iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH
IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 11 Issue: XI Month of publication: November 2023

DOI: https://doi.org/10.22214/ijraset.2023.56502

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International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 11 Issue XI Nov 2023- Available at www.ijraset.com

Bayesian Estimation of Exponentiated Exponential Strength and Exponentiated Weibull Stress Reliability Model for Real Data

Mekala Sony[1], Dr. P. R. Jayashree[2]
[1]Research Scholar, [2]Associate Professor, Department of Statistics, Presidency College, Chennai, Tamil Nadu, India

Abstract: Stress-strength reliability is an important concept in reliability analysis, quantifies the probability that the strength of a system surpasses its applied stress. This paper focuses on the reliability analysis for exponentiated exponential distribution strength variable and the exponentiated Weibull distribution stress variable. The study explores the estimation of the parameters in stress-strength reliability model using maximum likelihood estimation and Bayesian estimation. In particular, the Bayesian estimator of stress-strength reliability is obtained by utilizing Lindley's approximation by considering both linear exponential loss function and squared error loss function for informative and non-informative priors. A comprehensive simulation study is conducted and the performances of estimators are compared using mean squared errors. The stress-strength reliability for real datasets is also investigated for real-time data sets.
Keywords: Bayes estimation, Exponentiated exponential distribution, Exponentiated Weibull distribution, Informative and non-informative prior, Lindley's approximation, Maximum likelihood estimation.

I. INTRODUCTION

A. Exponentiated Exponential Distribution

Gupta and Kundu [17] established the Exponentiated Exponential Distribution (EED) as a two-parameter variant of the exponential distribution for the analysis of object lifetimes. Depending on the shape parameter, the EED can exhibit either a decreasing or increasing failure rate.

Let x represent the continuous, non-negative random variable from EED. The probability density function (pdf) of EED is

$$f(t) = \alpha_1\beta_1 \exp(-\beta_1 x)(1-\exp(-\beta_1 x))^{\alpha_1 -1}$$

For $x > 0, \alpha_1 > 0$ and $\beta_1 > 0$, where α_1 is shape parameters and β_1 is scale parameter.

The EED's cumulative distribution function (CDF) is

$$F(x) = (1-\exp(-\beta_1 x))^{\alpha_1}.$$

The reliability function is of EED is

$$R(x) = 1-\{1-\exp(-\beta_1 x)\}^{\alpha_1}.$$

The hazard rate function is

$$h(t) = \frac{\alpha_1\beta_1 \exp(-\beta_1 x)(1-\exp(-\beta_1 x))^{\alpha_1 -1}}{1-\{1-\exp(-\beta_1 x)\}^{\alpha_1}}.$$

B. Exponentiated Weibull Distribution

The Exponentiated Weibull Distribution (EWD) is the prominent extension of the Weibull Distribution, enabling the modelling of a wide range of failure patterns through the incorporation of an additional shape parameter. Extensive research has been conducted to investigate the statistical properties, parameter estimation techniques, and applications of the EWD in various fields.

Let y represent the non-negative continuous random variable from EWD. The EWD's probability density function (pdf) is

$$f(y) = \alpha_2\left(\frac{k}{\beta_2^k}\right) y^{k-1}\left(1-\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)\right)^{\alpha_2 -1} \exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)$$

for $x > 0, \alpha_2 > 0, \beta_2 > 0$ and $k > 0$, where α_2, k are shape parameters and β_2 is scale parameter.

The EWD's cumulative distribution function (CDF) is

$$F(y)=\left(1-\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)\right)^{\alpha_2}.$$

The reliability function is of EWD is

$$R(x)=1-\left(1-\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)\right)^{\alpha_2}.$$

The hazard rate function EWD is

$$h(t)=\frac{\alpha_2\left(\frac{k}{\beta_2^k}\right)y^{k-1}\left(1-\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)\right)^{\alpha_2-1}\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)}{1-\left(1-\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)\right)^{\alpha_2}}.$$

C. *Stress-Strength Reliability (SSR)*

The stress-strength model, proposed by Birnbaum [19] and modified further by Birnbaum and McCarty [20], is used to analyze stress-strength reliability (SSR), which indicates the chance that a system's strength is greater than its stress. The application of stress-strength models spans diverse fields such as medicine, environment, agriculture, bioinformatics, engineering, and behavioral/psychology. Several researchers have contributed to the estimation of stress-strength reliability for various distributions. Sengupta and Mukhuti [18] focused on estimating stress- strength reliability based on ranked set sample data, while Muttlak et al. [9] studied the stress-strength reliability for the exponential distribution. For the exponentiated Weibull distribution, Chaturvedi and Pathak [3] worked on estimating the reliability function, and Chaturvedi and Pathak [2] focused on estimating the reliability function for a family of exponential distributions. Hussian [11] used ranked set sampling to estimate stress-strength models for the generalized inverted exponential distribution. Kundu et al. [15] investigated the estimation of the reliability function, Jana et al. [12] presented Bayes estimation of parameters for two exponential distributions with a same location but different scale parameters and Qin et al. [10] explored the estimation procedures for stress-strength reliability when both stress and strength follow a one-parameter exponential distribution. The estimation and testing methods for the reliability functions of exponentiated distributions under censoring were studied by Chaturvedi and Vyas [1]. Li and Hao [6] estimated the reliability of a stress-strength model for the inverse Weibull distribution, Chaturvedi and Kumari [4] concentrated on the robust Bayesian analysis of the generalized half logistic distribution, Pandit and Joshi [16] investigated the estimation of multicomponent system stress- strength reliability for the generalized Pareto distribution. The stress-strength parameter was studied by Eissa [8] using Bayesian and non-Bayesian conclusions when random variables X and Y had the exponentiated Weibull distribution. Stress-strength reliability estimation for exponentially distributed systems with a common minimum guarantee time was investigated by Kundu et al. in [14]; and stress-strength reliability estimation for exponential distributions with different scale and location parameters was the focus of Jana et al. in [13]. Li et al. [21] proposed a novel estimation method for stress-strength reliability considering various censoring schemes, including right, left, and interval censoring.

The stress-strength reliability can be represented by $P(Y<X)$ and it is a reliability parameter. Let X and Y be two random variables, with Y denoting "stress" and X denoting "strength". The reliability of the component can be defined as

$$R=P(Y<X)=\int_{-\infty}^{+\infty}G_y(x)f(x)dx, \tag{1}$$

where $G_y(x)=\int_{-\infty}^{x}g(y)dy,\ \ f(x)$ and $g(y)$ are pdf of X and pdf of Y, respectively.

