remarks: In the present paper the numerical examples solved with given end conditions and it was observed that the proposed method is second order convergent but when the same examples are solved with improved end conditions which shows that method is fifth order convergent.

In paper [33] Geetha and Tamilselvan considered A differential equation with a small positive parameter ε (0 < ε <<1*)* multiplying the highest derivative term subject to boundary conditions belongs to a class of problems known as singular perturbation problems (SPPs). Singularly perturbed boundary value problems appear in many branches of applied mathematics, like fluid dynamics, quantum mechanics, turbulent interaction of waves and currents, elasticity, gas porous electrodes theory, lubrication theory etc. They considered the following Singularly perturbed turning point problems (SPTPPs)

$$Lu=\varepsilon u''(x)+a(x)u'(x)-b(x)u(x)=f(x),\qquad \forall x\in\Omega=(-1,1) \tag{136}$$

with Robin boundary conditions

$$B_1u(-1)=\beta_1u(-1)-\varepsilon\beta_2u'(-1)=A,\quad B_2u(1)=\gamma u(1)+\varepsilon\gamma_2u'(1)=B \tag{137}$$

where ε is a small positive parameter, *a(x)*, *b(x)* and *f (x)* are sufficiently smooth functions. The differential equation has been discretized by cubic spline function and The boundary conditions (34) are approximated as follows:

$$B_1u(x_0)=\beta_1u(x_0)-\varepsilon\beta_2D^0u(x_0)=A,\quad B_2u(x_N)=\gamma u(x_N)+\varepsilon\gamma_2D^0u(x_N)=B \tag{138}$$

After some calculation a tridiagonal system has been obtained that is solved by well-known methods

14) Remarks: In this method the differential equation is approximated by the cubic spline functions whereas the boundary conditions are discretized by the central difference operator. Two numerical experiments show that the proposed method is of second order convergence. Stability of the method has been discussed

In paper [34] a new numerical method for the numerical solution of quasi-linear elliptic boundary value problems (EBVPs) with significant first order partial derivatives has been presented. The proposed equation possesses the following form

$$A(x,y,U)\partial_x{}^2U+B(x,y,U)\partial_y{}^2U=G(x,y,U,\partial_xU,\partial_yU),\qquad (x,y)\in\Omega \tag{139}$$

where the values of $U(x,y)$ at the boundary $\partial\Omega$ are known. Such EBVPs frequently occur in combustion theory, plasma physics, steady state heat and mass transfer equation with volume reaction, steady transonic gas flow, mass transfer with a volume chemical reaction in translational shear fluid flow, stationary anisotropic diffusion equation etc. the non-polynomial spline $S_m(x)$ has been taken as

$$S_m(x)=a_i\sinh\omega(x-x_l)+b_i\cosh\omega(x-x_l)+c_i(x-x_l)+d_i \tag{140}$$

Satisfying the following properties

$$S_m(x_l) = U_{l,m}, \qquad S_m(x_{l-1}) = U_{l-1,m},$$
$$\left(\frac{d^2 S_m(x)}{dx^2}\right)_l = \mathrm{M}_{i,m}, \quad \left(\frac{d^2 S_m(x)}{dx^2}\right)_{l-1} = \mathrm{M}_{i-1,m}, \tag{141}$$

Some algebraic has been done to find the values of the unknowns.

15) Remarks: The present method has been implemented on examples whose results show that the method is better than cubic polynomial spline scheme in terms of iteration number and consequently the computing time. The proposed method can be extended to three-dimensional non-linear EBVPs. The proposed numerical method is the limiting case of non-polynomial spline parameter provides the cubic spline method. The convergence of the proposed method has been discussed using matrix theory.

In paper [35]authors used the method which is the combination of two methods (mixed decomposition method, spline collocation method) for obtaining a numerical solution of the following singular boundary-value problem

$$(x^{\alpha} y')' = f(x, y) \tag{142}$$

Subject to boundary conditions

$$y(0) = A, \qquad y(1)=\mathrm{B}. \tag{143}$$

where 0<α<1, 0<x<1 and A, B are finite real constants. To solve the above problem the author has used joint decomposition spline mixed technique that divide the domain Ω= [0,1] into two subintervals as $\Omega=\Omega_1 \cup \Omega_2$ =[0, δ]∪[δ,1]. In the domain which contain singularity modified decomposition method has been used and in the domain Ω_2 an adaptive spline collocation method over a uniform or non-uniform mesh has been utilized. The non-uniform fourth order spline collocation method is applicable only for linear problem so to apply the adaptive technique on the nonlinear equation (142) and (143) by using the Newton method the subsequent iteration has been obtained

$$\begin{aligned} &(x^{\alpha} y'_m)' - f_y(x, y_{m-1}) y_m = f(x, y_{m-1}) - f_y(x, y_{m-1}) y_{m-1} \\ &y(0) = A, \qquad \mathrm{y(1)=B} \end{aligned} \tag{144}$$

16) Remarks: The main advantage of the method is that it resolves the problem of singularity by merging the two methods, a modified decomposition technique applied on a small interval near the singularity and a fourth order spline collocation technique which is used on the remaining part of the domain of the problem.

