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Application of Third Order Taylor Method to Population Equations

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Abstract: In this paper, we implement the third order Taylor method for three different population initial value problems. The Taylor method is derived from the Taylor series expansion. The Taylor series method is one of the earliest methods for the approximate solution for initial value problems for ordinary differential equations. Taylor method is implemented to linear population equation, non-linear population equation and non-linear population equation with an oscillation. The method of solving three initial value problems is implemented using Python Programming.
Keywords: Differential equation, Taylor method, Population equation, Non-linear population equation, Oscillation, Python.

I. INTRODUCTION

The Taylor polynomial series approximation method is well known and is used in variety of applications. Important application of Taylor method is that it can be executed using interval arithmetic and hence allows us to obtain validated numerical methods for differential equations [12]. Taylor series expansion is an amazing concept not only in Mathematics but also in Optimization theory, Function approximation and Machine Learning [11]. It is widely applied in numerical computations when estimate of function values at different points are required [9]. Georg Fuchs et al. presented the application of Taylor series method for a practical mechanical engineering application. The performance of Taylor method is demonstrated by comparison to standard fixed step numerical integration methods [1]. Okan Ozer et al. applied Taylor expansion to determine the analytical expression for eigenfunctions. The results are obtained by simple algorithm produces excellent numerical results for eigenvalues [2]. Robert Bario investigated Taylor series method by using an efficient variable step variable order scheme [3]. Atefeh Armand et al. have proposed Taylor expansion for fuzzy valued functions. The effectiveness of the proposed method is verified by examples [4].

Marija Milosevic et al. have investigated the application Taylor series method for solving stochastic differential equations with time-dependent delay [5]. Vazquez-Leal H. et al. proposed the application of Taylor series method for solving non-linear differential equations on finite intervals. Their result shows that the Taylor series method is capable to generate easily computable and highly accurate approximations for non-linear equations [6]. Suchismita Ghosh et al. have applied Taylor series method to solve states of control systems. They have analyzed the states of the control system by Taylor series method and compared with exact solutions [7]. Eduardo Pasquetti and Paulo B.G. have proposed the application of Taylor expansion to solve non-linear ordinary differential equations with non-polynomial non-linearities [8]. This paper proposes the application of third order Taylor method for three different population initial value problems. The paper is organized as follows: Section II presents Taylor Method, Section III discusses the Population Equation, Section IV focuses on Implementation and Results and finally the Conclusion is presented in Section V.

II. TAYLOR METHOD

Taylor series is an expansion of a function into an infinite series of a variable x. The coefficients of the expansion or of the subsequent terms of the series involve the successive derivatives of the function [10]. The function to be expanded should have a n^{th} derivative in the interval of expansion. The function $f(x)$ has derivatives of all orders of a given intervals, the Taylor series is generated by $f(x)\ at\ x = a$ is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots$$

The general third order Taylor method for the first order differential equation

$y' = f(t, y)$ with the initial condition $y(t_0) = y_0 \ldots\ldots\ldots\ldots\ldots..(1)$

numerical approximates $y\ at\ time\ point\ t_i$ as y_i with the formula

$$y_{i+1} = y_i + h\left[f(t_i, y_i) + \frac{h}{2}f'(t_i, y_i) + \frac{h^2}{6}f''(t_i, y_i)\right] \ldots\ldots\ldots (2)$$

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$for\ i = 0,1,2, \dots \dots \dots , N-1$, where h is the step size.

With the local truncation error of $\tau = \frac{h^4}{4!} f''''(\mu_i) where\ \mu_i \epsilon [t_i\ , t_{i+1}]$

III. POPULATION EQUATION

The general form of population growth differential equation is

$y' = ky \dots \dots \dots \dots . (3)$ where k is the growth rate.

The initial population at time a is $y(a) = A,\ a \le t \le b$

Integrating equation (3) gives the analytic solution $y = Ae^{kx}$. We will use this equation to illustrate the application of the Taylor method.

