On a Special Type of Water Wave Problem Arising in Oceanography

Partha Agasti
Assistant Professor of Mathematics, SBSS Mahavidyalaya, Goaltore, Paschim Medinipur, Pin-721128, India

Abstract: A simple and straightforward reduction procedure (which is somewhat different from Packham's [1] approach) is employed here to find the linear solution for the normally incident incoming waves at the interface of two liquids where the liquids are bounded on the left by a rigid vertical cliff. Analytical expressions for velocity potentials in each of the two liquids are obtained here, assuming the lower liquid to be of uniform finite depth and the upper liquid to be of infinite height.
Keywords: linear theory, inviscid liquid, irrotational flow, vertical cliff, source potential.

I. INTRODUCTION

Problem of water waves on a beach which slopes at an angle $\pi/2$n with the horizontal, n being any integer is an important oceanic phenomenon. For n=1, i.e., when a vertical cliff exists on one side of the ocean, a sloping beach problem reduces to the problem involving a vertical cliff. The solution of the corresponding two dimensional as well as three dimensional problems involving a vertical cliff were obtained by Stoker [2,3], exploiting a powerful, though, complicated method essentially based on the complex variable theory, for deep water case.

However, existing literature on problems involving two liquids are, in general, complicated because of the coupled boundary conditions at the interface of the liquids. Since then, few attempts have been made to investigate this class of water wave problems associated with Laplace's equation and few of its generalizations by employing different mathematical techniques (cf. [4]-[8]). The present study is concerned with the two dimensional problem of incoming wave progressing towards a rigid vertical cliff, in two immiscible liquid, where the lower liquid is of finite constant depth h and the upper liquid is of finite constant height H. Allowing no reflection of waves by the cliff, the problem under consideration is attacked for solution, assuming linear theory, by a simple reduction procedure and analytical expressions for the velocity potentials, in each of the two liquid, are obtained. As a particular case, a known result for the problem involving a liquid of finite depth h is recovered (cf. [6]).

II. STATEMENT OF THE PROBLEM

Let us consider the two-dimensional irrotational motion of two inviscid, homogeneous liquid with densities ρ_1 and $\rho_2 (\rho_1 > \rho_2)$, of the lower and upper liquid respectively, under the action of gravity. A rectangular Cartesian co-ordinate system is chosen in which the y-axis is taken to be vertically downwards into the lower liquid, the plane $y = 0, x > 0$ is the mean position of the interface, $x = 0$ is the rigid wall, and the two liquid of densities ρ_1 and ρ_2 occupy the regions $x > 0,\ 0 < y \leq h$ and $x > 0$, $-H \leq y < 0$ respectively. The origin is taken at a point on the line of intersection where the mean interface and the wall meet.

III. MATHEMATICAL FORMULATION OF THE PROBLEM

Let The problem is to find the velocity potentials $\Phi_j(x, y, t)$ with $j = 1,2$ ($j = 1$ for the lower liquid and $j = 2$ for the upper liquid). For periodic motion, we can assume

$$\left.\begin{aligned} \Phi_1(x,y,t) &= Re[\phi_1(x,y)\exp(-i\sigma t)] \\ \Phi_2(x,y,t) &= Re[\phi_2(x,y)\exp(-i\sigma t)] \end{aligned}\right\} \tag{3.1}$$

where σ is the circular frequency.

The problem is to find Φ_1, Φ_2 which behave as $x \to \infty$ like progressive waves moving towards the cliff. Thus the problem under consideration can be investigated by way of determining the potentials $\phi_1(x, y), \phi_2(x, y)$ which satisfy the boundary value problems given below:

1) Two-dimensional Laplace's equations:

$$\left.\begin{aligned} \nabla^2\phi_1 &= 0 \\ \nabla^2\phi_2 &= 0 \end{aligned}\right\} \tag{3.2}$$

in the respective flow domain, where ∇^2 is the two dimensional Laplacian.

2) Linearised interface conditions:

$$\left.\begin{aligned} \phi_{1_y} &= \phi_{2_y} \\ k\phi_1 + \phi_{1_y} &= s\left(k\phi_2 + \phi_{2_y}\right) \end{aligned}\right\} \text{ on } y = 0, x > 0, \tag{3.3}$$

where $K = \sigma^2/g$ is the wave number, g is the acceleration due to gravity and $s = \rho_2/\rho_1$

3) Conditions at the rigid bottom and top :

$$\left.\begin{aligned} \phi_{1_y} &= 0 \ on\ y = h \\ \phi_{2_y} &= 0 \ on\ y = H \end{aligned}\right\} \tag{3.4}$$

4) Conditions on the rigid cliff $x = 0$:

$$\left.\begin{aligned} \phi_{1_x} &= 0,\ 0 < y \le h \\ \phi_{2_x} &= 0,\ -H \le y < 0 \end{aligned}\right\} \tag{3.5}$$

Our purpose is to obtain ϕ_1, ϕ_2 satisfying (3.2) to (3.5) and the condition that they behave at infinity as progressive waves moving towards the wall.