In paper [36] authors have considered following class of linear and non-linear singular boundary value problems:

$$y^{(2r)}(x) + \frac{k_1}{x} y'(x) + \frac{k_2}{x^2} y(x) = f(x, y(x), y'(x) \ldots\ldots y^{2r-1}(x)), \qquad 0 \le x \le 1 \tag{145}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = y'(0) = 0 \\ &y^{(j)}(0) = \alpha_j, \qquad j=2,3.....,r-1 \\ &y^{(i)}(1) = \beta_i, \qquad i=0,1,.........r-1. \end{aligned} \tag{146}$$

To solve above problem the equation (145) has been modified at singular point x=0 as

$$y^{(2r)}(0) + (k_1 + \frac{k_2}{2}) y''(0) = f(0, y(0), y'(0) \ldots\ldots\ldots y^{(2r-1)}(0)) \tag{147}$$

The approximate solution of (147) is given by

$$y(x_i) = \sum_{j=-d}^{n-1} c_j B_j(x_i), \tag{148}$$

where c_j are unknown real coefficients and $B_j(x)$ are the (2r+1)-degree B-spline functions. After some calculation a linear system of $n + 2r + 1$ equation of the $n + d$ unknown coefficients, has been obtained and solved this linear system by Q-R method. Same procedure has been applied to the nonlinear singular higher order boundary value problem.

The error analysis of the septic and nonic spline has been discussed. The numerical examples have been solved by the proposed method.

17) Remarks: The author presented a method for solving singular linear and nonlinear higher-order boundary value problem. This method is easy to implement and yields the desired accuracy and numerical results demonstrate this.

III.CONCLUSIONS

In this paper various spline based methods to solve different type of boundary value problem (ordinary as well partial differential equation) discussed in detail. the convergence of the methods discussed in various papers which shows that spline based methods are easy in implementation and gives better results than other methods used to solve boundary value problem such as finite difference method, variational iteration method Adomian Decomposition method etc. Convergence of the spline based methods is very good. On the basis of above discussion new researchers may get help to improve or modification in the methods to get more accuracy.

IV.ACKNOWLEDGMENT

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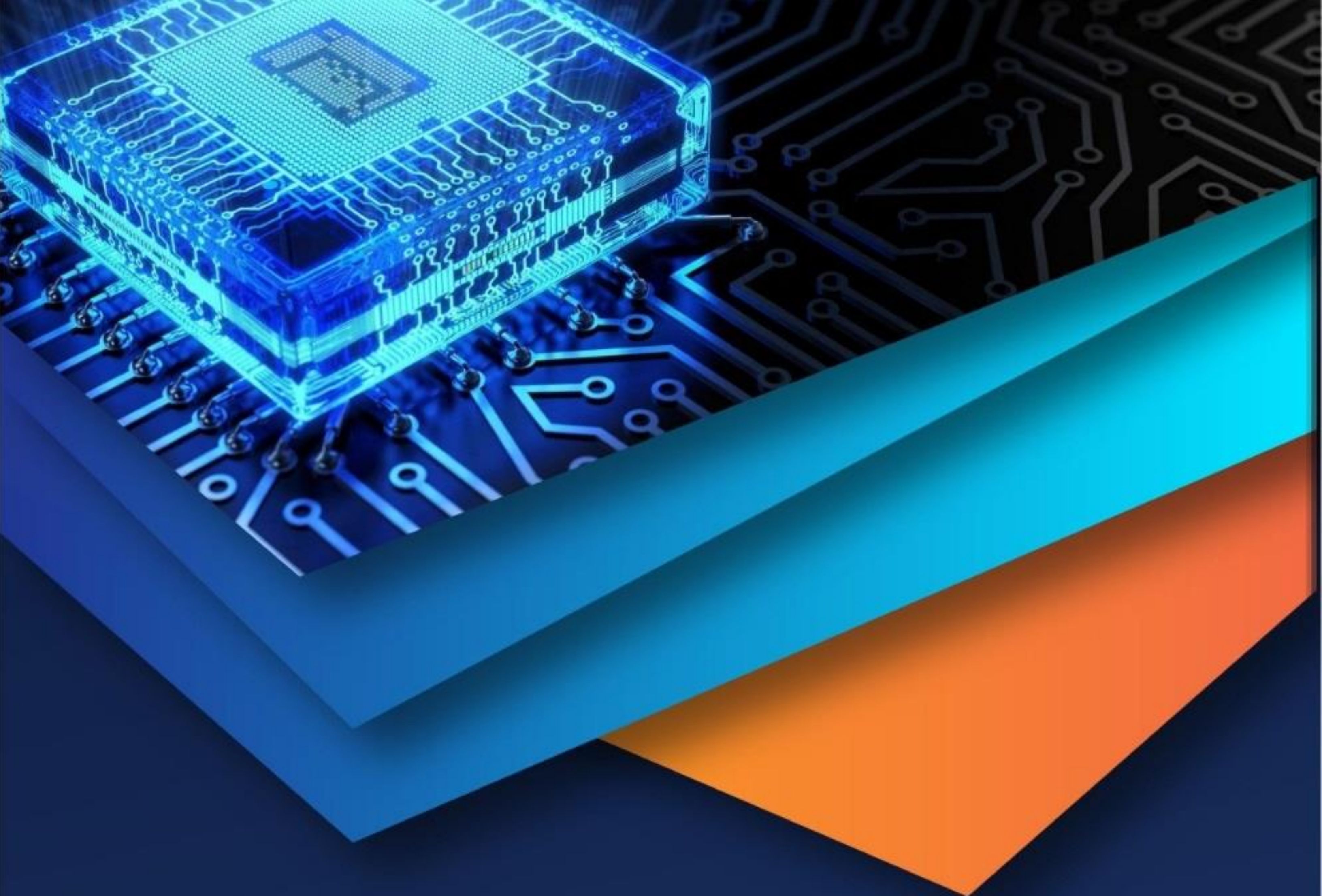

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