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On Reducibility of Certain q-Double Hypergeometric Series and Clausen Type Identities

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Abstract: In this paper, we have made use of certain known summations to establish transformations of q-double series in terms of single series. We have deduced Clausen type identities from these results.
Keywords : Hypergeometric functions, Summations, Transformation, Identities and Convergence.

I. INTRODUCTION

For α, real or complex and $|q| < 1$, we define the q-shifted factorials by

$$[\alpha; q]_n = \begin{cases} 1 & if\ n = 0 \\ (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}), & if\ n = 1,2,3,\dots \end{cases} \tag{1.1}$$

A basic hypergeometric function is defined as:

$${}_r\Phi_s \begin{bmatrix} a_1, a_2, \dots, a_r; q; z \\ b_1, b_2, \dots, b_s; q^\lambda \end{bmatrix}$$
$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n q^{\lambda n(n-1)/2}}{[[q, b_1, b_2, \dots, b_s; q]]_n}, \tag{1.2}$$

Where $[a_1, a_2, \dots, a_r; q]_n = [a_1 : q]_n\ [a_2; q]_n \dots [a_r; q]_n$.

The series ${}_r\Phi_s$ converges absolutely for all z if $\lambda > 0$ and for $|z| < 1$ if $\lambda = 0$. we shall use the following series identity to establish our results.

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} B(n,k) = \sum_{n,k=0}^{\infty} B(n+k,k), \tag{1.3}$$

Provided the series on both sides of (1.3) exist.

II. NOTATIONS AND DEFINITIONS

Notations and definitions appearing in this paper have their usual meaning. We shall use the following known summations of q-series in our analysis:

$${}_2\Phi_1 \begin{bmatrix} q^{-n}, a; q; zq^n/a \\ c \end{bmatrix} = \frac{[c/a; q]_n}{[c; q]_n}. \tag{2.1}$$

$${}_3\Phi_2 \begin{bmatrix} a, b, q^{-n} & ; q; q \\ c, abq^{1-n}/c & \end{bmatrix} = \frac{[c/a, c/b; q]_n}{[c, c/ab; q]_n} \tag{2.2}$$

$${}_2\Phi_1 \begin{bmatrix} x, q^{-n} & ; q; -q/x \\ q^{-n}/x & \end{bmatrix} = \frac{[q;q]_n [x^2q^2; q^2]_m}{[xq;q]_n [q^2;q^2]_m}, \tag{2.3}$$

Where m is the greatest integer $\leq n/2$.

$${}_4\Phi_3 \begin{bmatrix} q^{-n} & , -q^{-n}/xy, x, y & ; q; q \\ -xyq, q^{-n}/x, q^{-n}/y & & \end{bmatrix}$$
$$= \frac{[q, xyq; q]_n [x^2q^2, y^2q^2; q^2]_m}{[xq, yq; q]_n [q^2, x^2y^2q^2; q^2]_m}, \tag{2.4}$$

Where m is the greatest integer $\leq n/2$

$${}_2\Phi_1 \begin{bmatrix} x, q^{-n}; q; -1/x \\ q^{-n}/x \end{bmatrix} = \frac{[q;q]_n [x^2q^2;\ q^2]_m x^{n-2m}}{[xq;q]_n [q^2;q^2]_m}, \tag{2.5}$$