Further, as no reflection of waves, by the wall, is allowed which can be justified by assuming a source/sink type behavior in the potential functions at the origin, in the absence of surface tension (cf. [9]), which lead to the conditions

$$\phi_1, \phi_2 \to \ln r \quad as\ r = (x^2 + y^2)^{1/2} \to 0. \tag{3.6}$$

Noting the conditions (3.3) and (3.4), and following Gorgui and Kassem [10], we can assume that

$$\left.\begin{aligned} \phi_1 &\to \frac{\cosh k_0(h - y)}{\sinh k_0 h} \exp(-ik_0 x) \\ \phi_2 &\to -\frac{\cosh k_0(H + y)}{\sinh k_0 H} \exp(-ik_0 x) \end{aligned}\right\} as\ x \to \infty. \tag{3.7}$$

IV. SOLUTION OF THE PROBLEM

To solve the problem, mathematically, we reduce the boundary value problem, described by (3.2) to (3.6) and the infinity requirement given by (3.7), to another boundary value problem. To do this, let us introduce two new functions ψ_1 and ψ_2 of x, y by the following relations:

$$\left.\begin{aligned} \phi_1 &= \frac{2\cosh k_0(h - y)}{\sinh k_0 h} \cos k_0 x + \psi_1(x, y) \\ \phi_2 &= -\frac{2\cosh k_0(H + y)}{\sinh k_0 H} \cos k_0 x + \psi_2(x, y) \end{aligned}\right\} \tag{4.1}$$

Where ψ_1, ψ_2 satisfy the boundary value problems described by (3.2) to (3.6) together with the infinity requirement

$$\left.\begin{aligned} \psi_1 &\to -\frac{\cosh k_0(h - y)}{\sinh k_0 h} \exp(ik_0 x) \\ \psi_2 &\to \frac{\cosh k_0(H + y)}{\sinh k_0 H} \exp(ik_0 x) \end{aligned}\right\} as\ x \to \infty\ . \tag{4.2}$$

It should be mentioned here that, as $x \to \infty$, ψ_1, ψ_2 defined by (4.1) represent outgoing wave, however, ϕ_1, ϕ_2 represent incoming wave, though ψ_1, ψ_2 and ϕ_1, ϕ_2 satisfy the same boundary value problems described by (3.2) to (3.6). Thus if ψ_1, ψ_2 are known, the time independent potential functions ϕ_1, ϕ_2 can be derived by using (4.1).

Alternative representation for ψ_1, ψ_2 satisfying (3.2) to (3.6), are given by

$$\left.\begin{aligned} \psi_1(x, y) &= c\int_0^\infty \frac{\cosh k(h - y) \sinh kH}{\Delta(k)} \cos kx\, dk \\ \psi_2(x, y) &= -c\int_0^\infty \frac{\cosh k(H + y) \sinh kh}{\Delta(k)} \cos kx\, dk \end{aligned}\right\} \tag{4.3}$$

Where $\Delta(k) = k(1 - s)\sinh kh \sinh kH - K(\cosh kh \sinh kH + s\cosh kH \sinh kh)$.

Here $\Delta(k)$ has a simple pole at $k = k_0 > 0$ (say), a simple pole at $k = k' < 0$ (say), and an infinite number of complex poles with positive real part, of the form $s\xi_n \pm ik_n$ (cf. [8]). Here the contour is indented below the pole at $k = k_0$ to account for the outgoing nature of ψ_1, ψ_2 as $x \to \infty$, and c is a constant to be determined such that the conditions at infinity given by (4.2) are satisfied. It can be easily shown that ψ_1, ψ_2 given by (4.3) satisfy the interface conditions (3.3) and for small r they behave like $\ln r$ (cf. [10]). It is to be noted here that in the absence of the upper liquid (i.e when $s = 0$), $\Delta(k) = 0$ has a simple real root $k_0 > 0$, and an infinite number of purely imaginary roots of the form $\pm ik_n$ (cf. [11]).

