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Analytical Approximate Solution of a Coupled Two Frequency Hill's Equation

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Abstract: A coupled two frequency Hill's equation is solved. Analytically approximate solution correct up-to first order is derived using modified Lindstedt-Poincare perturbation method. For a wide range of controlling parameters, we compare the numerical and analytical solutions. The solution is the first step towards developing a comprehensive understanding of the electrodynamics of charged particles in a combinational ion trap utilizing both electrostatic DC and RF fields along with a constant static magnetic field with prospects of confining antimatter such as anti-hydrogen for a reasonably long durations of time.
Keywords: Hill's equation, Perturbation methods, Modified Lindstedt Poincare method, Paul trap, Penning trap

I. INTRODUCTION

Hill's equation [1] is a second order differential equation with periodic coefficients. The equation can be described as

$$\ddot{x} + f(t)x = 0\#(1)$$

where $f(t)$ is a period function, often a combination of several cosine and sine functions. Hill's equations find application in several diverse areas of applied sciences. The differential equation appears in several settings, such as, in the analysis of lunar stability [2], modeling of quadrupole mass spectrometer [3], the dynamics of an electron in a crystal using one dimensional Schrodinger equation [4], in a two level system in quantum optics [5] and electromagnetic ion traps [6] in which electrostatic DC and RF fields are used to confine charged particles in a limited space in a perturbation free environment.

A well-known equation arising out of Eq. (1) is the Mathieu equation:

$$\ddot{x} + (a - 2q_1\cos(2t))x = 0\#(2)$$

Controlling parameters, a, q_1 determine the stability of the solution of Eq. (2). For example, if $a = 0$, the solutions up-to $q_1 = 0.9$ are stable. Stability of Eq. (2) is well documented in the literature. The equation governs the dynamics of charged particles inside an electromagnetic ion trap, namely, Paul trap, wherein charged particles are under the influence of electrostatic DC and RF fields only. The coefficients a and q_1 are proportional to the applied voltage strengths. In this paper, we attempt to derive analytical approximate solution for a coupled two frequency Hill's equation [7] which can be written as

$$\ddot{x} - p\dot{y} + (a - 2q_1\cos(2\eta^{-1}t) - 2q_2\cos(2t))x = 0 \qquad (3)$$

Such a coupled system has recently gained importance to study the electrodynamics of charge particles relevant to particle confinement using two radio frequencies [7,15] in a combinational trap utilizing features of both Paul and Penning trap. In context to such a trapping, coefficients a, q_1, q_2 are proportional to the applied electrostatic DC and RF (radio frequency) and p is the proportional to the applied magnetic field.

To get a better understanding of how Eq. (3) relates to the trapping of particles inside a combinational trap, consider the quadrupole potential in a dual frequency Paul trap given by

$$\Phi(x,y,z,t) = \left(U_0 + V_1\cos(\omega_1 t) + V_2\cos(\omega_2 t)\right)\left((x^2 + y^2 - 2z^2)/r_0^2\right)\#(4)$$

The electric field generated by this potential is $\vec{E}(x,y,z,t) = -\vec{\nabla}\Phi(x,y,z,t)$. Since there exists a magnetic field $\vec{B} = B_0\hat{k}$ due the features attributed to a Penning trap, the net force experienced by a charged particle of charge Q and mass M moving with a velocity $\vec{v}$ is given by the Lorentz force equation $\vec{F} = -Q\vec{\nabla}\Phi + Q(\vec{v} \times \vec{B})$. If $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$, the components of force in the three orthogonal directions, i. e., F_x, F_y, F_z are given by:

$$F_x = -\left(U_0 + V_1\cos(\omega_1 t) + V_2\cos(\omega_2 t)\right)(2x/r_0^2) + Qv_yB_0\#(5)$$

$$F_y = -\left(U_0 + V_1\cos(\omega_1 t) + V_2\cos(\omega_2 t)\right)(2y/r_0^2) - Qv_xB_0\#(6)$$

$$F_z = \left(U_0 + V_1\cos(\omega_1 t) + V_2\cos(\omega_2 t)\right)(4z/r_0^2)\#(7)$$