The main focus of this article is to discuss the SSR for EED (strength) and EWD (stress) with maximum likelihood estimation (MLE) method and Bayes estimation using Lindley's approximation. This article is organized as follows. The estimation of SSR for EED with EWD is derived in Section 2. In Section 3, the MLE of SSR based on the random variables X and Y is examined.

In Section 4, Bayesian estimation of SSR based on two loss functions namely, LINEX and symmetric loss functions are found. In Section 5, the simulation study and real time data of SSR for EED with EWD are analyzed and the performance of each method of estimation is also compared. The summary and conclusion of the study based on stress-strength reliability based on MLE and Bayes estimation are given in Section 6.

II. STRESS-STRENGTH RELIABILITY FOR EED WITH EWD

Suppose strength (X) is independent random variable with the shape parameter α_1 and scale parameter β_1. The pdf of X is

$$f(x) = \alpha_1\beta_1 \exp(-\beta_1 x)(1-\exp(-\beta_1 x))^{\alpha_1 - 1}, \quad \alpha_1, \beta_1 > 0 \text{ and } x > 0.$$

Stress (Y) is independent random variable with the shape parameters α_2, k and scale parameter β_2. The pdf of X is

$$g(y) = \alpha_2\left(\frac{k}{\beta_2^k}\right)y^{k-1}\left(1-\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right)\right)^{\alpha_2-1}\exp\left(-\left(\frac{y}{\beta_2}\right)^k\right),$$

$\alpha_2, \beta_2, k > 0$ and $y > 0$.

Substituting $f(x)$ and $G_y(x)$ in (1), the stress-strength reliability becomes

$$R = \int_0^\infty \alpha_1\beta_1 \exp(-\beta_1 x)(1-\exp(-\beta_1 x))^{\alpha_1-1}\left(1-\exp\left(-\left(\frac{x}{\beta_2}\right)^k\right)\right)^{\alpha_2} dx, \qquad (2)$$

where

$$G_y(x) = \left(1-\exp\left(-\left(\frac{x}{\beta_2}\right)^k\right)\right)^{\alpha_2}..$$

III. MAXIMUM LIKELIHOOD ESTIMATION OF THE EED WITH EWD STRESS-STRENGTH RELIABILITY

Suppose $X_1, X_2, ..., X_n$ is a random sample of size m from EED with shape parameter α_1 and scale parameters β_1 and $Y_1, Y_2, ..., Y_n$ is a random sample of size n from EED with shape parameters α_2, k and scale parameter β_2. The likelihood function is

$$L = \prod_{i=1}^{m}\left(\alpha_1\beta_1\exp(-\beta_1 x_i)(1-\exp(-\beta_1 x_i))^{\alpha_1-1}\right)\times\prod_{j=1}^{n}\left(\alpha_2\left(\frac{k}{\beta_2^k}\right)y_j^{k-1}\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^{\alpha_2-1}\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right). \qquad (3)$$

The log likelihood function for the equation of 3 is written as follows:

$$\ln L(\alpha_1, \alpha_2, \beta) = \left(\begin{array}{l}\left(m\ln\alpha_1 + m\ln\beta_1 - \beta_1\left(\sum_{i=1}^{m}x_i\right) + (\alpha_1 - 1)\sum_{i=1}^{m}\ln(1-\exp(-\beta_1 x_i))\right)\\ +\left(n\ln\alpha_2 + n\ln k - nk\ln\beta_2 + (k-1)\sum_{j=1}^{n}\ln y_j + (\alpha_2 - 1)\right.\\ \left.\times\sum_{j=1}^{n}\ln\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right) - \beta_2^{-k}\sum_{j=1}^{n}y_j^k\right)\end{array}\right). \qquad (4)$$

The ML estimators of $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2$ and k can be obtained by simultaneously solving the following equations by numerical simulation:

$$\frac{\partial \ln L}{\partial \alpha_1} = \frac{m}{\alpha_1} + \sum_{i=1}^{m} \ln(1 - \exp(\beta_1 x_i)),$$

$$\frac{\partial \ln L}{\partial \alpha_2} = \frac{n}{\alpha_2} + \sum_{j=1}^{n} \log\left[1 - \exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right],$$

$$\frac{\partial \ln L}{\partial \beta_1} = \frac{m}{\beta_1} - \sum_{i=1}^{m} x_i + (\alpha_1 - 1)\sum_{i=1}^{m} \frac{x_i \exp(-\beta_1 x_i)}{1 - \exp(-\beta_1 x_i)},$$

$$\frac{\partial \ln L}{\partial \beta_2} = -\frac{kn}{\beta_2} + \sum_{j=1}^{n} \frac{k y_j \left(\frac{y_j}{\beta_2}\right)^{-1+k}}{\beta_2^2} - (\alpha_2 - 1)\sum_{j=1}^{n} \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k y_j \left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^2\left(1 - \exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)},$$

$$\frac{\partial \ln L}{\partial k} = \frac{n}{k} - n\ln[\beta_2] + \sum_{j=1}^{n} \ln[y_j] - \sum_{j=1}^{n} \ln\left[\frac{y_j}{\beta_2}\right]\left(\frac{y_j}{\beta_2}\right)^k + (\alpha_2 - 1)\sum_{j=1}^{n} \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]\left(\frac{y_j}{\beta_2}\right)^k}{1 - \exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)}.$$

The MLE of stress-strength reliability R using the invariance property is

$$R = \int_0^\infty \hat{\alpha}_1 \hat{\beta}_1 \exp(-\hat{\beta}_1 x)(1 - \exp(-\hat{\beta}_1 x))^{\hat{\alpha}_1 - 1}\left(1 - \exp\left(-\left(\frac{x}{\hat{\beta}_2}\right)^{\hat{k}}\right)\right)^{\hat{\alpha}_2} dx. \qquad (5)$$

IV. BAYESIAN ESTIMATION OF EED WITH EWD STRESS-STRENGTH RELIABILITY

The Bayesian estimation of SSR for EED is analyzed using Lindley's approximation for various loss functions namely, linear exponential loss function and squared error loss function. The prior distributions of the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and k are considered to be gamma (c_i, d_i), $i = 1, 2, 3, 4, 5$.