Where m is the greatest integer $\leq n/2$.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, -q^{-n}/xy, xq, yq ; q; q \\ -xyq, q^{1-n}/x, q^{1-n}/y \end{matrix}\right] = \frac{(-)^n[q;q]_n[xyq;q]_n[x^2q^2,y^2q^2;q^2]_m}{q^n[x,y;q]_n[q^2,x^2y^2q^2;q^2]_m}, \tag{2.6} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, -q^{-n}/x^2, y, -y ; q; q \\ q^{-n}/x, -q^{-n}/x, y^2 q \end{matrix}\right] = \frac{[q;q]_n[x^2y^2q^2;q^2]_n[x^2q^2,y^2q^2;q^2]_m}{[x^2q^2;q^2]_n[y^2q;q]_n[q^2,x^2y^2q^2;q^2]_m}, \tag{2.7} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0; q; q \\ q^{-n}/x, -q^{-n}/x \end{matrix}\right] = \frac{[q;q]_n[x^2q^2;q^2]_m}{[x^2y^2;q^2]_n[q^2;q^2]_m}, \tag{2.8} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0; q; 1 \\ q^{-n}/x, -q^{-n}/x; q \end{matrix}\right] = \frac{[q;q]_n[x^2q^2;q^2]_m q^{n(n+1)/2}x^{2n-2m}}{[x^2y^2;q^2]_n[q^2;q^2]_m}, \tag{2.9} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, q^{-n}/x^2, yq, -yq; q; q \\ q^{1-n}/x, q^{1-n}/x, y^2q \end{matrix}\right] \frac{(-)^n[q;q]_n[x^2y^2q^2;q^2]_n[x^2q^2,y^2q^2;q^2]_m}{q^2[x^2;q^2]_n[y^2q;q]_n[q^2,x^2y^2q^2;q^2]_m}, \tag{2.10} $$

where m is the greatest integer $\leq n/2$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0; q; q \\ q^{1-n}/x, q^{1-n}/x, \end{matrix}\right] = \frac{(-)^n[q;q]_n[x^2q^2;q^2]_m}{q^n[x^2;q^2]_n[q^2;q^2]_m}, \tag{2.11} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0; q; q^2 \\ q^{1-n}/x, q^{1-n}/x; q \end{matrix}\right] = \frac{(-)^n[q;q]_n[x^2q^2;q^2]_m q^{n(n-1)/2}x^{2n-2m}}{[x^2;q^2]_n[q^2;q^2]_m}, \tag{2.12} $$

Where m is the greatest integer $\leq n/2$.

Putting yq^{-n} for y in [Verma and Jain 1; (2.20) P.1027] we get the following summation formula:

$$ {}_4\Phi_3\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 1/xy, -1/xy; q; q \\ q^{1-n}/x, -q^{-n}/x, 1/x^2y^2 \end{matrix}\right] = \frac{(-)^n(xq)^{-n}[q;q]_n[1/y^2;q^2]_n[x^2q^2;q^2]_m[y^2q^2;q^2]_{m-n}}{[x,xq;q]_n[1/x^2y^2;q]_n[q^2;q^2]_m[x^2y^2q^2;q^2]_{m-n}} \tag{2.13} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, q^{-n}/x^2, 0; q; q \\ q^{1-n}/x, -q^{-n}/x; \end{matrix}\right] = \frac{(-)^n[q;q]_n[x^2q^2;q^2]_m x^{n-2m}}{[x,-xq;q]_n[q^2;q^2]_m}, \tag{2.14} $$

Where m is the greatest integer $\leq n/2$.

$$ {}_3\Phi_2 \begin{bmatrix} q^{-n}, q^{-n}/x^2, 0; q; q \\ q^{1-n}/x, -q^{-n}/x; q \end{bmatrix} = \frac{(-)^n x^n q^{n(n-1)/2}[q;q]_n [x^2q^2;q^2]_m}{[x,-xq;q]_n [q^2;q^2]_m}, \quad (2.15) $$

Where m is the greatest integer $\leq n/2$.