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Here, $U_0, V_{1,2}$ are the applied DC and RF voltages respectively, ω_1, ω_2 are the primary and secondary RF frequencies, respectively, and r_0 is the trap dimension. Upon substituting $\omega_2 t = 2\tau, F_x = M\ddot{x}, F_y = M\ddot{y}, a = 8QU_0/Mr_0^2$, $q_{1,2} = -4QV_{1,2}/Mr_0^2, p = 2QB_0/\omega_2 M, v_x = \dot{x}, v_y = \dot{y}$ and $\omega_2/\omega_1 = \eta$ in Eq. (5), Eq. (6) and rearranging the terms, one obtains the two coupled equations given in Eq. (3). Since τ is a dummy variable, without loss of generality, it can be replaced by t in the subsequent equations. The confinement in the $x - y$ plane is through a set of coupled differential equations given by Eq. (3), whereas, along the z axis, the trapping is on account of a combination of DC and RF voltages, exactly like it is in a dual frequency Paul trap. The Lorentz force due to the magnetic field acts inwards. This increases the stability of the charged particles simultaneously being trapped by the application of a static and a dynamic electric field in combination with a constant magnetic field. In recent years, the trap employing dual frequency has gained importance since it is being viewed as a promising option to trap anti-hydrogen. In general, charged particles with varied charge to mass ratio can be trapped effectively inside a dual frequency Paul trap [7].
To produce anti-hydrogen, positron and antiproton are to be trapped and a magnetic field is required to trap the resulting neutral particle, anti-hydrogen. The limitation of a conventional single frequency Paul trap in trapping two species with different charge to mass ratio is that the weakly confined species is pushed away from the trap center [8]. The ALPHA experiment [9,10] and ATRAP experiment [11,12] rely on a variation of Penning trap using static magnetic field for their initial confinement. However, it is not possible to trap oppositely charged particles in a Penning trap on account of the presence of only DC electric field along with a static magnetic field. Hence a combinational trap inheriting features of both a dual frequency Paul trap and a Penning trap holds a lot of potential in confinement of oppositely charged species with a large charge to mass variation and will most certainly be a significant improvement when compared to earlier methods utilizing both electric and magnetic fields in a conventional single frequency Paul trap [13,14]. Dynamics governed by the differential equations given in Eq. (3) is therefore of great interest. It offers a starting point to the understanding of the electrodynamics that will emerge inside a combinational trap. In Sec. 2, we derive the time evolution of position of the confined particle in x and y direction. In Sec. 3, comparison of the analytical approximate solution with the numerical solution for a wide range of control parameters shows the robustness of the solution to depict the particle dynamics. Sec. 4 contains a conclusion and a discussion on the importance of the analytical solution.

II. ANALYTICAL APPROXIMATE SOLUTION

We begin expressing the equations in a concise form by writing $A(t) = a - 2q_1\cos(2\eta^{-1}t) - 2q_2\cos(2t)$. The coupled differential equations in Eq. (3) can now be written as

$$\ddot{x} - p\dot{y} + A(t)x = 0 \# (8)$$

$$\ddot{y} + p\dot{x} + A(t)y = 0 \# (9)$$

Multiplying Eq. (8) by imaginary j and adding to Eq. (9) gives

$$\ddot{z} - jp\dot{z} + A(t)z = 0 \# (10)$$

Where $z = y + jx$. Let $z = w(t)\exp(jpt/2)$. The function $w(t)$ is a complex function which can further be substituted as $w = X + jY$. Hence upon substituting $z = (X + jY)\exp(jpt/2)$ and after some basic manipulations, Eq. (10) can be written as

$$\ddot{w} + (A(t) + p^2/4)w = 0 \# (11)$$

here, $\ddot{w} = \ddot{X} + j\ddot{Y}$. Writing $a1 = a + p^2/4, q_2 = q_r q_1, \Omega_1 = 2\eta^{-1}, \Omega_2 = 2$, Eq. (11) can be expressed as

$$\ddot{w} + \left(a1 - 2q_1\cos(\Omega_1 t) - 2q_r q_1\cos(\eta\Omega_1 t)\right)w = 0 \# (12)$$

