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Certain Transformation Formulae for Basic Hypergeometric Series

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***Abstract**: In this paper, general transformation formulae for basic hypergeometric series of two variables have been established. Special cases have also been studied.*
***Keywords**: Basic hypergeometric series, Transformation formulae, Summation, Basic Analogue, Parameters*

I. INTRODUCTION

Jeugt, Pitre and Srinivasa Rao [1] obtain certain summation theorems for double and triple hypergeometric functions. The following interesting summation formula for double hypergeometric function has been established $F_{1;1}^{0;3}\left[\begin{matrix} :\delta-\alpha\beta+\gamma,-p;\alpha-\delta,\beta+p,-\gamma;1,1 \\ \beta: \quad \beta+\gamma \quad ; \quad \alpha+p \end{matrix}\right]=$

$$\frac{(\alpha)_p(\delta)_r}{(\partial)_p(\alpha)_r} \tag{1.1}$$

The basic analogue of (1.1) has been mentioned as

$$F_{1;1}^{0;3}\left[\begin{matrix} :\delta/\alpha,\beta q^r,q^{-p};\alpha/\delta,\beta q^p,q^{-r};q,q \\ \beta: \quad \delta q^r \quad ; \quad \alpha q^p \end{matrix}\right]=\left(\frac{\delta}{\alpha}\right)^{p-r}\frac{(\alpha;q)_p(\delta;q)_r}{(\alpha;q)_p(\alpha;q)_r} \tag{1.2}$$

II. DEFINITIONS AND NOTATIONS

The Gauss hypergeometric function is represented as:

$${}_2F_1\left[\begin{matrix} a,b;z \\ c \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n\, n!}, \tag{2.1}$$

Where,

$$(a)_n=a(a+1).....(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}, \quad (a)_0=1 \tag{2.2}$$

The generalised hypergeometric function is defined as:

$${}_AF_B\left[\begin{matrix} (a);z \\ (b) \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{[(a)]_n z^n}{[(b)]_n\, n!} \tag{2.3}$$

Where (a) stands for A-parameters of the form $a_1,a_2,.........a_A$. A double hypergeometric function is defined by

$$F_{C:D;D'}^{A:B;B'}\left[\begin{matrix} (a):(b);(b');x,y \\ (c):(d);(d') \end{matrix}\right]=\sum_{m,n=0}^{\infty}\frac{[(a)]_{m+n}[(b)]_m[(b')]_n x^m y^n}{[(c)]_{m+n}[(d)]_m[(d')]_n\, m!n!} \tag{2.4}$$

And in case of B=B', D=D', we simply write the function as,

$$F_{C:D}^{A:B}\left[\begin{matrix} (a):(b);(b');x,y \\ (c):(d);(d') \end{matrix}\right]$$

The basic analogue of (2.3) known as generalized basic hypergeometric function is defined by,

$${}_AF_B\left[\begin{matrix} (a);z \\ (b);q^\lambda \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{[(a)]_n z^n q^{\lambda n(n-1)/2}}{[(b)_n]_n (q)_n}, \tag{2.5}$$

Where (a) stands for A-parameters of the form $a_1, a_2, ...a_A$: and

$$(a)_n = (a;q)_n = (1-a)(1-aq)....(1-aq^{n-1}); (a;q)_0 = 1.$$

The basic double hypergeometric function is defined as:

$$F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b');x,y\\(c):(d);(d');q^{\lambda},q^{\mu}\end{matrix}\right] = \sum_{m,n=0}^{\infty}\frac{[(a)]_{m+n}[(b)]_m[(b')]_n x^m y^n}{[(c)]_{m+n}[(d)]_m[(d')]_n (q)_m (q)_n} q^{\lambda m(m-1)/2+\mu n(n-1)/2} \tag{2.6}$$

III.MAIN RESULTS

A. Analytic Proof of (1.2)

In this section we shall give analytic proof of (1.2).Les us represent the left hand side of (1.2) by Ω, then

$$\Omega = \sum_{m,n=0}^{p,r}\frac{\left(\frac{\delta}{\alpha}\right)_m (\beta q^r)_m (q^{-p})_m \left(\frac{\alpha}{\delta}\right)_n (\beta q^p)_n (q^{-r})_n q^{m+n}}{(\beta)_{m+n}(\delta q^r)_m (\alpha q^p)_n (q)_m (q)_n}$$

$$= \sum_{m=0}^{p}\frac{(\delta/\alpha)_m (\beta q^r)_m (q^{-p})_m q^m}{(\beta)_m (\delta q^r)_m (q)_m} {}_3\Phi_2\left[\begin{matrix}\alpha/\delta, \beta q^p, q^{-r}; q\\ \beta q^m \quad \alpha q^p\end{matrix}\right] \tag{3.1}$$