ψ_1, ψ_2 given by (4.3), may also be represented as follows :

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor:6.887
Volume 5 Issue IX, September 2017- Available at www.ijraset.com

$$\left.\begin{aligned}\psi_1 &= \frac{2\pi ic}{1-s}\left[\frac{g(k_0)}{f(k_0)}\cosh k_0(h-y)\sinh k_0H\exp(ik_0x) + \sum\frac{g(\gamma)}{f(\gamma)}\cosh\gamma(h-y)\sinh\gamma H\exp(i\gamma x)\right.\\ &\qquad\left.-\sum\frac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma\,(h-y)\sinh\bar\gamma\,Hexp(-i\bar\gamma x)\right]\\ \psi_2 &= -\frac{2\pi ic}{1-s}\left[\frac{g(k_0)}{f(k_0)}\cosh k_0(H+y)\sinh k_0h\exp(ik_0x) + \sum\frac{g(\gamma)}{f(\gamma)}\cosh\gamma(H+y)\sinh\gamma h\exp(i\gamma x)\right.\\ &\qquad\left.-\sum\frac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma\,(H+y)\sinh\bar\gamma\,h\,exp(-i\bar\gamma x)\right]\end{aligned}\right\}\tag{4.4}$$

Where $f(x) = \sinh 2xh\sinh^2 xH + 2xH\sinh^2 xH + s\sinh^2 xh\sinh 2xH + 2sxH\sinh^2 xh,$
$g(x) = \cosh xh\sinh xH + s\cosh xH\sinh xh,$
$\gamma = s\xi_n + ik_n,\quad \bar\gamma = s\xi_n - ik_n,\quad n = 1,2,3,\ldots\ldots\ldots$

Infinity requirements, given by (4.2), are satisfied by choosing

$$c = \frac{i(1-s)D}{2\pi}\tag{4.5}$$

where

$D = \left[\frac{g(k_0)}{f(k_0)}\sinh k_0h\sinh k_0H\right]^{-1}$ ZUsing (4.5) into (4.4), the functions ψ_1, ψ_2 can be found, which are given by

$$\left.\begin{aligned}\psi_1(x,y) &= -\frac{\cosh k_0(h-y)}{\sinh k_0h}\exp(ik_0x) - D\sum\frac{g(\gamma)}{f(\gamma)}\cosh\gamma(h-y)\sinh\gamma H\exp(i\gamma x)\\ &\quad+D\sum\frac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma\,(h-y)\sinh\bar\gamma H\,exp(-i\bar\gamma x)\\ \psi_2(x,y) &= \frac{\cosh k_0(H+y)}{\sinh k_0H}\exp(ik_0x) + D\sum\frac{g(\gamma)}{f(\gamma)}\cosh\gamma(H+y)\sinh\gamma h\exp(i\gamma x)\\ &\quad-D\sum\frac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma\,(H+y)\sinh\bar\gamma h\;\;exp(-i\bar\gamma x)\end{aligned}\right\}\tag{4.6}$$

Exploiting (4.6) into (4.1), the solutions ϕ_1, ϕ_2 for the original boundary value problems described by (3.2) - (3.6) together with conditions at infinity given by (3.7), are obtained, which are given by

$$\left.\begin{aligned}\phi_1(x,y) &= \frac{\cosh k_0(h-y)}{\sinh k_0h}\exp(-ik_0x) - D\sum\frac{g(\gamma)}{f(\gamma)}\cosh\gamma(h-y)\sinh\gamma H\exp(i\gamma x)\\ &\quad+D\sum\frac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma\,(h-y)\sinh\bar\gamma H\,exp(-i\bar\gamma x)\\ \phi_2(x,y) &= -\frac{\cosh k_0(H+y)}{\sinh k_0H}\exp(-ik_0x) + D\sum\frac{g(\gamma)}{f(\gamma)}\cosh\gamma(H+y)\sinh\gamma h\exp(i\gamma x)\\ &\quad-D\sum\frac{g(\bar\gamma)}{f(\bar\gamma)}\cosh\bar\gamma\,(H+y)\sinh\bar\gamma h\;\;exp(-i\bar\gamma x)\end{aligned}\right\}\tag{4.7}$$

Making use of (4.7) into (3.1), the velocity potentials $\Phi_1(x,y,t), \Phi_2(x,y,t)$ have been found (see Appendix-I). The explicit expressions for Φ_1, Φ_2 are given by

$$\left.\begin{aligned}\Phi_1(x,y,t) &= \frac{\cosh k_0(h-y)}{\sinh k_0h}\cos(k_0x+\sigma t) - 2D\sin\sigma t\sum u(k_n)\exp(-k_nx)\\ \Phi_2(x,y,t) &= -\frac{\cosh k_0(H+y)}{\sinh k_0H}\cos(k_0x+\sigma t) + 2D\sin\sigma t\sum v(k_n)\exp(-k_nx)\end{aligned}\right\}\tag{4.8}$$

Φ_1, Φ_2 represented by (4.8) are the velocity potentials for incoming water waves against a rigid vertical cliff in two immiscible liquids.