Applying Modified Lindstedt-Poincare method [16] in Eq. (12), we begin by writing:

$$a1 = \nu^2 + q_1\alpha_1 + q_1^2\alpha_2$$

Substituting the values of $a1$ and x from Eq. (13) in Eq. (12) and solving equations, one at a time for $\mathcal{O}(q_1^0), \mathcal{O}(q_1^1), \mathcal{O}(q_1^2)$, we get

$$X = D_1\phi(t) + E_1\psi(t) \# (14)$$

$$Y = D_2\phi(t) + E_2\psi(t) \# (15)$$

here $D_{1,2}$ and $E_{1,2}$, are real constants that depend on the initial positions and velocities of the charged particle, i.e., x_0, y_0, v_{x0}, v_{y0}. Moreover, one can express $\phi(t)$ and $\psi(t)$, correct up-to first order as

$\phi(t) = \cos(\nu t) + a_1\cos(\nu - \Omega_1)t + a_2\cos(\nu + \Omega_1)t + a_3\cos(\nu - \eta\Omega_1)t + a_4\cos(\nu + \eta\Omega_1)t$

$\psi(t) = \sin(\nu t) + a_1\sin(\nu - \Omega_1)t + a_2\sin(\nu + \Omega_1)t + a_3\sin(\nu - \eta\Omega_1)t + a_4\sin(\nu + \eta\Omega_1)t$

here $a_1 = q_1/(\nu^2 - (\nu - \Omega_1)^2), a_2 = q_1/(\nu^2 - (\nu + \Omega_1)^2), a_3 = q_r q_1/(\nu^2 - (\nu - \eta\Omega_1)^2), a_4 = q_r q_1/(\nu^2 - (\nu + \eta\Omega_1)^2)$ and ν is the slow frequency given by

$$\nu = \sqrt{[(a + p^2/4) + (2q_1^2/\Omega_1^2)(1 + q_r^2/\eta^2)]} \# (18)$$

In Eq. (13), the constants a1 and x are written up-to second order, even though independent solutions of Eq. (16), Eq. (17) are written up-to first order. This has been done to evaluate the slow frequency by deriving expressions for α_1 and α_2. The value of α_1, to eliminate secular terms for $\mathcal{O}(q_1^1)$ comes out to be $\alpha_1 = 0$. Similarly, the value of α_2, to eliminate secular terms for $\mathcal{O}(q_1^2)$ comes out to be $\alpha_2 = (-2/\Omega_1^2)(1 + q_r^2/\eta^2)$. Backtracking from X and Y, the time evolution of position $x(t)$ and $y(t)$ for the charged particle is