The general form of the non-linear sigmoidal population growth differential equation is

$$y' = \alpha y - \beta y^2 \dots \dots \dots \dots \dots \dots . (4)$$

and the non-linear sigmoidal population growth differential equation with oscillation is

$$y' = \alpha y - \beta y^2 + \sin(2\pi t) \dots \dots \dots \dots \dots \dots . (5)$$

where $\alpha\ is\ the\ growth\ rate\ and\ \beta\ is\ the\ death\ rate$. The initial population at time a is $y(a) = A,\ a \le t \le b$

1) *Specific Non-Linear Population Equation:* Given the growth rate $\alpha = 0.2\ and\ death\ rate\ \ \beta = 0.01$, giving the specific non-linear population differential equation

$y' = (0.2)y - (0.01)y^2$

and the specific non-linear population differential equation with oscillation

$y' = (0.2)y - (0.01)y^2 + \sin(2\pi t)$.

The initial population at time 2000 is $y(2000) = 6$, we are interested in the time period

$2000 \le t \le 2020$.

2) *Initial Condition:* To obtain a specific solution to a first order initial value problem, the initial population is 6 billion people and therefore the initial condition is considered as $y(2000) = 6$. In the year 2000 the world population was 6.1143 billion.

Let us consider three initial value problems to apply the third order Taylor method.

a) *Linear Population Equation*

Consider the linear population differential equation

$$y' = (0.1)y, \quad (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

b) *Non-linear Population Equation*

Consider the non-linear population differential equation

$$y' = (0.2)y - (0.01)y^2, \qquad (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

c) *Non-linear Population Equation with an Oscillation*

Consider the non-linear population differential equation with an oscillation

$$y' = (0.2)y - (0.01)y^2 + sin(2\pi t), \qquad (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

3) *Discrete Interval:* The continuous time interval $a \le t \le b$ is discretized into N interval separated by a constant step size $h = \frac{b-a}{N}$. Here the interval is $2000 \le t \le 2020$ with $N = 200$.

$$\therefore h = \frac{2020 - 2000}{200} = 0.1$$

This gives 201 discrete points with step size $h = 0.1$

$t_0 = 2000, t_1 = 2000.1, \dots \dots \dots , t_{200} = 2020$.

This is generalized to $t_i = 2000 + (0.1)i,\ i = 0,1,2, \dots \dots \dots \dots . . ,200$.

The graph below shows the discrete time points for $h = 0.1$

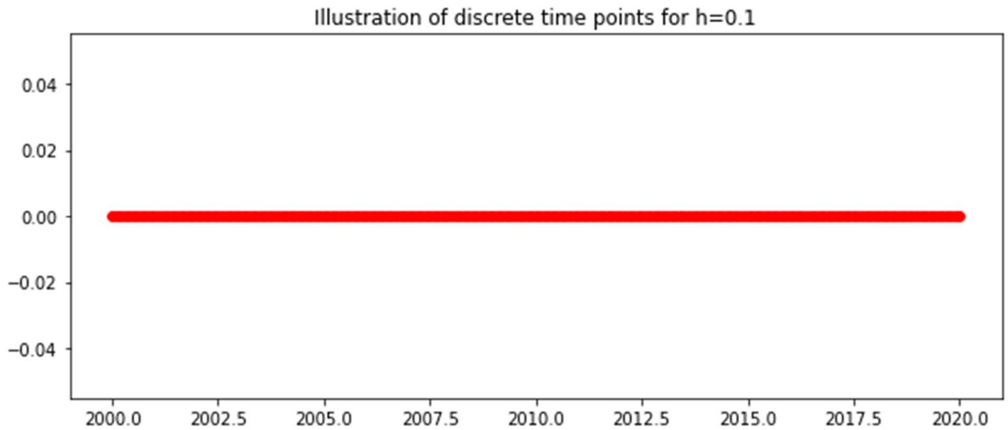

Fig. 1 Discrete Time points for $h = 0.1$

IV. IMPLEMENTATION AND RESULTS

A. Taylor Method to Linear Population Equation

The linear population differential equation

$$y' = (0.1)y, \qquad (2000 \leq t \leq 2020) \ldots\ldots\ldots (6)$$

with the initial condition $y(2000) = 6$ has analytic solution $y = 6e^{(0.1)(t-2000)}$

To write the specific third order Taylor method for the linear population equation

$f(t, y) = (0.1)y \ldots\ldots\ldots\ldots (7)$

Differentiating (7) with respect to t,

$f'(t, y) = (0.1)y' = (0.1)[(0.1)y] = 0.01y$

and the second derivative of f with respect to t,

$f'(t, y) = (0.01)y' = (0.01)(0.1)y = 0.001y$

1) *Linear Population third order Taylor Difference equation:* Substituting the derivatives of the linear population equation into the third order Taylor equation gives the difference equation

$$y_{i+1} = y_i + h\left[(0.1)y_i + \frac{h}{2}\left((0.01)y_i\right) + \frac{h^2}{6}\left((0.001)y_i\right)\right] \ldots\ldots\ldots (8)$$

$for\ i = 0,1,2, \ldots\ldots\ldots, 199, where\ y_i$ is the numerical approximation of $y\ at\ time\ t_i$, with the step size h and the initial condition $y_0 = 6$. The figure below shows the exact solution, y (squares) and the third order numerical approximation y_i (circles) for the linear population equation.