V. SPECIAL CASE

As a special case, if we make the assumption $s = 0$ (which leads to one fluid medium), we find (see Appendix-II):

$$D = \frac{2(2k_0h + \sinh 2k_0h)}{\sinh 2k_0h},\quad u(k_n) = -\frac{\cos k_n(h-y)\cos k_nh}{(2k_nh + \sin 2k_nh)}$$

so that

$$\Phi_1(x,y,t) = \frac{\cosh k_0(h-y)}{\sinh k_0h}\cos(k_0x+\sigma t) + \frac{4(2k_0h+\sinh 2k_0h)\sin\sigma t}{\sinh 2k_0h}\sum\frac{\cos k_n(h-y)\cos k_nh}{(2k_nh+\sin 2k_nh)}\exp(-k_nx)\tag{5.1}$$

The above expression for $\Phi_1(x, y, t)$ is the velocity potential for a two-dimensional progressive wave train moving towards a rigid cliff in a single liquid of uniform finite depth `h'.

To check, if one assumes

$$\phi_1 \to \frac{\cosh k_0(h-y)}{\sinh k_0 h} \exp(-ik_0 x), \text{ instead of}$$

$$\phi_1 \to \frac{\cosh k_0(h-y)}{\sinh k_0 h} \exp(-ik_0 x) \;\; as\; x \to \infty$$

In the calculations of Mandal and Kundu ([6]), the expression of $\Phi_1(x, y, t)$, for a single liquid given by (5.1) can be recovered.

VI. CONCLUSIONS

A relatively simple approach to find the solution of the two dimensional incoming waves at the interface of two superposed liquids and progressing towards a rigid vertical cliff is demonstrated here. Assuming linear theory, the explicit expression for the velocity potentials in each of the two liquid are obtained where the lower and upper liquids to be of finite depth and finite height respectively. The major advantage of the method described in this work is that the solution of the corresponding problem in the absence of upper liquid can be found simply by the substitution of $s = 0$. This problem is a simplified mathematical model of the well known sloping beach problem arising in oceanography.

Appendix-I:

$$u(k_n) = \frac{[H_1X_1Y_2 + H_2X_1Y_1 - H_1X_2Y_1 + H_2X_2Y_2]}{X_1^2 + X_2^2}, \qquad v(k_n) = \frac{[J_1X_1Y_2 + J_2X_1Y_1 - J_1X_2Y_1 + J_2X_2Y_2]}{X_1^2 + X_2^2},$$

$$H_1 = (G_1 \cos s\xi_n x - G_2 \sin s\xi_n x), \qquad H_2 = (G_1 \sin s\xi_n x + G_2 \cos s\xi_n x),$$

$$J_1 = (I_1 \cos s\xi_n x - I_2 \sin s\xi_n x), \qquad J_2 = (I_1 \sin s\xi_n x + I_2 \cos s\xi_n x),$$

$$X_i = A_i + B_i + C_i + D_i, \qquad Y_i = E_i + F_i, \; (i = 1,2)$$

$$A_1 = \frac{1}{2}[P(H)R(h) - Q(H)S(h) - R(h)], \qquad A_2 = \frac{1}{2}[P(H)S(h) + Q(H)R(h) - S(h)],$$

$$B_1 = [P(H) - 1]s\xi_n h - Q(H)k_n h, \qquad B_2 = [P(H) - 1]k_n h + Q(H)s\xi_n h,$$

$$C_1 = \frac{s}{2}[P(h)R(H) - Q(h)S(H) - R(H)], \qquad C_2 = \frac{s}{2}[P(h)S(H) + Q(h)R(H) - S(H)],$$

$$D_1 = s[\{P(h) - 1\}s\xi_n H - Q(h)k_n H], \qquad D_2 = s[\{P(h) - 1\}k_n H + Q(h)s\xi_n H],$$