Now, transforming the inner ${}_3\Phi_2$ series, we get

$$\Omega = \sum_{m=0}^{p}\frac{\left(\frac{\delta}{\alpha}\right)_m (\beta q^r)_m (q^{-p})_m q^m \left(\frac{\beta\delta q^m}{\alpha}\right)_r}{(\beta)_m (\partial q^r)_m (q)_m (\beta q^m)_r}\left(\frac{\alpha}{\delta}\right)^r \times {}_3\Phi_2\left[\begin{matrix}q^{-r}, \alpha/\delta, \alpha/\beta; q^{1+p-m}\\ \alpha q^p, \quad \alpha/\beta\delta \quad q^{1-m-r}\end{matrix}\right] \tag{3.2}$$

Summing the inner ${}_3\Phi_2$ series with the help of Saalschütz summation formula, we get,

$$\Omega = \frac{(\delta)_{p+r}(\beta)_{p+r}(\alpha)_p \left(\frac{\beta\delta}{\alpha}\right)_p \left(\frac{\beta\delta}{\alpha}\right)_r \left(\frac{\alpha}{\delta}\right)^r}{(\alpha)_{p+r}(\beta\delta/\alpha)_{p+r}(\delta)_p(\beta)_p(\beta)_r} \times {}_3\Phi_2\left[\begin{matrix}q^{-r}, \delta/\alpha, \beta\delta/\alpha, q^r, q\\ \delta q^r, \quad \beta\delta/\alpha\end{matrix}\right] \tag{3.3}$$

Again, applying the transformation formula and then summing the ${}_3\Phi_2$ series on the right hand side of (3.3) with the help of, we get the right hand side of (1.2).

B. General Transformation Formula

We shall establish the following general transformation formula:

$$\Phi^{A:B+1;B'+1}_{C:D+1;D'+1}2\left[\begin{matrix}(a):(b),\alpha;(b'),\delta;\delta z_1/\alpha, \alpha z_2/\alpha\\(c):(d),\alpha;(d'),\alpha;q,q\end{matrix}\right] = \sum_{m,n=0}^{\infty}\frac{[(a)]_{m+n}[(b)]_m[(b')]_n (\beta)_{m+n}(\alpha)_m(\delta/\alpha)_m(\alpha/\delta)_n(-z_1)^m(-z_2)^n}{[(c)]_{m+n}[(d)]_m[(d')]_n (\alpha)_{m+n}(\partial)_{m+n}(\beta)_m(\beta)_n(q)_m(q)_n}\times$$

$$\times\Phi^{A;B+2;B'+2}_{C;D+2;D'+2}\left[\begin{matrix}(a)q^{m+n}:(b)q^m, \beta q^{m+n}, \alpha q^m;(b')q^n, \beta q^{m+n}, \delta q^n; z_1, z_2\\(c)q^{m+n}:(d)q^m, \beta q^m, \alpha q^{m+r};(d')q^n, \beta q^m, \delta q^{m+n}; q, q\end{matrix}\right]$$

(3.4)

Proof :

Let us represent the left hand side of (3.4) by ∧, then

$$\wedge = \sum_{p,r}^{\infty}\frac{[(a)]_{p+r}[(b)]_p[(b')]_r z_1^p z_2^r q^{p(p-1)/2+r(r-1)/2}}{[(c)]_{p+r}[(d)]_p[(d')]_r (q)_p (q)_r}\left\{\frac{(\alpha)_p(\delta)_r}{(\delta)_p(\alpha)_r}\left(\frac{\delta}{\alpha}\right)^{p-r}\right\}.$$

Putting the value of $\left\{\frac{(\alpha)_p(\delta)_r}{(\delta)_p(\alpha)_r}\left(\frac{\delta}{\alpha}\right)^{p-r}\right\}$, in the form of double series from (1.2) we get,

$$\wedge = \sum_{p,r}^{\infty} \frac{[(a)]_{p+r}[(b)]_p[(b')]_r z_1^p z_2^r q^{p(p-1)/2+r(r-1)/2}}{[(c)]_{p+r}[(d)]_p[(d')]_r (q)_p (q)_r} \times \sum_{m=0}^{p}\sum_{n=0}^{r} \frac{(\delta/\alpha)_m (\beta q^r)_m (q^{-p})_m (\alpha/\delta)_n (\beta q^p)_n (q^{-r})_n}{(\beta)_{m+n} (\delta q^r)_m (\alpha q^p)_n (q)_m (q)_n} q^{m+n}$$

Now changing the order of summations and putting $p+m$, $r+n$ for p and r respectively, we get the right hand side of (3.4) after some simplifications.