$$x = Y\cos(pt/2) + X\sin(pt/2)\#(19)$$
$$y = X\cos(pt/2) - Y\sin(pt/2)\#(20)$$

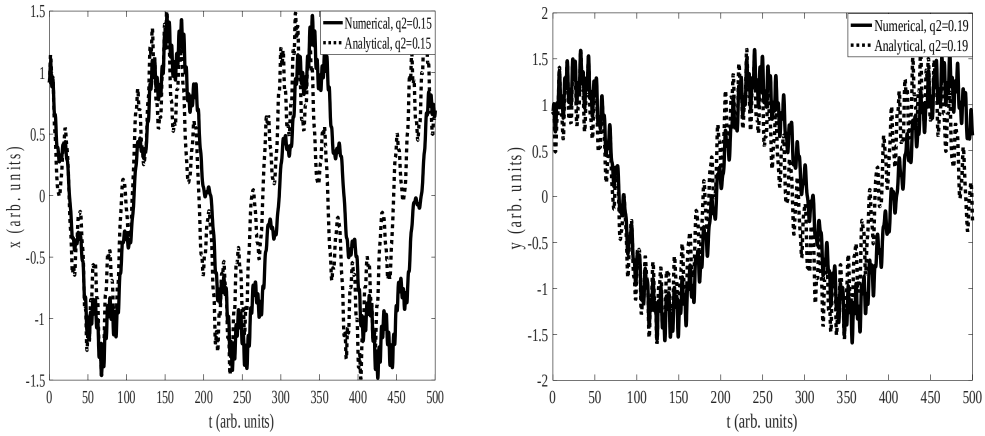
It is worth observing that $\dot{\phi}_0 = \dot{\phi}(t = 0) = 0$ and $\psi_0 = \psi(t = 0) = 0$. If one writes $\phi_0 = \phi(t = 0)$ and $\dot{\psi}_0 = \dot{\psi}(t = 0)$, the values of constants $D_{1,2}$ and $E_{1,2}$ come out to be, $D_1 = y_0/\phi_0, D_2 = x_0/\phi_0, E_1 = (v_{y0} + px_0/2)/\dot{\psi}_0$ and $E_2 = (v_{x0} - py_0/2)/\dot{\psi}_0$.

III. COMPARISON OF ANALYTICAL SOLUTION WITH NUMERICAL SOLUTION

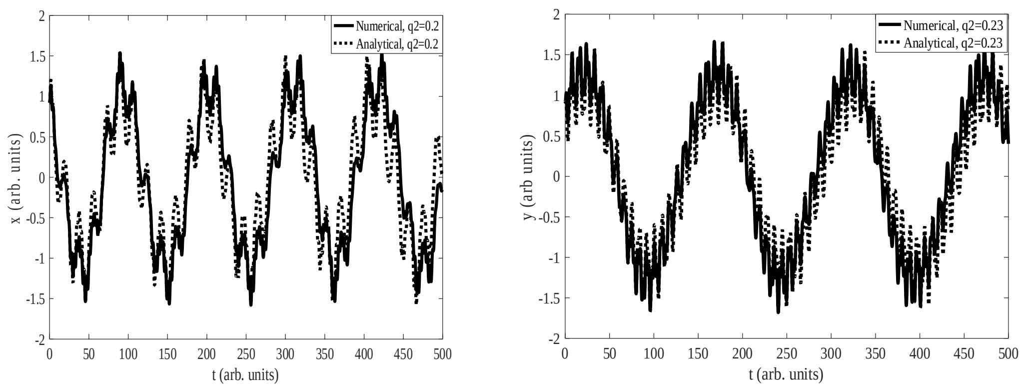
The solutions are obtained by varying the controlling parameters, namely, p, q_1, q_2 and η. In Fig. 1, a comparison of the numerical and analytical solution is shown with parameter values $p = 0.3, q_1 = 0.0011, \eta = 45$ in sub figures (a) $q_2 = 0.15$, (b) $q_2 = 0.2$, (c) $q_2 = 0.24$ and with parameter values $p = 0.7$, $q_1 = 0.002, \eta = 45$ in sub figures (d) $q_2 = 0.19$, (e) $q_2 = 0.23$, (f) $q_2 = 0.27$.
In Fig. 2, a comparison of the numerical and analytical solution is shown with parameter values $p = 0.9, q_1 = 0.002, \eta = 45$ in sub figures (a) $q_2 = 0.15$, (b) $q_2 = 0.17$, (c) $q_2 = 0.2$ and with parameter values $p = 0.3, q_1 = 0.0011$, $q_2 = 0.2$ in sub figures (d) $\eta = 5$, (e) $\eta = 35$, (f) $\eta = 55$. The values of q_1 and q_2 are proportional to the applied RF voltages V_1 and V_2 respectively, p is proportional to the applied magnetic field strength B_0 and η is the ratio of the secondary voltage frequency ω_2 and primary voltage frequency ω_1.

