International Journal for Research in Applied Science & Engineering Technology (IJRASET)
*ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor:6.887*
*Volume 5 Issue VII, July 2017- Available at www.ijraset.com*

# Solution of Fractional Legendre Equation

Rajesh Pandey
*Department of Applied Science, Institute of Engineering & Technology, Sitapur Road, Lucknow 226021 India.*

***Abstract**: In this paper we study Legendre conformable fractional differential equation. It turns out that in certain cases, similar to the classical case, certain solutions are fractional polynomials. Further, we study basic properties of such fractional polynomials.*
***Keywords**: Legendre Fractional Equation, Legendre Fractional Polynomials, Fractional Derivative, Conformable and Feasible*

## I. INTRODUCTION

The subject of fractional derivative is as old as calculus. In 1695, L, Hospital asked if the expression $\frac{d^{0.5}}{dx^{0.5}} f$ has any meaning. Since then, many researchers have been trying to generalize the concept of the usual derivative to fractional derivatives. These days, many definitions for the fractional derivative are available. Most of these definitions use an integral form.

## II. NOTATIONS AND DEFINITIONS

*A. The Most Popular Definitions are*

1) *Riemann- Liouville Definition:* If n is a positive integer and $\alpha \in [n-1, n)$, the $\alpha^{th}$ derivative of is given by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{\alpha}^{t} \frac{f(x)}{(t-x)^{a-n+1}}\, dx.$$

2) *Caputo Definitions.* For $\alpha \in [n-1, n)$, *the a derivative of f is*

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{\alpha}^{t} \frac{f^{n}(x)}{(t-x)^{a-n+1}}\, dx.$$

Now, all the definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all the definitions of fractional derivative is the linearity property. However the followings are setbacks of one definition or another:

a) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0 (D_a^{\alpha}(1) = 0$ for the Caputo derivative), if $\alpha$ is not a natural number.

b) All fractional Derivatives do not satisfy the known product rule:
$$D_a^{\alpha}(fg) = f D_a^{\alpha}(g) + g D_a^{\alpha}(f).$$

c) All fractional derivatives do not satisfy the known quotient :
$$D_a^{\alpha}(f/g) = \frac{g D_a^{\alpha}(f) - f D_a^{\alpha}(g)}{g^2}.$$

d) All fractional derivatives do not satisfy the chain rule :
$$D_a^{\alpha}(f \text{ o } g)(t) = f^{(\alpha)} g(t) g^{(\alpha)}(t).$$

e) All fractional derivatives do not satisfy: $D^{\alpha} D^{\beta} f = D^{\alpha+\beta} f$ in general.

f) Cauto definition assumes that the function $f$ is differentiable.

g) $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$

In a new definition called conformable fractional derivative was introduced. The new definition satisfies:

i) $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$ for all $a, b \epsilon R$.

ii) $T_{\alpha}(\lambda) = 0$, For all constant functions $f(t) = \lambda$.

Further for $\alpha \epsilon (0,1]$ and $f, g$ be $\alpha -$ differentiable at a point t, with $g(t) \neq 0$. then

iii) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$.

iv) $T_{\alpha}\left(\frac{f}{g}\right) = \frac{gT_{\alpha}(f) - ft_{\alpha}(g)}{g^2}$.

We list here the fractional derivatives of certain functions, for the purpose of comparing the result of new definition with the usual definition of the derivative.

$$T_{\alpha}(t^p) = p\, t^{p-\alpha}$$

$$T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right) = \cos\frac{1}{\alpha}t^\alpha.$$

$$T_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right) = -\sin\frac{1}{\alpha}t^\alpha$$
$$T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$$

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives. The conformable definition of fractional derivatives to introduce fractional Laplace transform, and factorial Taylor expansion.

The classical Legendre differential equation

$$(1-x^2)y'' - -2xy' + k(k+1)y = 0$$

The point $x = 0$ is an ordinary point for the equations. Solving the equation around $x = 0$, using series solution, and assuming that $k$ is a natural number, gives polynomial solution, called Legendre polynomials, given by the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

these sets of polynomials turned to be orthogonal polynomial in the sense $\int_{-1}^{1} P_n(x)P_m(x)dx = 0\ n \neq m$. such polynomial turned out to be particular interest in many problems in mathematical physics like heat distribution is spherical regions and in the structure of atoms.

