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Solutions of Three-Point Boundary Problems for Non-Linear Second-Order Differential Equations

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***Abstract:** The paper investigates the problem of existence of positive solution of non-linear third-order differential equations. Under the suitable conditions, the existence and multiplicity of positive solutions are established by using Krasnoselskii's fixed-point theorem of cone*
***Keywords:** Boundary value problem, Integral equation, Positive Solution, Third order, Cone*

I. INTRODUCTION

Most of the recent results on the positive solutions are concerned with single equation and simple boundary condition; there are few results on the symmetric positive solutions. Consider the following boundary value problem:

$$u''(t) + \alpha(t)f\big(t, u(t)\big) = 0, 0 < t < 1,$$
$$u(0) = u(1-t), u'(0) - u'(1) = u\left(\frac{1}{2}\right) \quad (1.1)$$

by using Krasnoselskii's fixed-point theorem, the existence of symmetric positive solutions is shown under certain conditions on f. Yang and Sun considered the boundary value problem of differential equations

$$-u''(x) = f(x, v),$$
$$-v''(x) = g(x, u),$$
$$u(0) = u(1) = 0, \quad (1.2)$$
$$v(0) = v(1) = 0.$$

using the degree theory, the existence of positive solutions of (1.2) is established. Motivated by the work of Sun and Yang, we concern with the existence of symmetric positive solutions of the boundary value problems.

$$-u''(t) = f(t, v),$$
$$-v''(t) = g(t, u),$$
$$u(t) = u(1-t), \alpha u'(0) - \beta u'(1) = \gamma u\left(\frac{1}{2}\right), \quad (1.3)$$
$$v(t) = v(1-t), \alpha v'(0) - \beta v'(1) = \gamma v\left(\frac{1}{2}\right),$$

Where $f, g\,; [0,1] \times R^{+} \to R^{+}$ are continuous, both $f(.\,, u)$ and $g(.\,, u)$are symmetric on $[0,1], f\ (x, 0) \equiv 0,\ \ g(x, 0) \equiv 0, |\beta - \alpha| \leq \left|\frac{\gamma}{2}\right|, \beta + \alpha \geq 2\gamma, \alpha, \beta \geq 0, \gamma\ \neq 0$.The arguments for establishing the symmetric positive solutions of (1.3) involve properties of the function Lemma that play a key role in defining certain cones. A fixed point theorem due to Krasnoselskii is applied to yield the existence of symmetric positive solutions of (1.3).

II. NOTATIONS AND DEFINITIONS

We present some necessary definitions and preliminary lemmas that will be used in the proof of the results.

Definitions 1.Let E be a real Banach space. A nonempty closed set $P\ \subset E$ is called a cone of E if it satisfies the following conditions:

(i) $x\ \in P, \lambda > 0$ implies $\lambda x\ \in P$;

(ii) $x\ \in P, -x\ \in P$ implies $x = 0$.

Definition 2.The function u is said to be concave on $[0,1]$ if $(rt_1 + (1-r)t_2) \geq ru(t_1) + (1-r)u(t_2), r, t_1, t_2\ \in [0,1]$.

Definition 3. The function u is said to be symmetric on $[0,1] if\ u(t) = u(1-t), t\ \in [0,1]$

Definition 4. The function (u, v) is called a symmetric positive solution of the BVP (1.3) if u and v are symmetric and positive on $[0,1]$, and satisfy the BVP (1.3)

We shall consider the real Banach space $C[0,1]$, equipped with norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Denote $C^+ [0,1] = \{u \in C[0,1]: u(t) \ge 0, t \in [0,1]\}$.

III.MAIN RESULT

Lemma 1. Let $y \in C[0,1]$, be symmetric on $[0,1]$, then the three-point BVP

$$u''(t) + y(t) = 0, 0 < t < 1$$
$$u(t) = u(1-t), \alpha u'(0) - \beta u'(1) = \gamma u\left(\frac{1}{2}\right), \quad (3.1)$$

has a unique symmetric solution $u(t) = \int_0^1 G(t,s)y(s)ds$, where $G(t,s) = G_1(t,s) + G_2(s)$, here

$$G_1(t,s) = \begin{cases} t(1-s), 0 \le t \le s \le 1 \\ s(1-t), 0 \le s \le t \le 1 \end{cases}$$

$$G_2(s) = \begin{cases} \left(\frac{1}{2} - s\right) - \frac{1}{2}(1-s) + \frac{(\alpha-\beta)(1-s)}{\gamma} + \frac{\beta}{\gamma}, 0 \le s \le \frac{1}{2} \\ -\frac{1}{2}(1-s) + \frac{(\alpha-\beta)(1-s)}{\gamma} + \frac{\beta}{\gamma}, \frac{1}{2} \le s \le 1 \end{cases}$$