IV. CONCLUSION AND DISCUSSION

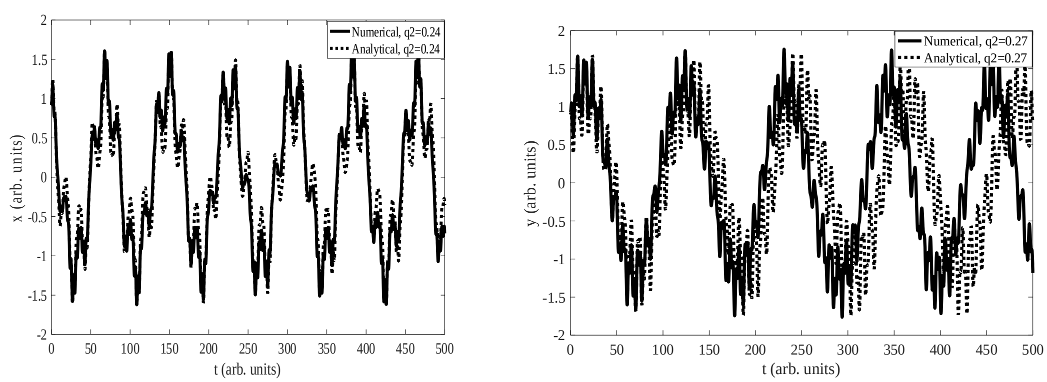
For particle trajectory in $x - y$ plane, the analytical approximate solution correct up-to first order, is derived for the coupled two frequency Hill's equation using modified Lindstedt-Poincare method. The analytical solution matches well with the numerical solution obtained by numerical simulating the system of coupled differential equations given in Eq. (3). The analytical solution has a limited number of harmonic terms, i.e., $(\nu \pm \Omega_1)$ and $(\nu \pm \eta\Omega_1)$ terms, whereas the numerical solution encompasses the effect of all the harmonic terms which make up the complete solution. Therefore, the matching is observed for some range of controlling parameters only.
If the order of the analytical solution is increased, the range of operating parameters for which the two solutions coincide will widen. However, the derivation of such higher order terms will be mathematically challenging. It is important to see that the solution described by Eq. 16 and Eq. 17 will blow up when $\eta \sim 1$.
To obtain single frequency solutions one can simply substitute $q_r = 0$ and keep η away from unity. In most of the practical settings [7,15], the value of η is substantially higher than unity, a regime wherein the analytical solutions are a good match to the numerical solutions. Experience guides us that analytical solution correct up-to first and second order are usually sufficient to provide deeper insights to both individual particle as well as collective dynamics inside the trap [17-19].
The relevance of an analytical solution cannot be understated when one must study the collective dynamics inside such combinational traps. Since the fields are spatially linear in this set up, one must see if a distribution function can be constructed for the particles by the method of inversion [17]. It is well known that RF heating on account of applied RF fields will increase the temperature of the charged particles.
The analytical tracking of temperature variation for each species inside such a trap is therefore important [18-19]. Temperature can be evaluated as the second order moment of the distribution function. To the best of my knowledge, such analytical work on collective dynamics for combinational traps has not been undertaken. Going ahead in this direction will require us to choose some operating parameters for stable configuration. The analytical expressions for particle dynamics derived in this work assumes importance as a vital starting point.
Imperfections in electrode geometry of the trap introduce deviations from the quadrupole potential. It would be interesting to see if analytical solutions can be derived for particle dynamics in such a scenario. Study of nonlinear resonances, deviation in the values of secular frequencies, changes in the stability regimes of the dynamics are all very interesting problems that could be taken up as future work.



(a) (d)