IV. SPECIAL CASES OF (3.4) AND RESULTS:

A. Putting A = C = 0, B = B' = D = D' = 1, $b_1 = \delta, d_1 =$ and $d_1' = \delta$ in (3.4) we get,

$$\sum_{u,v=0}^{\infty} \frac{q^{u(u-1)/2+v(v-1)/2}}{(q)_u (q)_v} \left(\frac{\delta}{\alpha}\right)^{u-v} z_1^u z_2^v$$

Certain transformation formulae for basic hypergeometric series

$$= \sum_{m,n=0}^{\infty} \frac{(\delta)_m (\alpha)_n (\beta)_{m+n} (\delta/\alpha)_n (\alpha/\delta)_n (-z_1)^m (-z_2)^n}{(\alpha)_{m+n} (\delta)_{m+n} (\beta)_m (q)_m (q)_n} {}_2\Phi_2\left[\begin{matrix}\delta q^m, \beta q^{m+n}; z_1\\ \beta q^m; \alpha q^{m+n}; q\end{matrix}\right] {}_2\Phi_2\left[\begin{matrix}\alpha q^n, \beta q^{m+n}; z_2\\ \beta q^n; \delta q^{m+n}; q\end{matrix}\right] \quad (3.5)$$

Taking $z_1 = z_2$ and then equating the coefficients of z^{u+v} of both sides we get the following summation formula:

$$\sum_{r=0}^{u}\sum_{s=0}^{v} \frac{(q^{-u})_r (q^{-v})_s \left(\frac{q^{1-u-v}}{\beta}\right)_{r+s} \left(\frac{q^{1-u-v}}{\alpha}\right)_s \left(\frac{q^{1-u-v}}{\delta}\right)_r (\alpha q^{u+v})^r (\delta q^{u+v})^s}{\left(\frac{q^{1-u-v}}{\beta}\right)_r \left(\frac{\alpha}{\delta} q^{1-u}\right)_r \left(\frac{q^{1-u-v}}{\beta}\right)_s \left(\frac{\alpha}{\delta} q^{1-u}\right)_s (q)_r (q)_s q^{rs}}$$

$$= = \frac{(\alpha)_{u+v} (\delta)_{u+v} (\beta)_u (\beta)_v (-)^{u+v} q^{(u/2)+(v/2)} (\delta/\alpha)^{u+v}}{(\beta)_{u+v} (\delta)_u (\partial/\alpha)_u (\alpha)_v (\alpha/\delta)_v} \quad (3.6)$$

B. PUTTING A = C = 0, B = B' = D = D' = 1, $b_1 = \beta, d_1 = \alpha, b_1' = \beta, d_1' = \alpha$ IN (3.4) WE GET :

$${}_1\Phi_1\left[\begin{matrix}\beta; z_1 \quad \delta/\alpha\\ \partial; q\end{matrix}\right] {}_1\Phi_1\left[\begin{matrix}\beta; z_2 \quad \alpha/\delta\\ \alpha; q\end{matrix}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n} (\delta/\alpha)_m (\alpha/\delta)_n (-z_1)^m (-z_2)^n}{(\delta)_{m+n} (\alpha)_{m+n} (q)_m (q)_n} \times {}_1\Phi_1\left[\begin{matrix}\beta q^{m+n}; z_1\\ \alpha q^{m+n}; q\end{matrix}\right] {}_1\Phi_1\left[\begin{matrix}\beta q^{m+n}; z_2\\ \delta q^{m+n}; q\end{matrix}\right] \quad (3.7)$$

Taking $z_1 = -\alpha/\beta$, and $z_2 = -\delta/\beta$, in (3.7) and summing ${}_1\Phi_1$ series of both sides we get an identity:

$$\sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n} (\delta/\alpha)_m (\alpha/\delta)_n (\alpha/\beta)^m (\delta/\beta)^n}{(q)_m (q)_n} = 1 \quad (3.8)$$