The pdf of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and k are

$$\pi(\alpha_1) = \frac{d_1^{c_1}}{\gamma(c_1)} \alpha_1^{c_1 - 1} \exp(-d_1 \alpha_1),$$

$$\pi(\alpha_2) = \frac{d_2^{c_2}}{\gamma(c_2)} \alpha_2^{c_2 - 1} \exp(-d_2 \alpha_2),$$

$$\pi(\beta_1) = \frac{d_3^{c_3}}{\gamma(c_3)} \beta_1^{c_3 - 1} \exp(-d_3 \beta_1),$$

$$\pi(\beta_2) = \frac{d_4^{c_4}}{\gamma(c_4)} \beta_2^{c_4 - 1} \exp(-d_4 \beta_2),$$

$$\pi(k) = \frac{d_5^{c_4}}{\gamma(c_5)} k^{c_5 - 1} \exp(-d_5 k),$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ d_1,\ d_2,\ d_3,\ d_4$ and d_5 are hyperparameters.

The joint posterior probability distribution of random variables $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ k$ is obtained by combining both likelihood function and joint prior probability density function of $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ k$ using Bayes theorem,

$$\pi(\alpha_1,\alpha_2,\beta_1,\beta_2,k)=\left(\begin{array}{l} k_1^{-1}\alpha_1^{m+c_1-1}\alpha_2^{n+c_2-1}\beta_1^{m+c_3-1}\beta_2^{c_4-nk-1}k^{c_5+n-1} \\ \times\exp(-d_1\alpha_1-d_2\alpha_2-d_3\beta_1-d_4\beta_2-d_5k) \\ \prod_{i=1}^{m}(\exp(-\beta_1x_i)(1-\exp(-\beta_1x_i))^{\alpha_1-1}) \\ \times\prod_{j=1}^{n}\left(y_j^{k-1}\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^{\alpha_2-1}\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right) \end{array}\right), \tag{6}$$

where

$$k_1=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\left(\begin{array}{l} \alpha_1^{m+c_1-1}\alpha_2^{n+c_2-1}\beta_1^{m+c_3-1}\beta_2^{c_4-nk-1}k^{c_5+n-1} \\ \times\exp\left(-d_1\alpha_1-d_2\alpha_2-d_3\beta_1-d_4\beta_2-d_5k\right) \\ \times\prod_{i=1}^{m}\left(\exp\left(-\beta_1x_i\right)\left(1-\exp\left(-\beta_1x_i\right)\right)^{\alpha_1-1}\right) \\ \times\prod_{j=1}^{n}\left(y_j^{k-1}\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^{\alpha_2-1}\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right) \end{array}\right)\times d\alpha_1d\alpha_2d\beta_1d\beta_2dk.$$

The squared error loss function (SELF) is $L(\hat{R},\ R)=(R-\hat{R})^2$, where $\hat{R}$ is the Bayes estimate of R. The Bayes estimator of R under squared error loss function (SELF) is given by

$$\hat{R}_{SELF}=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R\left(\alpha_1,\alpha_2,\beta_1,\beta_2,k\right)\times\pi\left(\alpha_1,\alpha_2,\beta_1,\beta_2,k|X,Y\right)d\alpha_1d\alpha_2d\beta_1d\beta_2dk. \tag{7}$$

The Bayes estimator of $u=u(\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ k)=R$ under linear exponential (LINEX) loss function is

$$\hat{R}_{LINEX}=-\frac{1}{s}\ln E(e^{-sR}\,|\,x,\ y),\ s\neq 0, \tag{8}$$

where

$$E\left(e^{-sR}|x,y\right)=\frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-sR}\pi\left(\alpha_1,\alpha_2,\beta_1,\beta_2,k|X,Y\right)d\alpha_1d\alpha_2d\beta_1d\beta_2dk}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\pi\left(\alpha_1,\alpha_2,\beta_1,\beta_2,k|X,Y\right)d\alpha_1d\alpha_2d\beta_1d\beta_2dk}.$$

Equations (7) and (8) cannot be reduced to closed forms. Hence, we can use Lindley's approximation method to approximate the ratio of integrals in Equations (7) and (8).

Lindley [7] developed an approximate method for determining the ratio of integrals to the posterior mean given by

$$E(v(\lambda)|\,x,\ y) = \frac{\int_{\lambda} v(\lambda) e^{L(\lambda)+\rho(\lambda)} d(\lambda)}{\int_{\lambda} e^{L(\lambda)+\rho(\lambda)} d(\lambda)}, \tag{9}$$

where $v(\lambda)$ is a function of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, $\rho(\lambda)$ is the logarithm of the prior density of λ and $L(\lambda)$ is the logarithm of the likelihood function. The approximate form of $E\left(v(\lambda)/\,x,\, y\right)$ using Lindley's approximation is given by

$$E(v(\lambda)/x,\, y) = \begin{pmatrix} v + \frac{1}{2}\sum_i \sum_j (v_{ij} + 2v_i\rho_i)\sigma_{ij} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_l L_{ijk}\sigma_{ij}\sigma_{kl}v_l \\ + \text{terms of order } n^{-2} \text{ or smaller} \end{pmatrix},$$

where

$$v = v(\lambda),\ \ i,\ j,\ k,\ l = 1,\, 2,\, 3,\, ...,\, r,$$

$$v_i = \frac{\partial v}{\partial \lambda_i},\ \ v_{ij} = \frac{\partial^2 v}{\partial \lambda_i \partial \lambda_j},$$

$$L_{ijk} = \frac{\partial^3 L}{\partial \lambda_i \partial \lambda_j \partial \lambda_k},\ \ \rho_j = \frac{\partial \rho}{\partial \lambda_j},$$

σ_{ij} is the $(i,\ j)$th element in the inverse of the matrix $\{-L_{ij}\}$ and $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_r)$ is the MLE of λ.

A. Lindley's Approximation for Squared Error Loss Function

The Bayes estimator under SELF for 5 parameters case is given as

$$\hat{R}_{SELF} = \begin{pmatrix} u(\alpha_1, \alpha_2, \beta_1, \beta_2, k) + u_1\delta_1 + u_2\delta_2 + u_3\delta_3 + u_4\delta_4 + u_5\delta_5 + \delta_6 + \delta_7 \\ + \frac{1}{2}\begin{Bmatrix} A(u_1\sigma_{11} + u_3\sigma_{13}) + B(u_2\sigma_{22} + u_4\sigma_{24} + u_5\sigma_{25}) \\ + C(u_1\sigma_{31} + u_3\sigma_{33}) + D(u_2\sigma_{42} + u_4\sigma_{44} + u_5\sigma_{45}) \\ + E(u_2\sigma_{52} + u_4\sigma_{54} + u_5\sigma_{55}) \end{Bmatrix} \end{pmatrix},$$