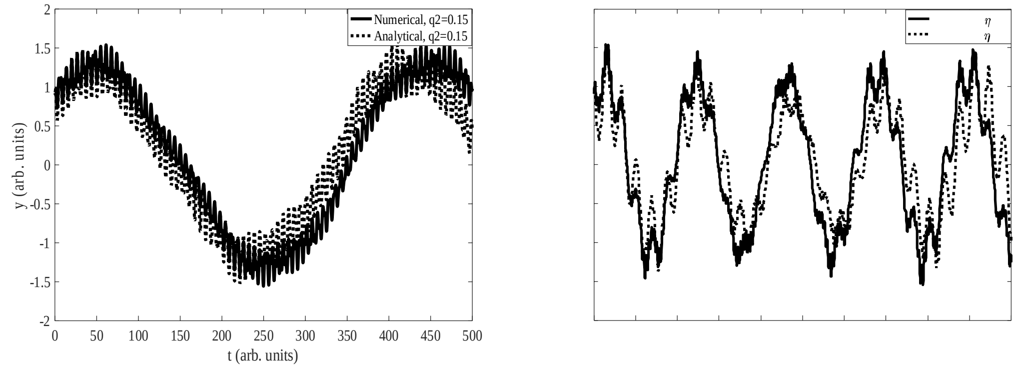


(b) (e)

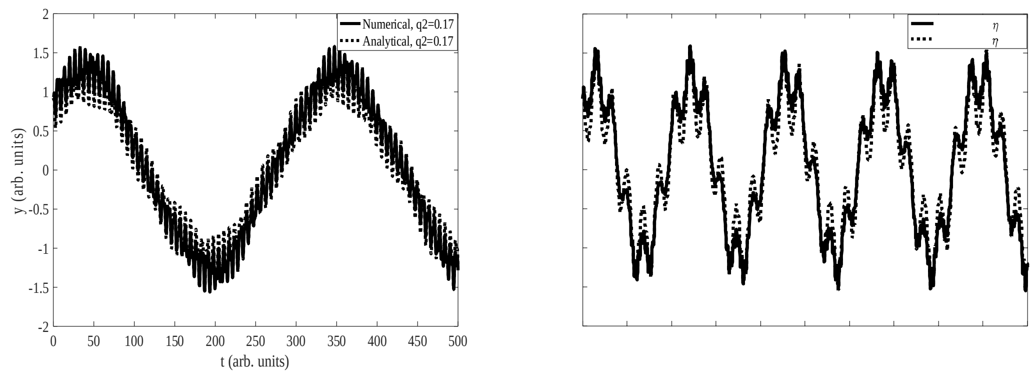


(c) (f)

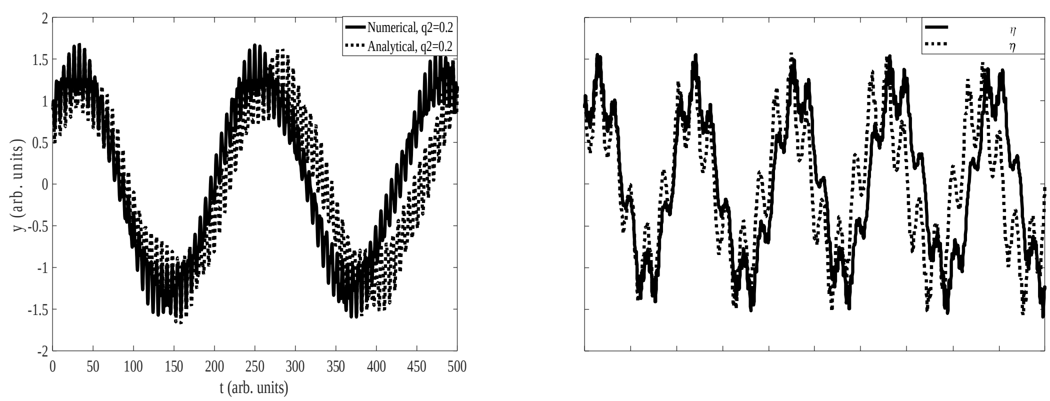
Fig 1. The plots (a), (b), (c), show a comparison of the numerical and the analytical solutions for parameter values $p = 0.3, q_1 = 0.0011, \eta = 45$. Plots (d), (e), (f) show a comparison of the numerical and the analytical solutions for $p = 0.7, q_1 = 0.002, \eta = 45$.



(a) (d)



(b) (e)



(c) (f)

Fig 2. The plots (a), (b), (c), show a comparison of the numerical and the analytical solutions for parameter values $p = 0.9, q_1 = 0.002, \eta = 45$ Plots (d), (e), (f) show a comparison of the numerical and the analytical solutions for $p = 0.3, q_1 = 0.0011, q_2 = 0.2$.