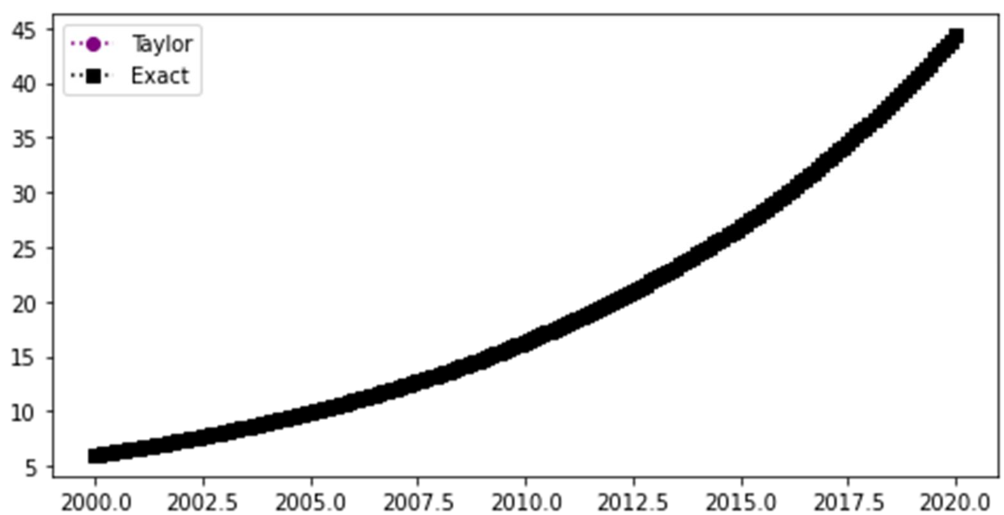

Fig. 2 Exact solution and Taylor approximation for Linear Population equation

Table 1 below shows the time, the third order numerical approximation y_i, the exact solution y and the exact error $|y(t_i) - y_i|$ for the linear population equation.

	time t_i	Taylor	Exact (y)	Exact Error
0	2000.0	6.000000	6.000000	0.000000e+00
1	2000.1	6.060301	6.060301	2.500000e-09
2	2000.2	6.121208	6.121208	5.100000e-09
3	2000.3	6.182727	6.182727	7.700000e-09
4	2000.4	6.244865	6.244865	1.030000e-08
5	2000.5	6.307627	6.307627	1.300000e-08
6	2000.6	6.371019	6.371019	1.580000e-08
7	2000.7	6.435049	6.435049	1.860000e-08
8	2000.8	6.499722	6.499722	2.150000e-08
9	2000.9	6.565046	6.565046	2.440000e-08

Table 1.Taylor approximation to linear population equation

B. Taylor method to Non-Linear Population Equation

Consider the non-linear population differential equation

$$y' = (0.2)y - (0.01)y^2, \qquad (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

To write the specific third order Taylor difference equation for the initial value problem we need to find the first derivative of

$$f(t,y) = (0.2)y - (0.01)y^2$$

with respect to t, we obtain $f'(t,y) = 0.2y' - 0.02y'y$

$$= 0.2(0.2y - 0.01y^2) - 0.02(0.2y - 0.01y^2)y$$

$$= (0.2 - 0.02y)(0.2y - 0.01y^2)$$

$$f'(t,y) = (0.2 - 0.02y)f(t,y)$$

and the second derivative with respect to t,

$$f''(t,y) = -0.02y'(0.2y - 0.01y^2) + (0.2 - 0.02y)(0.2y' - 0.02y'y)$$

$$= -0.02[(0.2y - 0.01y^2)]^2 + (0.2 - 0.02y)^2(0.2y - 0.01y^2)$$

1) *Non-Linear Population third order Taylor Difference equation:* Substituting the derivatives of the non-linear population equation into the third order Taylor equation gives the difference equation

$$y_{i+1} = y_i + h[(0.2y_i - 0.01{y_i}^2) + \frac{h}{2}((0.2 - 0.02y_i)((0.2y_i - 0.01{y_i}^2)) +$$

$$\frac{h^2}{6}(-0.02(0.2y_i - 0.01{y_i}^2)^2 + (0.2 - 0.02y_i)^2(0.2y_i - 0.01{y_i}^2)]$$