Throughout this paper, we let $D^\alpha y$ denote the conformable fractional derivative of $y$, where $\alpha \epsilon$ (0, 1]. The second $\alpha -$ derivative of $y$ will be demoted by $D^\alpha D^\alpha y$. A series is called a fractional power series if it can be written in the form $\sum_{n=0}^{\infty} a_n x^{n\alpha}$, for $\alpha \epsilon$ (0, 1]. Further, we let $I_\alpha^0(f)x = \int_0^x \frac{f(t)}{t^{1-\alpha}}dt$,. for the basic structure of conformable fractional derivatives, integrals and fractional power series.

## III. SOLUTION OF FRACTIONAL LEGENDRE EQUATIONS

Consider the equation

$$(1\text{-}x^{2\alpha})D^\alpha D^\alpha y - 2\alpha x^\alpha D^\alpha y + \alpha^2 k(k+1)y = 0 \qquad (1)$$

Where $\alpha \epsilon$ (0, 1], and $k$ is any real number. Clearly, if $\alpha = 1$, then equation (1) is just the classical Legendre equation. We will use series solution around $x = 0$ to get polynomial solutions for equation (1), when $k$ is assumed to be a natural number.

Now $x = 0$ is an ordinary point for the equation. Using the fractional power expansion and for $x > 0$, we let

$$y = \sum_{n=0}^{\infty} a_n x^{n\alpha}$$

So

$$D^\alpha y = \sum_{n=1}^{\infty} \alpha n a_n x^{n\alpha-\alpha}$$

$$D^\alpha D^\alpha y = \sum_{n=2}^{\infty} \alpha^2 n(n-1)a_n x^{n\alpha-2\alpha}$$

Substitute these in equations (1) we get:

$$\sum_{n=2}^{\infty} \alpha^2 n(n-1)a_n x^{n\alpha-2\alpha} - x^{2\alpha}\sum_{n=2}^{\infty} \alpha^2 n(n-1)a_n x^{n\alpha-2\alpha} - 2ax^a \sum_{n=1}^{\infty} \alpha n a_n x^{n\alpha-\alpha} + a^2 k(k+1)\sum_{n=0}^{\infty} a_n x^{n\alpha} = 0$$

Substitute these in equation (2), replacing $n$ by $n+2$ we get:

$$\sum_{n=0}^{\infty} a_{n+2}\,\alpha^2(n+2)(n+1)x^{n\alpha} \tag{3}$$

The second term sum needs no change of variables and it is equal to:

$$-\sum_{n=2}^{\infty} \alpha^2 n(n-1)\,a_n x^{n\alpha} \tag{4}$$

The third term sum needs no change of variables and it is equal to:

$$-\sum_{n=1}^{\infty} 2\alpha^2 n a_n\, x^{n\alpha} \tag{5}$$

The fourth term sum needs no change of variables and it is equal to:

$$\sum_{n=0}^{\infty} \alpha^2\, k(k+1)a_n x^{n\alpha} \tag{6}$$

Now, unifying all summations to start from $n = 2$ and put them in one summation we get:

$$(2\alpha^2 a_2 + \alpha^2 k(k+1)a_0) + (6\alpha^2 a_3 - 2\alpha^2 a_1 + \alpha^2 k(k+1)a_1)x$$
$$+\sum_{n=2}^{\infty}[a_{n+2}\alpha^2(n+2)(n+1) - \alpha^2 n\,(n-1)a_n - 2\alpha^2 n a_n + \alpha^2 k(k+1)a_n]x^{n\alpha} = 0$$

From which we get

$$a_2 = -\frac{k(k+1)}{2}a_0 \tag{7}$$

$$a_3 = \frac{2-k(k+1)}{6}\,a_1 \tag{8}$$

And

$$a_{n+2} = \frac{n(n+1)-k(k+1)}{(n+1)(n+2)}a_n \tag{9}$$

Hence there are two independent solutions $y_1$ which is the sum over the odd terms and $y_2$ which is the sum over the ever terms. However, such solutions do not have finite value at 1 and at -1, which means that such solutions are not physically feasible. Thus the only series solutions of interest are those that terminate after finitely many steps. That means polynomial solutions. Thus in equation (9) if $k$ is a natural number, then $a_{k+2} = 0$ and so $a_{k+2n} = 0$. However one may still have $a_{k(2n+1)} \neq 0$ for all $n$. Hence by choosing one of $a_0$ (or $a_1$ ) equal to zero, we can make all even-numbered coefficients (or all odd- numbered coefficients) equal to zero.