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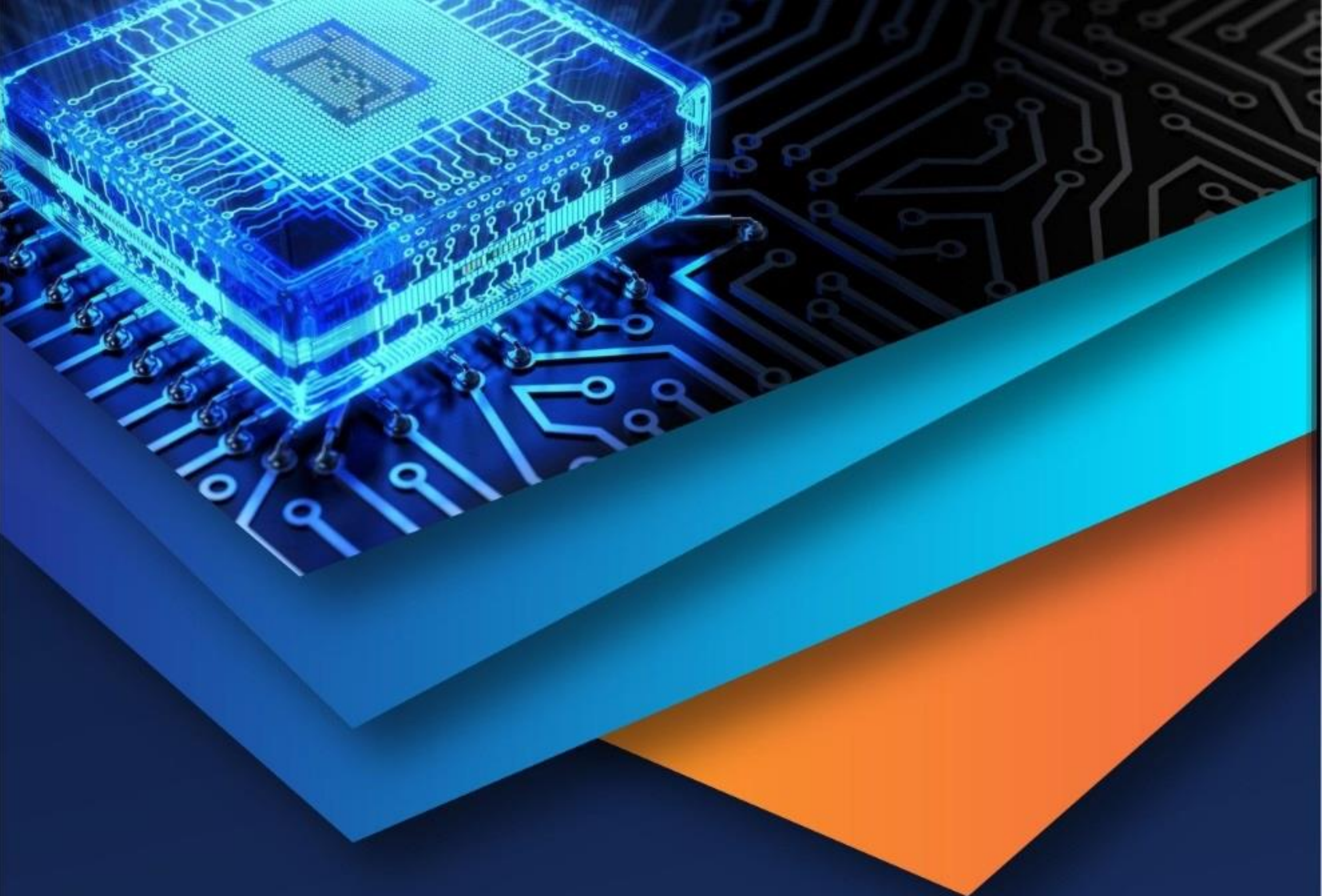

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