Putting wq^m for w in [Alsalam and Verma 1; (4.3) P.420] we get the summation formula:

$$ {}_4\Phi_3 \begin{bmatrix} a, aq, a^2\, q^{2-2m}/w^2, q^{-2m}; q^2, q^2 \\ a^2q^2, aq^{1-2m}/w, aq^{2-2m}\,/w \end{bmatrix} = \frac{[w;q]_{2m}[w/a.-q;q]_m}{[w/a;q]_{2m}[w,-aq;q]_m}, \quad (2.16) $$

$$ {}_4\Phi_3 \begin{bmatrix} q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}-n}; q; q^2 \\ \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{1/2} \end{bmatrix} = \frac{[cd;q]_n [c,d,-q^{1/2};q^{1/2}]_n}{[c,d;q]_n [cd;q^{1/2}]_n}, \quad (2.17) $$

$$ {}_4\Phi_3 \begin{bmatrix} q^{-n}, c, d, \frac{1}{cd} q^{\frac{3}{2}-n}; q; q \\ \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{1/2} \end{bmatrix} = \frac{[cdq^{-1/2};q^{1/2}]_{2n}[c,d;q^{1/2}]_n [q;q]_n}{[cdq^{-1/2};q^{1/2}]_n [cdq^{1/2};q]_n [c,d;q]_n [q^{1/2};q^{1/2}]_n}. \quad (2.18) $$

$$ {}_4\Phi_3 \begin{bmatrix} q^{-n}, c, d, \frac{1}{cd} q^{\frac{1}{2}-n}; q; q \\ \frac{1}{c} q^{1-n}, \frac{1}{d} q^{1-n}, cdq^{-1/2} \end{bmatrix} = \frac{[q,cd;q]_n [c,d;q^{1/2}]_n q^{-n/2}}{[c,d;q]_n [cdq^{-1/2};q^{1/2}]_n [q^{1/2};q^{1/2}]_n}, \quad (2.19) $$

III. MAIN RESULTS

In this section we shall establish certain transformations of double series in the term of single series.

(i) Multiplying both sides of (2.1) by an arbitrary sequence B_n, summing over n from 0 to ∞, applying the identity (1.3) and then replacing B_n by $\frac{z^n}{[q;q]_n} A_n$, where A_n is another arbitrary sequence, we get:

$$ \sum_{n,k=0}^{\infty} A_{n+k} \frac{[a;q]_k (-cz/a)^k z^n q^{k(k-1)/2}}{[c;q]_k [q;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n \frac{[c/a;q]_n z^n}{[q,c;q]_n}. \quad (3.1) $$

This is a transformation which reduces a double series in terms of a single series.

Similarly, one can easily establish the following results:

$$ (ii) \quad \sum_{n,k=0}^{\infty} A_{n+k} \frac{[a,b;q]_k [c/ab;q]_n (cz/ab)^k z^n}{[q,c;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n \frac{[c/a,c/b;q]_n z^n}{[q,c;q]_n}. \quad (3.2) $$

(Using (2.2) with $B_n = \frac{[c/ab;q]_n z^n}{[q;q]_n} A_n$)

$$ (iii) \quad \sum_{n,k=0}^{\infty} A_{n+k} \frac{[x;q]_k [xq;q]_n (-zq)^k z^n}{[q;q]_k [q;q]_n} = \sum_{n=0}^{\infty} A_n \frac{[x^2q^2;q^2]_m z^n}{[q^2;q^2]_m}. \quad (3.3) $$

Where m is the greatest integer $\leq n/2$.

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(Using (2.3) with $B_n = \frac{[xq;q]_n z^n}{[q;q]_n} A_n$)

(iv) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,y;q]_k [xq,yq;q]_n (-zq)^k z^n}{[q,-xyq;q]_k [q,-xyq;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[xyq;q]_n [x^2q^2, y^2q^2; q^2]_m z^n}{[-xyq;q]_n [q^2, x^2y^2q^2; q^2]_m} \qquad (3.4)$$

Where m is the greatest integer $\leq n/2$.

(Using (2.4) with $B_n = \frac{[xq,yq;q]_n z^n}{[q,-xyq;q]_n} A_n$)

(v) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x;q]_k [xq;q]_n (-z)^k z^n}{[q;q]_k [q;q]_n}$$

$$= \sum_{n=0}^{\infty} A_n z^n \frac{[x^2q^2;q^2]_m x^{n-2m}}{[q^2;q^2]_m} \qquad (3.5)$$

Where m is the greatest integer $\leq n/2$.