Simplifying the recurrence formula in (9), to get

$$a_{m+2} = -\frac{(k-n)(n+k+1)}{(n+1)(n+2)}\,a_n \tag{10}$$

Notice, if one substitute in the recurrence formula (10), $n = 0$, we get $a_2 = -\frac{k(k+1)}{2}a_0$, which in equation (7). If $n = 2$ we get $a_4 = \frac{k(k-2)(k+3)(k+1)}{4!}a_0$. And so on to get

$$a_{2n} = (-1)^n \frac{k(k-2)\ldots\big(k-2(n-1)\big).(k+2n-1)(k+2n-3)\ldots(k+1)}{(2n)!}a_0 \tag{11}$$

So if $k$ is even natural number, one can simplify (11) to get

$$a_{2n} = (-1)^n \frac{(k+2n)!\left[\frac{k}{2}!\right]^2}{(2n)!\,k!\left(\frac{k}{2}+n\right)!\left(\frac{k}{2}-n\right)!} a_0 \qquad (12)$$

The constant $a_0$ is usually chosen so that the polynomial solution at $x = 1$ equals 1. So the value to be given to $a_0$ is $a_0 = (-1)^{\frac{k}{2}} \frac{k!}{\left[\left(\frac{k}{2}\right)\right]^2}$. So

For $k = 0$, we get the polynomial $P_0(x) = 1$. for $k = 2$, we get the polynomial $P_2(x) = \frac{1}{2}(3x^{2\alpha} - 1)$. similarly, for $n = 4,6,8, \ldots$ There are similar formulas and structure if $k$ is an odd natural number: $P_1(x) = x^\alpha, P_3(x) = \frac{1}{5}(5x^{3\alpha} - 3x^\alpha) \ldots$

Let $D^{n\alpha} = D^\alpha D^\alpha \ldots D^\alpha, n$ −times. With such notations and using the known formula, $D^\alpha x^p = px^{p-1}$, we can have a nice closed formula for all such polynomials, as follows:

$$P_n(x) = \frac{1}{\alpha^n 2^n n!} D^{n\alpha}(x^{2\alpha} - 1)^n \qquad (13)$$

## IV. CONCLUSIONS

Thus we can conclude that n order to study the orthogonality of fractional Legendre polynomials on an interval [-1, 1] the definition of $D^\alpha f(x)$ must be extended such that it includes the negative values of $x$.These polynomials are very useful in solving problems involving heat distribution in spherical region and in the structure of atoms.

## V. ACKNOWLEDGMENT

My thanks are due to Dr. G.C Chaubey Ex Associate Professor & Head department of Mathematics TDPG College Jaunpur, Professor B. Kunwar Department of Mathematics IET, Lucknow and Dr. S.P Singh Head department of Mathematics TDPG College, Jaunpur for their encouragement and for providing necessary support. I am extremely grateful for their constructive support.

## REFERENCES

[1] K.S. Miller. An introduction to fractional calculus and fractional differential equations, J. Wiley, and Sons, New York (1993).
[2] I. Prodlubny. Fractional differential equations. Academic Press, U.S.A. (1999).
[3] A. Kilbas, H. Shrivastava, and J. Trujillo. Theory and applications of fractional differential equations. Math. Studies. Northholland, New York 2006.
[4] K. Oldham, and J. Spanier, The fractional calculus, theory and applications of differential and integration of arbitrary order. Academic Press, U.S.A. (1974).
[5] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh. A new defination of fractional derivative. Journal of computational applied Mathematics, 264(2014)65-70
[6] K. Sarveswara Rao, Engineering Mathematics University Press (2012).
[7] S. Pal Engineering Mathematics Oxford University Press (2015).

 