$for\ i = 0,1,2,\dots\dots\dots,199,$ $where\ y_i$ is the numerical approximation of $y\ at\ time\ t_i$, with the step size h and the initial condition $y_0 = 6$.

The Figure below shows the third order Taylor numerical approximation for the non-linear population equation.

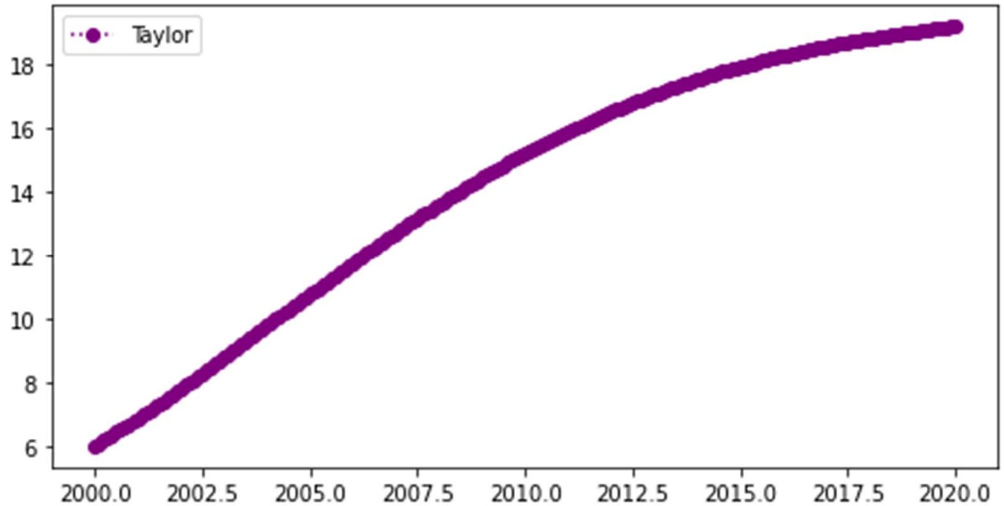

Fig. 3.Taylor approximation for non-linear population equation

The Table below shows the time and the third order numerical approximation for the non-linear population equation.

	time t_i	Taylor
0	2000.0	6.000000
1	2000.1	6.084335
2	2000.2	6.169332
3	2000.3	6.254983
4	2000.4	6.341279
5	2000.5	6.428207
6	2000.6	6.515760
7	2000.7	6.603924
8	2000.8	6.692689
9	2000.9	6.782043

Table 2. Taylor approximation for non-linear population equation

C. Taylor method to Non-Linear Population Equation with an oscillation

Consider the non-linear population differential equation with an oscillation

$$y' = (0.2)y - (0.01)y^2 + sin(2\pi t), \qquad (2000 \le t \le 2020)$$

with the initial condition $y(2000) = 6$.

To write the specific third order Taylor difference equation for the initial value problem we need to find the first derivative of

$$f(t,y) = (0.2)y - (0.01)y^2 + \sin(2\pi t),$$

with respect to t, we obtain

$$\begin{aligned} f'(t,y) &= 0.2y' - 0.02y'y + 2\pi\cos(2\pi t) \\ &= (0.2 - 0.02y)y' + 2\pi\cos(2\pi t) \\ &= (0.2 - 0.02y)((0.2)y - (0.01)y^2 + \sin(2\pi t)) + 2\pi\cos(2\pi t) \end{aligned}$$

and the second derivative with respect to t,

$$f''(t,y) = (-0.02y')[(0.2)y - (0.01)y^2 + \sin(2\pi t)] + (0.2 - 0.02y)(0.2y' - 0.02y'y + 2\pi\cos(2\pi t)) - (2\pi)^2(\sin(2\pi t))$$

$$f''(t,y) = -0.02((0.2)y - (0.01)y^2 + 2\pi\sin(2\pi t))^2 + (0.2 - 0.02y)[(0.2 - 0.02y)((0.2)y - (0.01)y^2 + \sin(2\pi t))] + 2\pi\cos(2\pi t) - (2\pi)^2(\sin(2\pi t)$$