$$E_1 = T(h)V(H) - U(h)W(H), \qquad E_2 = T(h)W(H) + U(h)V(H),$$

$$F_1 = s[T(H)V(h) - U(H)W(h)], \qquad F_2 = s[T(H)W(h) + U(H)V(h)],$$

$$G_1 = T(h-y)V(H) - U(h-y)W(H), \qquad G_2 = T(h-y)W(H) + U(h-y)V(H),$$

$$I_1 = T(H+y)V(h) - U(H+y)W(h), \qquad I_2 = T(H+y)W(h) + U(H+y)V(h),$$

$$P(x) = \cosh 2s\xi_n x \cos 2k_n x, \qquad Q(x) = \sinh 2s\xi_n x \sin 2k_n x,$$

$$R(x) = \sinh 2s\xi_n x \cos 2k_n x, \qquad S(x) = \cosh 2s\xi_n x \sin 2k_n x,$$

$$T(x) = \cosh s\xi_n x \cos k_n x, \qquad U(x) = \sinh s\xi_n x \sin k_n x,$$

$$V(x) = \sinh s\xi_n x \cos k_n x, \qquad W(x) = \cosh s\xi_n x \sin k_n x.$$

Appendix-II: Upon substitution $s = 0$, we have

$$P(x) = \cos 2k_n x, \quad Q(x) = R(x) = 0, \quad S(x) = \sin 2k_n x, \quad A_1 = 0, \quad A_2 = -\sin^2 k_n H \sin 2k_n h,$$

$$B_1 = 0, \quad B_2 = -2k_n h \sin^2 k_n H, \quad C_1 = C_2 = D_1 = D_2 = 0, \quad X_1 = 0, X_2 = -(2k_n h + \sin 2k_n h)\sin^2 k_n H$$

$$T(x) = \cos k_n x, \quad U(x) = V(x) = 0, \quad W(x) = \sin k_n x, \quad E_1 = 0, \; E_2 = \cos k_n h \sin k_n H, \; F_1 = F_2 = 0$$

$$Y_1 = 0, \; Y_2 = \cos k_n h \sin k_n H, \; G_1 = 0, \; G_2 = \cos k_n(h-y)\sin k_n H, \; H_1 = 0, \; H_2 = \cos k_n(h-y)\sin k_n H,$$

$$f(k_0) = (2k_0 h + \sinh 2k_0 h)\sinh^2 k_0 H, \qquad g((k_0) = \cosh k_0 h \sin k_0 H,$$

$$D = \frac{2(2k_0 h + \sinh 2k_0 h)}{\sinh 2k_0 h}, \qquad u(k_n) = -\frac{\cos k_n(h-y)\cos k_n h}{(2k_n h + \sin 2k_n h)}.$$

REFERENCES

[1] B. A .Packham, “Capillary-gravity waves against a vertical cliff ,” Proc Camb. Phil. Soc., vol. 64, pp. 827–832, 1968.
[2] J. J. Stoker, “Surface waves in water of variable depth,” Quart. Appl. Math.., vol. 5, pp. 1–54, 1947.
[3] J. J. Stoker, “Water waves,” Interscience, New York, 1957.
[4] A. Chakrabarti, “Capillary-gravity waves against a corrugated vertical cliff,” Appl. Sci. Res., vol. 45, pp. 303-317, 1988.
[5] A. Chakrabarti, “Obliquely incident water waves against a vertical cliff,” Appl. Math. Lett., vol. 5, pp. 13-17, 1992.
[6] B. N. Mandal, and P. K. Kundu, “Incoming water waves against a vertical cliff in an ocean,” Proc. Indian. Natn. Sci. Acad., vol. 55A, pp. 643-654, 1989.
[7] P. Agasti, and P. K. Kundu, “Incoming waves against a vertical cliff in two liquids,” Advances and Applications in Fluid mechanics, vol. 7(1), pp. 71-81, 2010.
[8] R. N. Chakrabarti, and B. N. Mandal, “Singularities in a two fluid medium,” Internat. J. Math and Math. Sci., vol. 6, pp. 737-754, 1983.
[9] Y. S. Yu, and F. Ursell, “Surface waves generated by an oscillating circular cylinder on water of finite depth: theory and experiment,” J. Fluid. Mech., vol. 11, pp. 529-551, 1961.
[10] M. A. Gorgui, and S. E. Kassem, “Basic singularities in the theory of internal waves,” Quart. J. Mech. Appl. Math.., vol. 31, pp. 31-48, 1978.
[11] R. C. Thorne, “Multipole expansions in the theory of surface waves,” Proc Camb. Phil. Soc., vol. 49, pp. 707-715, 1953.