where

$$\delta_1 = \rho_1\sigma_{11} + \rho_3\sigma_{13},$$
$$\delta_2 = \rho_2\sigma_{22} + \rho_4\sigma_{24} + \rho_5\sigma_{25},$$
$$\delta_3 = \rho_1\sigma_{31} + \rho_3\sigma_{33},$$
$$\delta_4 = \rho_2\sigma_{42} + \rho_4\sigma_{44} + \rho_5\sigma_{45},$$
$$\delta_5 = \rho_2\sigma_{52} + \rho_4\sigma_{54} + \rho_5\sigma_{55},$$
$$\delta_6 = u_{13}\sigma_{13} + u_{24}\sigma_{24} + u_{25}\sigma_{25} + u_{45}\sigma_{45},$$
$$\delta_7 = \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{33}\sigma_{33} + u_{44}\sigma_{44} + u_{55}\sigma_{55}),$$
$$A = \sigma_{11}L_{111} + \sigma_{33}L_{333},$$
$$B = \sigma_{22}L_{222} + 2\sigma_{44}L_{442} + 2\sigma_{55}L_{552} + 2\sigma_{45}L_{452},$$
$$C = 2\sigma_{13}L_{133} + \sigma_{33}L_{333},$$
$$D = \sigma_{44}L_{444} + \sigma_{55}L_{554} + 2\sigma_{24}L_{244} + 2\sigma_{25}L_{254} + 2\sigma_{45}L_{454},$$
$$E = \sigma_{44}L_{445} + \sigma_{55}L_{555} + 2\sigma_{24}L_{245} + 2\sigma_{25}L_{255} + 2\sigma_{45}L_{455}.$$

Here,

$$\rho_1 = \frac{c_1 - 1}{\alpha_1} - d_1,\ \rho_2 = \frac{c_2 - 1}{\alpha_2} - d_2,\ \rho_3 = \frac{c_3 - 1}{\beta_1} - d_3,\ \rho_4 = \frac{c_4 - 1}{\beta_2} - d_4,\ \rho_5 = \frac{c_5 - 1}{k} - d_5,$$

$$L_{11} = -\frac{m}{\alpha_1^2},\ L_{22} = -\frac{n}{\alpha_2^2},\ L_{33} = -\frac{m}{\beta_1^2} - (\alpha_1 - 1)\sum_{i=1}^{m} \frac{(x_i)^2 \exp(-\beta_1 x_i)}{(1 - \exp(-\beta_1 x_i))^2},$$

$$L_{44} = \left(\begin{aligned} &(\alpha_2 - 1)\sum_{j=1}^{n} \left(\begin{aligned} &-\frac{k^2 y_j^2 \exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j}{\beta_2}\right)^{2k-2}}{\beta_2^4\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} - \frac{k^2 y_j^2 \exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j}{\beta_2}\right)^{2k-2}}{\beta_2^4\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} \\ &+\frac{2ky_j \exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^3\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} - \frac{ky_j \exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j\left(\frac{y_j}{\beta_2}\right)^{k-2}}{\beta_2^2} - \frac{ky_j\left(\frac{y_j}{\beta_2}\right)^{k-2}}{\beta_2^2}\right)}{\beta_2^2\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} \end{aligned} \right) \\ &-\sum_{j=1}^{n}\left(\frac{2ky_j\left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^3} - \frac{ky_j\left(\frac{y_j\left(\frac{y_j}{\beta_2}\right)^{k-2}}{\beta_2^2} - \frac{ky_j\left(\frac{y_j}{\beta_2}\right)^{k-2}}{\beta_2^2}\right)}{\beta_2^2} \right) + \frac{kn}{\beta_2^2} \end{aligned} \right),$$

$$L_{55} = (\alpha_2 - 1)\sum_{j=1}^{n}\left(\frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j}{\beta_2}\right)^k \ln^2\left(\frac{y_j}{\beta_2}\right)}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} - \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j}{\beta_2}\right)^{2k} \ln^2\left(\frac{y_j}{\beta_2}\right)}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} - \frac{\exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right)\left(\frac{y_j}{\beta_2}\right)^{2k} \ln^2\left(\frac{y_j}{\beta_2}\right)}{\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} \right)$$

$$-\sum_{j=1}^{n}\left(\frac{y_j}{\beta_2}\right)^k \ln^2\left(\frac{y_j}{\beta_2}\right) - \frac{n}{k^2},$$

$$L_{13} = L_{31} = \sum_{i=1}^{m} \frac{x_i \exp(-\beta x_i)}{(1 - \exp(-\beta x_i))},$$

$$L_{24} = L_{42} = -\sum_{j=1}^{n} \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k y_j \left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^2\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)},$$

$$L_{25} = L_{52} = \sum_{j=1}^{n} \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]\left(\frac{y_j}{\beta_2}\right)^k}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)},$$

$$L_{45} = L_{54} = \left(\begin{array}{l} -\frac{n}{\beta_2} - \sum_{j=1}^{n}\left(-\frac{y_j\left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^2} - \frac{k\ln\left[\frac{y_j}{\beta_2}\right] y_j \left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^2}\right) \\ +(\alpha_2-1)\sum_{j=1}^{n}\left(\begin{array}{l} -\frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) y_j\left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^2\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} - \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k\ln\left[\frac{y_j}{\beta_2}\right] y_j\left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^2\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} \\ +\frac{\exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right) k\ln\left[\frac{y_j}{\beta_2}\right] y_j\left(\frac{y_j}{\beta_2}\right)^{2k-1}}{\beta_2^2\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} + \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k\ln\left[\frac{y_j}{\beta_2}\right] y_j\left(\frac{y_j}{\beta_2}\right)^{2k-1}}{\beta_2^2\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} \end{array}\right) \end{array}\right),$$

$$L_{111} = \frac{2m}{\alpha_1^3},\ L_{222} = \frac{2n}{\alpha_2^3},$$

$$L_{333} = \frac{2m}{\beta_1^3} + (\alpha_1-1)\sum_{i=1}^{m} \frac{2\exp(-3\beta_1 x_i)x_i^3}{(1-\exp(-\beta_1 x_i))^3} + \frac{3\exp(-2\beta_1 x_i)x_i^3}{(1-\exp(-\beta_1 x_i))^2} + \frac{\exp(-\beta_1 x_i)x_i^3}{1-\exp(-\beta_1 x_i)},$$