Proof. From (3.1), we have $u''(t) = -y(t)$. For $t \in [0,1]$ integrating from 0 to t we get

$$u'(t) = -\int_0^t y(s)ds + A_1 \quad (3.2)$$

Since $u'(t) = -u'(1-t)$, we can find that $-\int_0^t y(s)ds + A_1 = -\int_0^{1-t} y(s)ds - A_1$,

Which leads to $A_1 = \frac{1}{2}\int_0^t y(s)ds - \frac{1}{2}\int_0^{1-t} y(s)ds = \frac{1}{2}\int_0^t (s)ds + \frac{1}{2}\int_0^{1-t} y(1-s)d(1-s) = \frac{1}{2}\int_0^t y(s)ds + \frac{1}{2}\int_t^1 y(s)ds = \frac{1}{2}\int_0^1 y(s)ds = \int_0^1 (1-s)y(s)ds$.

Integrating again we get,

$$u(t) = -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds + A_2.$$

From (3.1) and (3.2) we have

$$(\alpha-\beta)A_1 + \beta\int_0^1 y(s)ds = \gamma\left(-\int_0^{\frac{1}{2}}\left(\frac{1}{2} - s\right)ds + \frac{1}{2}\int_0^1 (1-s)y(s)ds + A_2\right)$$

Thus

$$A_2 = \int_0^{\frac{1}{2}}\left[\left(\frac{1}{2} - s\right) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]y(s)ds + \int_{\frac{1}{2}}^1\left[\frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]y(s)ds.$$

From above we can obtain the BVP (3.1) has a unique symmetric solution

$$u(t) = -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds$$
$$+\int_0^{\frac{1}{2}}\left[\left(\frac{1}{2} - s\right) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]y(s)ds + \int_{\frac{1}{2}}^1\left[\frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]y(s)ds.$$
$$= \int_0^1 G_1(t,s)y(s)ds + \int_0^1 G_2(s)y(s)ds$$
$$= \int_0^1 [G_1(t,s) + G_2(s)]\,y(s)ds.$$

This completes the proof.

Lemma 2. The function $G(t,s)$ satisfies $\frac{3}{4}G(s,s) \le G(t,s) \le G(s,s)$ for $t, s \in [0,1]$if α, β, γ are defined in (1.3)

Proof. For any $t \in [0,1]$ and $s \in \left[0, \frac{1}{2}\right]$, we have

$$G(t,s) = G_1(t,s) + G_2(s) \geq G_2(s) = \frac{1}{4}G_2(s) + \frac{3}{4}G_2(s)$$

$$= \frac{1}{4}\left[\left(\frac{1}{2} - s\right) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right] + \frac{3}{4}\left[\left(\frac{1}{2} - s\right) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]$$

$$\geq s(1-s)G_2(s) + \frac{3}{4}G_2(s).$$

Note that $|\beta - \alpha| \leq \left|\frac{\gamma}{2}\right|, \alpha + \beta \geq 2\gamma, \gamma \neq 0$, we obtain $G_2(s) \geq \frac{3}{4}$. Thus $G(t,s) \geq \frac{3}{4}\,[G_1(s,s) + G_2(s)] = \frac{3}{4}G\,(s,s)$. As in the same way we can conclude $G\,(t,s) \geq \frac{3}{4}G(s,s)$ for any $t\, \in [0,1]$ and $s\, \in [\frac{1}{2},1]$. It is obvious that $G(s,s) \geq G(t,s)$ for $t,s\, \in [0,1]$. The proof is complete.

Lemma 3. Let $y\, \in\, C^{+}[0,1]$, then the unique symmetric solution $u\,(t)$ of the BVP (1.3) is nonnegative on $[0,1]$

Proof. Let $y\, \in\, C^{+}[0,1]$. From the fact that $u''(t) = -y(t) \leq 0, t \in [0,1]$, we know that the graph of $u(t)$ is concave on $[0,1]$. From (3.1) We have that

$$u(0) = u(1) = \int_0^{\frac{1}{2}}\left[\left(\frac{1}{2} - s\right) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]y(s)ds\; + \int_{\frac{1}{2}}^{1}\left[\frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)\right]y(s)ds \geq 0$$

Note that $u(t)$ is concave , thus $u(t) \geq 0$ for $t\, \in [0,1]$. This completes the proof.