Taking $\beta \to \infty$ in (3.8) we get :

$$\sum_{m,n=0}^{\infty} \frac{(\delta/\alpha)_m (\alpha/\delta)_n (\alpha)^m (\delta)^n (-)^{m+n} (q)^{(m+n)(m+n-1)/2}}{(q)_m (q)_n} = 1 \quad (3.9)$$

Replacing q by q^2 in (3.9) and then taking $\alpha = \delta q$ and finally putting $\delta = 1$, we get:

$$\sum_{m,n=0}^{\infty} \frac{(q^{-1}; q^2)_m (q; q^2)_n q^m (-)^{m+n} q^{(m+n)(m+n-1)}}{(q^2; q^2)_m (q^2; q^2)_n} = 1 \quad (3.10)$$

C. PUTTING $z_1/\beta, z_2/\beta$ FOR Z_1 AND Z_2 IN (3.7) AND THEN TAKING $\beta \to \infty$ WE GET :

$${}_0\Phi_1\left[\begin{matrix}-; -z_1\delta/\alpha\\ \delta; q^2\end{matrix}\right] {}_0\Phi_1\left[\begin{matrix}-; -z_2\alpha/\delta\\ \delta; \quad q^2\end{matrix}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(\delta/\alpha)_m (\alpha/\delta)_n z_1^m z_2^n q^{(m+n)(m+n-1)/2} (-)^{m+n}}{(\delta)_m (\alpha)_{m+n} (q)_m (q)_n} \times {}_0\Phi_1 \left[\begin{matrix} -; -z_1 q^{m+n} \\ \alpha q^{m+n}; q^2 \end{matrix} \right] {}_0\Phi_1 \left[\begin{matrix} -; -z_2 q^{m+n} \\ \delta q^{m+n}; \; q^2 \end{matrix} \right] \quad (3.11)$$

Again taking $\delta = q^{1+v_1}, \alpha = q^{1+v_2}, z_1 = \frac{x^2}{4} q^{1+v_2}, z_2 = \frac{y^2}{4} q^{1+v_1}$ in (3.11) , we get:

$${}_0\Phi_1 \left[\begin{matrix} -; \frac{x^2}{4} q^{1+v_1} \\ q^{1+v_1}; \; q^2 \end{matrix} \right] {}_0\Phi_1 \left[\begin{matrix} -; \frac{y^2}{4} q^{1+v_2} \\ q^{1+v_2}; \; q^2 \end{matrix} \right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(q^{v_1-v_2}; q)_m (q^{v_2-v_1}; q)_n \left(\frac{x}{2}\right)^{2m} \left(\frac{y}{2}\right)^{2n} q^{(1+v_2)m} q^{(1+v_1)n}}{(q^{1+v_1}, q^{1+v_2}; q)_{m+n} (q)_m (q)_n} \times$$

$$q^{(m+n)(m+n-1)/2} {}_0\Phi_1 \left[\begin{matrix} -; \frac{x^2}{4} q^{1+v_2+m+n} \\ q^{1+v_2+m+n}; \; q^2 \end{matrix} \right] {}_0\Phi_1 \left[\begin{matrix} -; \frac{y^2}{4} q^{1+v_1+m+n} \\ q^{1+v_1+m+n}; \; q^2 \end{matrix} \right] \quad (3.12)$$

Changing ${}_0\Phi_1$ series into Bessel function of second kind defined by,

$$J_{v_1}^{(2)}(x;q) = \frac{(q^{1+v};q)_\infty (x/2)^v}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-)^n (x^2/4)^n (q^{1+v})^n q^{n^2-n}}{(q;q)_n (q^{1+v};q)_n}$$

We get :

$$J_{v_1}^{(2)}(x;q) J_{v_2}^{(2)}(y;q) = \sum_{m,n=0}^{\infty} \frac{(q^{v_1-v_2};q)_m (q^{v_2-v_1};q)_n (-)^{m+n} (x/y)^{m-n}}{(q)_m (q)_n}$$

$$\times \quad q^{(1+v_1)n+(1+v_2)m} J_{v_1+m+n}(y) J_{v_2+m+n}(x) \quad (3.13)$$

A number of similar other interesting results can also be deduced.

V. CONCLUSION

In this paper, an attempt has been made to give the analytic proof of (1.2) we shall also make use of (1.2) to establish a general transformation formula for basic hypergeometric series of two variables. Special cases have also been studied and some very interesting and new results have been obtained.

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