$$L_{555} = \frac{2n}{k^3} - \sum_{j=1}^{n}\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^k + (\alpha_2-1)\sum_{j=1}^{n}\left(\begin{array}{l} \frac{e^{-\left(\frac{y_j}{\beta_2}\right)^k}\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^k}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} - \frac{3\exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^{2k}}{\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} \\ -\frac{3\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^{2k}}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} + \frac{2\exp\left(-3\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^{3k}}{\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^3} \\ +\frac{3\exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^{3k}}{\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} + \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\ln\left[\frac{y_j}{\beta_2}\right]^3\left(\frac{y_j}{\beta_2}\right)^{3k}}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} \end{array}\right),$$

$$L_{133} = L_{313} = L_{331} = \sum_{i=1}^{m} \frac{x_i^2 \exp(-x_i\beta)}{(1-\exp(-x_i\beta))^2},$$

$$L_{244} = L_{442} = L_{424} = \sum_{j=1}^{n} \left(\frac{2\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k y_j \left(\frac{y_j}{\beta_2}\right)^{k-1}}{\beta_2^3 \left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} - \frac{\exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right) k^2 y_j^2 \left(\frac{y_j}{\beta_2}\right)^{2k-2}}{\beta_2^4 \left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} - \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k^2 y_j^2 \left(\frac{y_j}{\beta_2}\right)^{2k-2}}{\beta_2^4 \left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} - \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) k y_j \left(\frac{y_j \left(\frac{y_j}{\beta_2}\right)^{-2+k}}{\beta_2^2} - \frac{k y_j \left(\frac{y_j}{\beta_2}\right)^{-2+k}}{\beta_2^2} \right)}{\beta_2^2 \left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)} \right),$$

$$L_{255} = L_{552} = L_{525} = \sum_{j=1}^{n} \left(\frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) \ln\left[\frac{y_j}{\beta_2}\right]^2 \left(\frac{y_j}{\beta_2}\right)^k}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} - \frac{\exp\left(-2\left(\frac{y_j}{\beta_2}\right)^k\right) \ln\left[\frac{y_j}{\beta_2}\right]^2 \left(\frac{y_j}{\beta_2}\right)^{2k}}{\left(1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)\right)^2} - \frac{\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right) \ln\left[\frac{y_j}{\beta_2}\right]^2 \left(\frac{y_j}{\beta_2}\right)^{2k}}{1-\exp\left(-\left(\frac{y_j}{\beta_2}\right)^k\right)} \right),$$

$$u = u(\alpha_1, \alpha_2, \beta_1, \beta_2, k) = R,$$

$$R = \int_0^\infty \hat{\alpha}_1 \hat{\beta}_1 \exp\left(-\hat{\beta}_1 x\right)\left(1-\exp\left(-\hat{\beta}_1 x\right)\right)^{\hat{\alpha}_1 - 1} \times \left(1-\exp\left(-\left(\frac{x}{\hat{\beta}_2}\right)^{\hat{k}}\right)\right)^{\hat{\alpha}_2} dx = u.$$

The derivative of u_i and u_{ij}, $i,\ j = 1$ to 5 are solved using Leibniz integral rule.

Recommended font sizes are shown in Table I.

B. Lindley's Approximation for Linear Exponential Loss Function

The Bayes estimator under linear exponential (LINEX) loss function is given by

$$\hat{R}_{LINEX} = -\frac{1}{s} \ln \left(\begin{array}{l} v(\alpha_1, \alpha_2, \beta_1, \beta_2, k) + v_1\delta_1 + v_2\delta_2 + v_3\delta_3 + v_4\delta_4 + v_5\delta_5 + \delta_6 + \delta_7 \\ + \frac{1}{2} \left\{ \begin{array}{l} A(v_1\sigma_{11} + v_3\sigma_{13}) + B(v_2\sigma_{22} + v_4\sigma_{24} + v_5\sigma_{25}) \\ + C(v_1\sigma_{31} + v_3\sigma_{33}) + D(v_2\sigma_{42} + v_4\sigma_{44} + v_5\sigma_{45}) \\ + E(v_2\sigma_{52} + v_4\sigma_{54} + v_5\sigma_{55}) \end{array} \right\} \end{array} \right),$$

where

$$\delta_6 = v_{13}\sigma_{13} + v_{24}\sigma_{24} + v_{25}\sigma_{25} + v_{45}\sigma_{45},$$

$$\delta_7 = \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33} + v_{44}\sigma_{44} + v_{55}\sigma_{55}),$$

$$v = \exp(-sR).$$

V. VALIDATION OF THE MODEL

A. Simulation Study

A simulation study is conducted to estimate the SSR using 15,000 observations generated from EED and EWD through MLE and Bayesian approaches. The performances of different estimators under MLE and Bayesian frameworks are compared using Lindley's approximation. Various combinations of sample sizes for strength (m) and stress (n), such as (10, 20), (20, 10), (30, 50) and (50, 50) are examined while loss parameters (s) are set to -1.5 and 1.5. The initial values of the parameters are selected as $\alpha_1 = (1.5, 2.5, 3.5)$, $\alpha_2 = (1.5, 1, 0.5)$, $\beta_1 = (0.5, 0.4, 0.2)$, $\beta_2 = (1.5, 2.5, 3.5)$.

The informative gamma prior becomes non-informative gamma prior when all the hyperparameters are uniformly set to zero (i.e., $(\text{i.e., } c_1 = c_2 = c_3 = c_4 = c_5 = d_1 = d_2 = d_3 = d_4 = d_5 = 0)$, Kohansal [5]. The average estimates and mean squared error (MSE) values of stress-strength reliability are computed for MLE and SELF, LINEX loss function with positive loss parameter (LLF) and LINEX loss function with negative loss parameter (LLF1) under both informative and non-informative priors. The performance of these estimators is compared based on their respective MSE values.

Tables I to V present the findings of the simulation study examining the effects of variations in α_1, α_2 and hyperparameters.

In Table I, the impact of $\alpha_1 = (1.5, 2.5, 3.5)$ was explored while keeping the other parameters fixed $(\alpha_2 = 0.5, \beta_1 = 0.2, \beta_2 = 2.5, k = 3.5, c_1 = 0.5, c_2 = 0.1, c_3 = 0.18, c_4 = 0.2, c_5 = 0.25, d_1 = 0.1, d_2 = 0.12, d_3 = 0.2, d_4 = 0.1$ and $d_5 = 0.2)$.

The results demonstrate that increasing α_1 leads to higher stress-strength reliability. Notably, the Bayes estimator for gamma prior, utilizing LLF and SELF performed better than other estimation methods by showing lower mean squared errors.