(Using (2.5) with $B_n = \frac{[xq;q]_n z^n}{[q;q]_n} A_n$)

(vi) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,-xq;q]_n (-z)^k z^n}{[q,x^2q;q]_n [q;q]_k}$$

$$= \sum_{n=0}^{\infty} A_n (-z)^n \frac{[x^2q^2;q^2]_m x^{n-2m}}{[x^2q;q]_n [q^2;q^2]_m} \qquad (3.6)$$

Where m is the greatest integer $\leq n/2$.

(Using (2.6) with $B_n = \frac{[x-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

(vii) $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[x,-xq;q]_n (-z)^k z^n q^{k(k-1)/2}}{[q,x^2q;q]_n [q;q]_k}$$

$$= \sum_{n=0}^{\infty} A_n \frac{(-zx)^n q^{n(n-1)/2} [x^2q^2;q^2]_m}{[x^2q;q]_n [q^2;q^2]_m} \qquad (3.7)$$

Where m is the greatest integer $\leq n/2$.

(Using (2.7) with $B_n = \frac{[x,-xq;q]_n z^n}{[q,x^2q;q]_n} A_n$)

$(viii)$ $$\sum_{n,k=0}^{\infty} A_{n+k} \frac{[a,aq;q^2]_k [wq/a,w/a;q^2]_n (zq)^k z^n}{[q^2,a^2q^2;q^2]_k [q^2,w^2/a^2;q^2]_n}$$

$$= \sum_{n=0}^{\infty} A_n \frac{[w;q]_{2n} z^n}{[w;q]_n [q;q]_n [-aq,-w/a;q]_n}, \qquad (3.8)$$

(Using (2.8) with $B_n = \frac{[w/a,wq/a;q^2]_n z^n}{[q^2,w^2/a^2;q^2]_n} A_n$)

IV. CLAUSEN TYPE IDENTITIES

In this section, we deduce the Clausen type identities from the result established in section (3)

(i) Taking $A_n = 1$ in (3.2) we get

$$ {}_2\Phi_1 \left[\begin{matrix} a,b;qcz/ab \\ c \end{matrix}\right] {}_1\Phi_0 \left[\begin{matrix} c/ab;q;z \\ - \end{matrix}\right]$$

$$= {}_2\Phi_1 \left[\begin{matrix} c/a,c/b;q;z \\ c \end{matrix}\right], \qquad (4.1)$$

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Which is the basic analogue of Euler's transformation.

(ii) For $A_n = 1$, (3.4) yields the product formula:

$$ {}_2\Phi_1\begin{bmatrix} x, y; q; -zq \\ -xyq \end{bmatrix} {}_2\Phi_1\begin{bmatrix} xq.\, yq; q; z \\ -xyq \end{bmatrix} $$
$$ = {}_4\Phi_3\begin{bmatrix} xyq, xyq^2, x^2q^2, y^2q^2; q^2; z^2 \\ -xyq, -xyq^2, x^2y^2q^2 \end{bmatrix} $$

$$ + \frac{z(1-xyq)}{(1+xyq)}\, {}_4\Phi_3\begin{bmatrix} xyq^2, xyq^3, x^2q^2, y^2q^2; q^2; z^2 \\ -xyq^2, -xyq^3, x^2y^2q^2 \end{bmatrix} \tag{4.2} $$

Which is known result [Verma and Jain 1; (2.37)P.1031]

Similarly, taking $A_n = 1$ in (3.3) - (3.8) we have the following results respectively.