Lemma 4. Let $y\, \in\, C^{+}\,[0,1]$, the the unique symmetric solution $u(t)$ of BVP (1.3) satisfies

$$\min_{t\,\in[0,1]} u(t) \geq \frac{3}{4}\,\|u\| \tag{3.3}$$

Proof. For any $t\, \in [0,1]$, on the one hand, from lemma 2 we have that $u(t) = \int_0^1 G(t,s)y(s)ds \leq \int_0^1 G(s,s)y(s)ds$. Therefore,

$$\|u\| \leq \int_0^1 G(s,s)y(s)ds \tag{3.4}$$

On the other hand, for any $t\, \in [0,1]$, from lemma 2 we obtain that

$$u(t) = \int_0^1 G(t,s)y(s)ds \geq \frac{3}{4}\int_0^1 G(s,s)y(s)ds \geq \frac{3}{4}\|u\| \tag{3.5}$$

From (3.5) and (3.4) we find that (3.3) holds.

Obviously, $(u,v) \in\, C^2[0,1] \times C^2[0,1]$ is the solution of (1.3) if and only if $(u,v) \in C\,[0,1] \times C[0,1]$ is the solution of integral equations

$$\begin{cases} u(t) = \int_0^1 G(t,s)f\big(s,v(s)\big)ds, \\ v(t) = \int_0^1 G(t,s)g\big(s,u(s)\big)ds, \end{cases} \tag{3.6}$$

Integral equations (3.6) can be transferred to the non-linear integral equation

$$u(t) = \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds \tag{3.7}$$

Define an integral operator $A\colon C \to C$ by

$$Au(t) = \int_0^1 G(t,s)f(s\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds \tag{3.8}$$

It is easy to see that the BVP (1.3) has a solution $u = u(t)$if and only if u is a fixed point of the operator A defined by (3.8).

Let $P = \{u\, \in\, C^{+}\,[0,1]\colon u(t)$ is symmetric, concave on $[0,1]$ and $\min_{0\leq t\leq 1} u(t) \geq \frac{3}{4}\,\|u\|\}$. It is obvious that P is a positive cone in $C[0,1]$.

Lemma 5. If the operator A is defined as in (3.8) then $A : P \to P$ is completely continuous.

Proof. It is obvious that Au is symmetric on $[0,1]$. Note that $(Au)''(t) - f(t,v(t)) \leq 0$, so we have that Au is concave. Thus from lemma 2 and non negativity of f and g,

$$Au(t) = \int_0^1 G(t,s)f(s\int_0^1 G(s,\xi)g\big(\xi,u(\xi)\big)d\xi)ds$$

$$\leq \int_0^1 G(s,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds$$

Then

$$\|Au\| \leq \int_0^1 G(s,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds$$

On the other hand

$$Au \geq \frac{3}{4}\int_0^1 G(s,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \geq \frac{3}{4}\|Au\|$$

Thus, $A(P) \subset P$. Since $G(t,s), f(t,u) and\ g(t,u)$ are continuous, it is easy to see that $A : P \to P$ is completely continuous. The Proof is complete.

IV. CONCLUSION

The positive solution of non-linear third-order differential equation exists. Using Krasnoselskii's fixed point theorem of cone the existence and multiplicity of these positive solutions are established. With the help of existence of positive solutions, five lemmas are proved.

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REFERENCES

[1] J. Henderson, H.B. Thompson, Multiple symmetric positive solutions for a second order of boundary value problem, Proc.Amer.Math.Soc. 128 (2000) 2373-2379.
[2] L.H. Erbe, H.Y. Wang. On the existence of positive solution of ordinary differential equation. Proc.Amer.Math.Soc. 120(1994) 743-748.
[3] R.Y. Ma. Multiple nonnegative solutions of second –order system of boundary value problem. Non-linear Anal. 42 (2002) 1003-1010.
[4] L.H. Erbe, H.Y. Wang. On the existence of positive solutions of ordinary differential equations. Proc. Amer.Math.Soc.120 (1994) 743-748.
[5] L. Hu, L.L Wang. Multiple positive solutions of boundary value problems for system of linear second-order differential equations. J. Math. Anal Appl. 335(2007) 1052-1060.
[6] G. Infantea, M. Zima: Positive solutions of multi-point boundary value problems at resonance, Non-linear Anal. 69, 2458-2171 (2008).
[7] D.J. Guo, V. Lakshmikantham: Non- Linear Problems in Abstract Cones. Academic Press, New York (1988).
[8] Q. Yao, Y. Feng. The existence of the solution for a third-order two-point boundary value problem. Appl. Math. Lett. (2002) 15,227-32.
[9] S. Li. Positive solutions of non-linear single third-order two-point boundary value problem. J. Math. Anal. Appl. 323 (2006) 413-25.
[10] Moustafa El-Shahed. Positive solutions for non-linear singular third order boundary value problem. Communications in Non-linear Sci. and Numerical Simulation, 14(2009)424-429.