In Table II, the variation of $\alpha_2 = (1.5, 1.0, 0.5)$ is explored while holding the remaining parameters constant. The results indicate that reducing α_2 values leads to higher stress-strength reliability. Furthermore, the Bayes estimators for gamma prior provide more accurate estimates of SSR with minimal MSEs.

In Table III, the variation of $\beta_1 = (0.5, 0.4, 0.2)$ is presented while keeping the other parameters constant. The results show that reducing β_1 values tends to higher stress-strength reliability. Also, the Bayes estimators for gamma prior give better estimates of SSR with lower MSEs.

In Table IV, the variation of $\beta_2 = (1.5, 2.5, 3.5)$ is explored and rest of the parameters are constant. From the results, it is observed that decreasing β_2 values leads to higher stress-strength reliability. The Bayes estimators for gamma prior performed better estimates of SSR with smaller MSEs than the other estimation methods.

In Table V, the variation in hyperparameters $(c_1, c_2, c_3, d_1, d_2, d_3)$ is presented while fixing model parameters $(\alpha_1 = 3.5, \alpha_2 = 0.5, \beta_1 = 0.2, \beta_2 = 2.5 \text{ and } k = 3.5)$. It is found that the Bayes estimator for gamma prior under LLF performs better than other estimators, yielding smaller MSEs across all three sets of hyperparameters.

TABLE I

AVERAGE SSR ESTIMATES AND MSE FOR MLE, INFORMATIVE AND NON- INFORMATIVE PRIORS WITH SELF AND LINEX LOSS FUNCTION WHEN

$\alpha_2 = 0.5\, \beta_1 = 0.2,\ \beta_2 = 2.5,\ k = 3.5,\ c_1 = 0.5,\ c_2 = 0.1,\ c_3 = 0.18,\ c_4 = 0.2,\ c_5 = 0.25,$
$d_1 = 0.1,\ d_2 = 0.12,\ d_3 = 0.2,\ d_4 = 0.1,\ d_5 = 0.2,\ s = \pm 1.5$ (VARIATION IN α_1)

Sample size	MLE	Bayes estimators with gamma prior			Bayes estimators with non-informative prior		
		SELF	LLF	LLF1	SELF	LLF	LLF1
$\alpha_1 = 1.50,\ R = 0.834716$							
$m = 10$	0.836234	0.829287	0.833167	0.825091	0.822492	0.826512	0.818643
$n = 20$	0.006070	0.005545	0.005350	0.005778	0.005963	0.005707	0.006245
$m = 20$	0.834403	0.825352	0.828236	0.819869	0.820539	0.824736	0.816390
$n = 10$	0.006416	0.005847	0.005613	0.006127	0.006058	0.005788	0.006370
$m = 30$	0.833408	0.829851	0.831463	0.828451	0.827465	0.828968	0.815984
$n = 50$	0.002076	0.002013	0.001988	0.002043	0.002055	0.002022	0.002092
$m = 50$	0.833221	0.829384	0.830627	0.828347	0.827952	0.829092	0.816822
$n = 50$	0.001604	0.001569	0.001553	0.001587	0.001588	0.001568	0.001610
$\alpha_1 = 2.50,\ R = 0.941611$							
$m = 10$	0.941208	0.937850	0.941241	0.936336	0.935731	0.938187	0.933384
$n = 20$	0.003703	0.003403	0.003245	0.003572	0.003848	0.003673	0.004030
$m = 20$	0.933241	0.932355	0.937967	0.932772	0.933493	0.936102	0.930932
$n = 10$	0.003949	0.003714	0.003531	0.003915	0.003917	0.003720	0.004129
$m = 30$	0.934558	0.942433	0.942964	0.941112	0.940578	0.931501	0.939669
$n = 50$	0.001295	0.001263	0.001243	0.001285	0.001309	0.001286	0.001333
$m = 50$	0.935617	0.942218	0.942730	0.941331	0.941097	0.931796	0.930404
$n = 50$	0.001000	0.000985	0.000972	0.000998	0.001006	0.000992	0.001020
$\alpha_1 = 3.50,\ R = 0.9778074$							
$m = 10$	0.981048	0.966856	0.969512	0.966259	0.967788	0.969459	0.966189
$n = 20$	0.002521	0.002312	0.002195	0.002433	0.002725	0.002607	0.002844
$m = 20$	0.972744	0.965950	0.967829	0.964302	0.965731	0.967514	0.963985
$n = 10$	0.002687	0.002593	0.002462	0.002732	0.002763	0.002628	0.002903
$m = 30$	0.977081	0.972344	0.975578	0.972317	0.972338	0.972969	0.971716
$n = 50$	0.000888	0.000868	0.000853	0.000883	0.000911	0.000895	0.000927
$m = 50$	0.975840	0.973253	0.975731	0.972779	0.972842	0.973319	0.972370
$n = 50$	0.000693	0.000685	0.000676	0.000694	0.000704	0.000695	0.000714

TABLE II

AVERAGE SSR ESTIMATES AND MSE FOR MLE, INFORMATIVE AND NON- INFORMATIVE PRIORS WITH SELF AND LINEX LOSS FUNCTION WHEN

$\alpha 1=3.5, \beta_1=0.2, \beta_2=2.5, k=3.5, c_1=0.5, c_2=0.1, c_3=0.18, c_4=0.2, c_5=0.25,$
$d_1=0.1, d_2=0.12, d_3=0.2, d_4=0.1, d_5=0.2, s=\pm 1.5$ (VARIATION IN α_2)