(iii)
$$ {}_2\Phi_1\begin{bmatrix} xq.\, yq; q; -z/q \\ -xyq \end{bmatrix} {}_2\Phi_1\begin{bmatrix} x.\, y; q; z \\ -xyq \end{bmatrix} $$
$$ = {}_4\Phi_3\begin{bmatrix} xyq, xyq^2, x^2q^2, y^2q^2; q^2; z^2/q^2 \\ -xyq, -xyq^2, x^2y^2q^2 \end{bmatrix} $$

$$ -\frac{z(1-xyq)}{q(1+xyq)}\, {}_4\Phi_3\begin{bmatrix} xyq^2 xyq^3, x^2q^2, y^2q^2; q^2; z^2/q^2 \\ -xyq^2, -xyq^3, x^2y^2q^2 \end{bmatrix} \tag{4.3} $$

(iv)
$$ {}_2\Phi_1\begin{bmatrix} y, -y; q; -zq \\ y^2q \end{bmatrix} {}_2\Phi_1\begin{bmatrix} xq, -xq; q; z \\ x^2q \end{bmatrix} $$
$$ {}_4\Phi_3\begin{bmatrix} xyq, -xyq, xyq^2, -xyq^2; q^2; z^2 \\ x^2q, y^2q, x^2y^2q^2 \end{bmatrix} $$
$$ + \frac{z(1-x^2y^2q^2)}{(1-x^2q)(1-y^2q)}\, {}_4\Phi_3\begin{bmatrix} xyq^2, -xyq^2, xyq^3, -xyq^3; q^2; z^2 \\ , x^2q^3, y^2q^3, x^2y^2q^2 \end{bmatrix} \tag{4.4} $$

(v)
$$ {}_2\Phi_1\begin{bmatrix} xq, -yq; q; z \\ x^2q \end{bmatrix} $$
$$ = [-zq; q]_\infty\, {}_0\Phi_1\begin{bmatrix} -; q^2; z^2 \\ x^2q \end{bmatrix} + \frac{z[-zq; q]_\infty}{(1-x^2q)}\, {}_0\Phi_1\begin{bmatrix} -; q^2; z^2 \\ x^2q^3 \end{bmatrix} \tag{4.5} $$

(vi)
$$ {}_2\Phi_1\begin{bmatrix} c, d \quad ; q; zq^{1/2} \\ cdq^{1/2} \end{bmatrix} {}_2\Phi_1\begin{bmatrix} c, d \quad ; q; z \\ cdq^{1/2} \end{bmatrix} $$
$$ = {}_4\Phi_3\begin{bmatrix} c, d, \sqrt{cd}, -\sqrt{cd} \quad ; q^{1/2}; z \\ cd, q^{1/4}\sqrt{cd}, -q^{1/4}\sqrt{cd} \end{bmatrix} \tag{4.6} $$

(vii)
$$ {}_2\Phi_1\begin{bmatrix} c, d \quad ; q; zq^{1/2} \\ cdq^{1/2} \end{bmatrix} {}_2\Phi_1\begin{bmatrix} c, d \quad ; q; z \\ cdq^{-1/2} \end{bmatrix} $$
$$ = {}_4\Phi_3\begin{bmatrix} c, d, \sqrt{cd}, -\sqrt{cd} \quad ; q^{1/2}; z \\ cdq^{-1/2}, q^{1/4}\sqrt{cd}, -q^{1/4}\sqrt{cd} \end{bmatrix} \tag{4.7} $$

$(viii)$
$$ {}_2\Phi_1\begin{bmatrix} c, d \quad ; q; zq^{-1/2} \\ cdq^{-1/2} \end{bmatrix} {}_2\Phi_1\begin{bmatrix} c, d \quad ; q; z \\ cdq^{1/2} \end{bmatrix} $$
$$ = {}_4\Phi_3\begin{bmatrix} c, d, \sqrt{cd}, -\sqrt{cd} \quad ; q^{1/2}; zq^{-1/2} \\ cdq^{-1/2}, q^{1/4}\sqrt{cd}, -q^{1/4}\sqrt{cd} \end{bmatrix} \tag{4.8} $$

V. CONCLUSION

In this paper a new method has been developed to establish certain transformation of double q-series in terms of a single series. These results lead to certain Clausen type identities. With the help of these results it is also possible to establish certain continued fraction representation involving q- series.

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