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1) *Non-Linear Population with oscillation third order Taylor Difference equation:* Substituting the derivatives of the non-linear population equation with oscillation into the third order Taylor equation gives the difference equation

$$y_{i+1} = y_i + h[(0.2y_i - 0.01{y_i}^2 + \sin(2\pi t_i))$$
$$+\frac{h}{2}((0.2 - 0.02y_i)(\,0.2y_i - 0.01{y_i}^2 + \sin(2\pi t_i)) + 2\pi\cos(2\pi t_i))$$
$$+\frac{h^2}{6}(-0.02(0.2y_i - 0.01{y_i}^2 + 2\pi\sin(2\pi t_i))^2 + (0.2 - 0.02y_i)[((0.2)y_i - (0.01){y_i}^2 + \sin(2\pi t_i))] + 2\pi\cos(2\pi t_i)$$
$$-(2\pi)^2(\sin(2\pi t_i)$$

$for\ i = 0,1,2, \ldots\ldots\ldots, 199,\ where\ y_i$ is the numerical approximation of $y\ at\ time\ t_i$, with the step size h and the initial condition $y_0 = 6$.

The Figure below shows the third order numerical approximation for the non-linear population equation with oscillation.

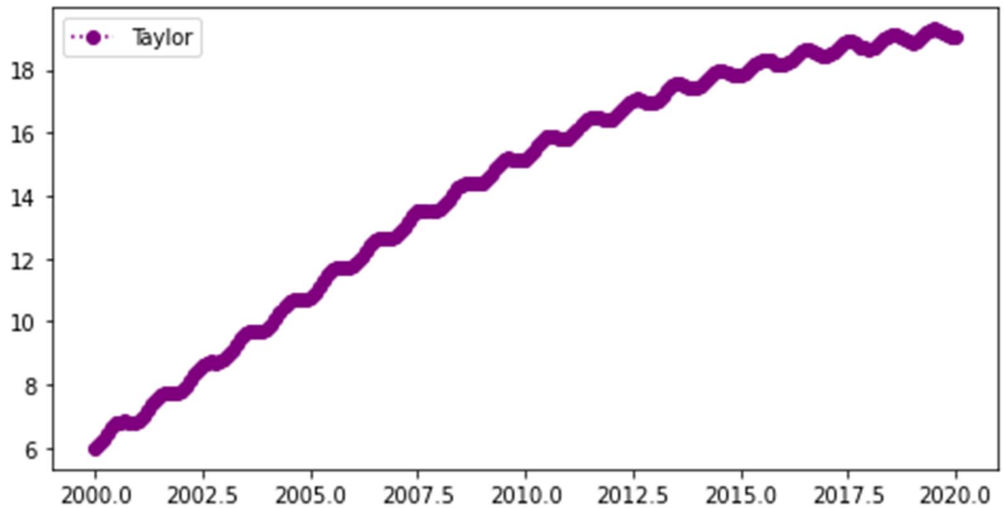

Fig. 4 Taylor approximation for non-linear population equation with oscillation

The Table below shows the time and the third order numerical approximation for the non-linear population equation with oscillation.

	time t_i	Taylor
0	2000.0	6.000000
1	2000.1	6.115834
2	2000.2	6.285332
3	2000.3	6.476682
4	2000.4	6.649942
5	2000.5	6.772316
6	2000.6	6.830702
7	2000.7	6.836694
8	2000.8	6.822138
9	2000.9	6.826948

Table 3. Taylor approximation for non-linear population equation with oscillation

V. CONCLUSION

We first introduced third order Taylor series expansion to the first order differential equation to obtain the numerical approximation of y at time t. We have proposed three different population initial value problems for linear population equation, non-linear population equation and non-linear population equation with an oscillation. To obtain the exact solution for the population equations, we have presented specific third order Taylor difference equation for the initial value problem. The time interval is discretized into N points by a constant step size. The solution is obtained by implementing Python programming for three initial value problems. The results are shown in Figure 2 for linear population equation and Table 1 shows the exact solution. Figure 3 and Table 2 shows the solution for non-linear population equation. Figure 4 and Table 3 shows the solution for non-linear population equation with an oscillation. In all three initial value problems, we observe that for Taylor approximation at time t shows the different population. The difference between population for linear population equation and non-linear population equation is 0.216997 billion approximately and that of linear population equation and non-linear population equation with an oscillation is 0.261902 billion approximately. The difference between population for non-linear population equation and non-linear population equation with an oscillation is 0.044905 billion approximately.