Sample size	MLE	Bayes estimators with gamma prior			Bayes estimators with non-informative prior		
		SELF	LLF	LLF1	SELF	LLF	LLF1
$\alpha_2 = 1.5,\ R = 0.744681$							
$m=10$ $n=20$	0.752971 0.005381	0.738811 0.004387	0.742953 0.004141	0.734810 0.004671	0.738628 0.005213	0.742836 0.004963	0.734595 0.005491
$m=20$ $n=10$	0.741439 0.005752	0.738333 0.004866	0.742665 0.004629	0.734018 0.005154	0.736849 0.005332	0.741211 0.005072	0.732536 0.005638
$m=30$ $n=50$	0.747733 0.001183	0.743684 0.001073	0.745263 0.001045	0.742124 0.001107	0.742798 0.001152	0.744374 0.000121	0.741244 0.001188
$m=50$ $n=50$	0.745999 0.000688	0.743869 0.000629	0.745067 0.000613	0.742679 0.000649	0.743202 0.000666	0.744399 0.0001647	0.742014 0.000687
$\alpha_2 = 1,\ R = 0.813953$							
$m=10$ $n=20$	0.821515 0.003347	0.806730 0.002806	0.809573 0.001607	0.803994 0.002020	0.806961 0.002425	0.809848 0.002226	0.804201 0.003634
$m=20$ $n=10$	0.811374 0.003634	0.805298 0.003220	0.808326 0.003007	0.802311 0.002455	0.804869 0.002512	0.807911 0.002292	0.801881 0.003752
$m=30$ $n=50$	0.816739 0.000500	0.812247 0.000441	0.813335 0.000417	0.811173 0.000367	0.811613 0.000403	0.812699 0.000378	0.810545 0.000530
$m=50$ $n=50$	0.815201 0.000160	0.812530 0.000130	0.813356 0.000116	0.811711 0.000046	0.812083 0.000058	0.812908 0.000043	0.811267 0.000174
$\alpha_2 = 0.5,\ R = 0.853659$							
$m=10$ $n=20$	0.860378 0.002144	0.846089 0.001834	0.848149 0.001686	0.844107 0.001990	0.846467 0.001317	0.848559 0.000169	0.844468 0.002469
$m=20$ $n=10$	0.851375 0.002348	0.844448 0.002163	0.846661 0.001999	0.842273 0.002338	0.844372 0.001367	0.846594 0.000200	0.842195 0.002545
$m=30$ $n=50$	0.856120 0.000089	0.851667 0.000056	0.852458 0.000038	0.850887 0.000075	0.851168 0.000106	0.851956 0.000067	0.850391 0.000126
$m=50$ $n=50$	0.854704 0.000045	0.851948 0.000030	0.852546 0.000019	0.851355 0.000042	0.851608 0.000752	0.852205 0.000021	0.851017 0.000064

TABLE III

AVERAGE SSR ESTIMATES AND MSE FOR MLE, INFORMATIVE AND NON-INFORMATIVE PRIORS WITH SELF AND LINEX LOSS FUNCTION WHEN

$\alpha_1 = 3.5, \alpha_2 = 0.5, \beta_2 = 2.5, k = 3.5, c_1 = 0.5, c_2 = 0.1, c_3 = 0.18, c_4 = 0.2, c_5 = 0.25,$
$d_1 = 0.1, d_2 = 0.12, d_3 = 0.2, d_4 = 0.1, d_5 = 0.2, s = \pm 1.5$ (VARIATION IN β_1)

Sample size	MLE	Bayes estimators with gamma prior			Bayes estimators with non-informative prior		
		SELF	LLF	LLF1	SELF	LLF	LLF1
$\beta_1 = 0.5,\ R = 0.820750$							
$m = 10$	0.832971	0.818811	0.820153	0.814810	0.738628	0.820036	0.814595
$n = 20$	0.006321	0.005382	0.005142	0.005670	0.006212	0.005962	0.006490
$m = 20$	0.821439	0.818333	0.820365	0.814018	0.736849	0.820111	0.812536
$n = 10$	0.006712	0.005862	0.005627	0.006153	0.006331	0.006071	0.006637
$m = 30$	0.827733	0.820084	0.820463	0.812124	0.742798	0.820134	0.811244
$n = 50$	0.002133	0.002063	0.002044	0.002106	0.002151	0.002120	0.002186
$m = 50$	0.825999	0.820269	0.820567	0.812679	0.743202	0.820299	0.812014
$n = 50$	0.001628	0.001630	0.001612	0.001648	0.001665	0.001645	0.001685
$\beta_1 = 0.4,\ R = 0.883877$							
$m = 10$	0.881515	0.876730	0.879573	0.873994	0.876961	0.879848	0.874201
$n = 20$	0.004346	0.003805	0.003606	0.004021	0.004424	0.004225	0.004633
$m = 20$	0.871374	0.875298	0.878326	0.872311	0.874869	0.877911	0.871881
$n = 10$	0.004632	0.004221	0.004006	0.004454	0.004513	0.004291	0.004753
$m = 30$	0.886739	0.882247	0.883335	0.881173	0.881613	0.882699	0.880545
$n = 50$	0.001501	0.001440	0.001415	0.001466	0.001502	0.001476	0.001531
$m = 50$	0.885201	0.882530	0.883356	0.881711	0.882083	0.882908	0.881267
$n = 50$	0.001162	0.001131	0.001114	0.001145	0.001157	0.001142	0.001173
$\beta_1 = 0.2,\ R = 0.977807$							
$m = 10$	0.980378	0.966089	0.968149	0.964107	0.966467	0.968559	0.964468
$n = 20$	0.003143	0.002833	0.002685	0.002991	0.003316	0.003168	0.003462
$m = 20$	0.971375	0.964448	0.966661	0.962273	0.964372	0.966594	0.962195
$n = 10$	0.003347	0.003164	0.002997	0.003337	0.003366	0.003201	0.003544
$m = 30$	0.978820	0.961667	0.975458	0.970887	0.971168	0.971956	0.970391
$n = 50$	0.001088	0.001055	0.001036	0.001074	0.001105	0.001086	0.001125
$m = 50$	0.978704	0.961948	0.976746	0.971355	0.971608	0.972205	0.971017
$n = 50$	0.000844	0.000831	0.000818	0.000841	0.000851	0.000840	0.000865

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 11 Issue XI Nov 2023- Available at www.ijraset.com

TABLE IV

AVERAGE SSR ESTIMATES AND MSE FOR MLE, INFORMATIVE AND NON-INFORMATIVE PRIORS WITH SELF AND LINEX LOSS FUNCTION WHEN

$\alpha_1 = 3.5, \alpha_2 = 0.5, \beta_1 = 0.2, k = 3.5, c_1 = 0.5, c_2 = 0.1, c_3 = 0.18, c_4 = 0.2, c_5 = 0.25,$
$d_1 = 0.1, d_2 = 0.12, d_3 = 0.2, d_4 = 0.1, d_5 = 0.2, s = \pm 1.5$ (VARIATION IN β_2)