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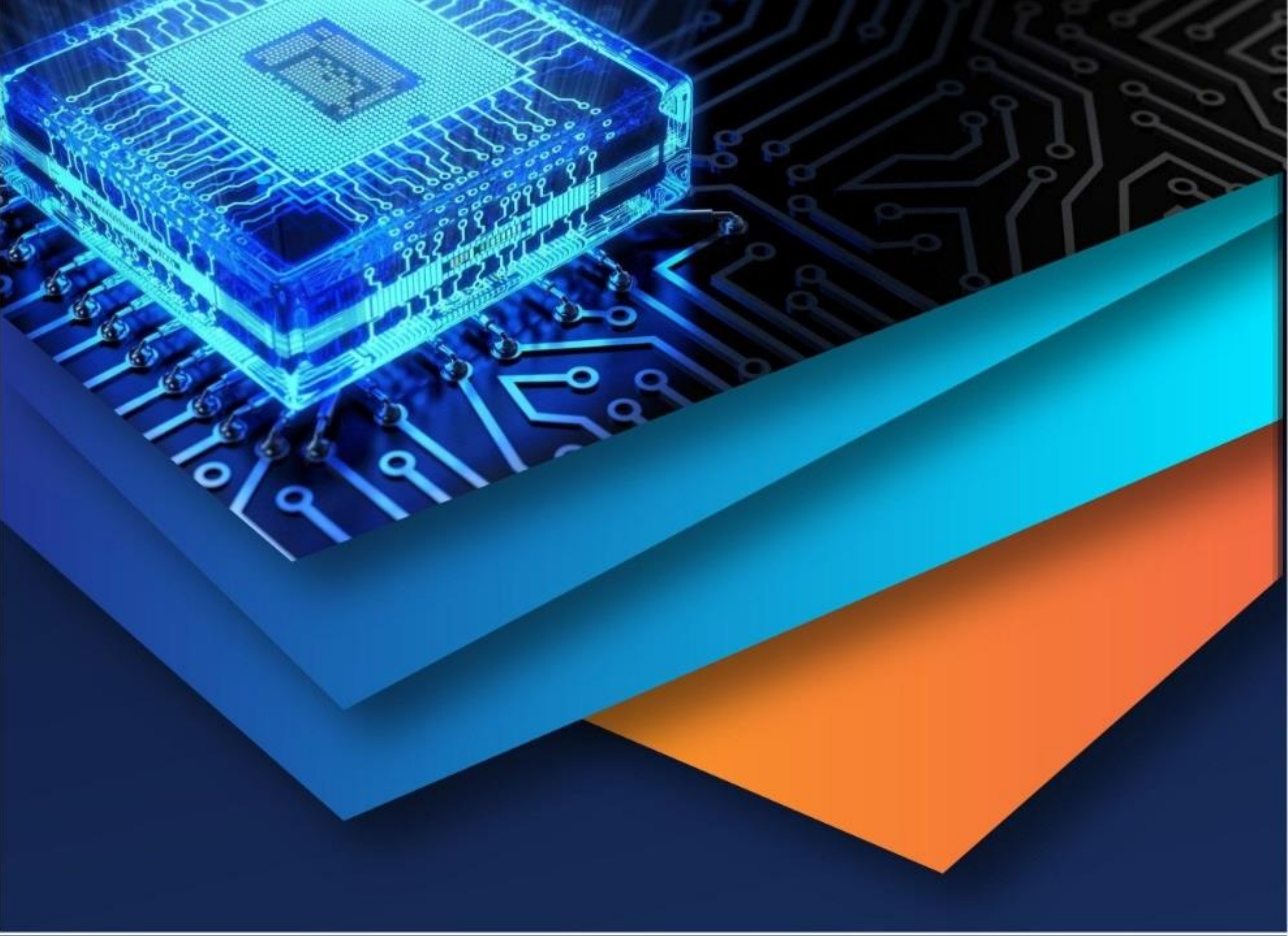

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