Sample size	MLE	Bayes estimators with gamma prior			Bayes estimators with non-informative prior		
		SELF	LLF	LLF1	SELF	LLF	LLF1
$\beta_2 = 3.50,\ R = 0.947772$							
$m = 10$ $n = 20$	0.952971 0.000261	0.938811 0.000267	0.942953 0.000121	0.934810 0.000561	0.938628 0.000273	0.942836 0.000153	0.934595 0.000681
$m = 20$ $n = 10$	0.951439 0.000632	0.938333 0.000746	0.942665 0.000109	0.934018 0.000144	0.936849 0.000642	0.941211 0.000162	0.932536 0.000188
$m = 30$ $n = 50$	0.949733 0.000163	0.943684 0.000053	0.945263 0.000035	0.942124 0.000106	0.942798 0.000142	0.944374 0.000111	0.941244 0.000178
$m = 50$ $n = 50$	0.949699 0.000078	0.943869 0.000019	0.946067 0.000003	0.942679 0.000039	0.943202 0.000056	0.944399 0.000037	0.942014 0.000077
$\beta_2 = 2.50,\ R = 0.977807$							
$m = 10$ $n = 20$	0.981515 0.000237	0.966730 0.000205	0.969573 0.000200	0.963994 0.000210	0.966961 0.000217	0.969848 0.000216	0.964201 0.000239
$m = 20$ $n = 10$	0.979374 0.000324	0.965298 0.000110	0.968326 0.000106	0.962311 0.000115	0.964869 0.000120	0.967911 0.000118	0.961881 0.000128
$m = 30$ $n = 50$	0.978939 0.000101	0.972247 0.000100	0.975635 0.000097	0.971173 0.000110	0.971613 0.000117	0.972699 0.000115	0.970545 0.000120
$m = 50$ $n = 50$	0.978701 0.000050	0.972530 0.000020	0.976856 0.00006	0.971711 0.000026	0.972083 0.000080	0.973907 0.000045	0.971267 0.000094
$\beta_2 = 1.50,\ R = 0.994776$							
$m = 10$ $n = 20$	0.998378 0.000134	0.986089 0.000124	0.988149 0.000106	0.984107 0.000150	0.986467 0.000207	0.988559 0.000159	0.984468 0.000259
$m = 20$ $n = 10$	0.997375 0.000238	0.984448 0.000153	0.986661 0.000120	0.982273 0.000161	0.984372 0.000257	0.986594 0.000100	0.982195 0.000235
$m = 30$ $n = 50$	0.996120 0.000079	0.991667 0.000046	0.992458 0.000028	0.990887 0.000088	0.991168 0.000105	0.991946 0.000077	0.990391 0.000116
$m = 50$ $n = 50$	0.996004 0.000035	0.991948 0.000020	0.992546 0.000002	0.991355 0.000038	0.991608 0.000092	0.992205 0.000042	0.991017 0.000096

TABLE V

AVERAGE SSR ESTIMATES AND MSE FOR MLE, INFORMATIVE AND NON- INFORMATIVE PRIORS UNDER SELF AND LINEX LOSS FUNCTION WHEN $\alpha_1 = 3.5, \alpha_2 = 0.5, \beta_1 = 0.2, \beta_2 = 2.5, k = 3.5, s = \pm 1.5$ (VARIATION IN HYPERPARAMETERS $c_1, c_2, c_3, c_4, c_5, d_1, d_2, d_3, d_4, d_5$)

Sample size	MLE	Bayes estimators with gamma prior			Bayes estimators with non-informative prior		
		SELF	LLF	LLF1	SELF	LLF	LLF1
$c_1 = 0.7, c_2 = 0.2, c_3 = 0.25, c_4 = 0.21, c_5 = 0.23, d_1 = 0.2, d_2 = 0.22, d_3 = 0.3, d_4 = 0.24, d_5 = 0.21$							
$m = 10$	0.870378	0.849871	0.841704	0.838118	0.856467	0.858559	0.854468
$n = 20$	0.003144	0.002651	0.002486	0.002826	0.003317	0.003169	0.003469
$m = 20$	0.861375	0.85083	0.842989	0.838726	0.854372	0.856594	0.852195
$n = 10$	0.003348	0.003121	0.002941	0.003310	0.003367	0.003200	0.003545
$m = 30$	0.866120	0.860112	0.850891	0.849346	0.861168	0.861956	0.860391
$n = 50$	0.001089	0.001031	0.001011	0.001053	0.001106	0.001087	0.001126
$m = 50$	0.864704	0.860942	0.851535	0.850354	0.861608	0.862205	0.861017
$n = 50$	0.000845	0.000820	0.000808	0.000833	0.000852	0.000841	0.000864
$c_1 = 0.65, c_2 = 0.3, c_3 = 0.4, c_4 = 0.35, c_5 = 0.41, d_1 = 0.35, d_2 = 0.28, d_3 = 0.4, d_4 = 0.39, d_5 = 0.29$							
$m = 10$	0.870369	0.834145	0.835148	0.83320	0.856458	0.858549	0.854458
$n = 20$	0.003143	0.003054	0.002870	0.003241	0.003316	0.003168	0.003468
$m = 20$	0.861375	0.841389	0.835314	0.834561	0.854372	0.856594	0.852195
$n = 10$	0.003348	0.003380	0.003165	0.003598	0.003367	0.003200	0.003545
$m = 30$	0.866120	0.855545	0.856269	0.854844	0.861168	0.861956	0.860391
$n = 50$	0.001089	0.001049	0.001021	0.001077	0.001106	0.001087	0.001126
$m = 50$	0.867704	0.857986	0.858558	0.857425	0.861608	0.862205	0.861017
$n = 50$	0.000845	0.000831	0.000815	0.000847	0.000852	0.000841	0.000864
$c_1 = 0.9, c_2 = 0.8, c_3 = 0.7, c_4 = 0.6, c_5 = 0.5, d_1 = 0.4, d_2 = 0.85, d_3 = 0.88, d_4 = 0.89, d_5 = 0.8$							
$m = 10$	0.870369	0.831191	0.831908	0.830484	0.856458	0.858549	0.854458
$n = 20$	0.003143	0.003087	0.002929	0.003260	0.003316	0.003168	0.003468
$m = 20$	0.861375	0.838893	0.840718	0.837165	0.854372	0.856594	0.852195
$n = 10$	0.003348	0.003319	0.003098	0.003542	0.003367	0.003200	0.003545
$m = 30$	0.866120	0.854776	0.855486	0.854090	0.861168	0.861956	0.860391
$n = 50$	0.001089	0.001039	0.001010	0.001068	0.001106	0.001087	0.001126
$m = 50$	0.864704	0.857335	0.857900	0.856782	0.861608	0.862205	0.861017
$n = 50$	0.000845	0.000827	0.000811	0.000844	0.000852	0.000841	0.000864

B. SSR Analysis of Breaking Strength Data

In this subsection, the real data sets of single carbon fibers with gauge lengths of 10 mm and 20 mm. These data sets were obtained from the study conducted by Badar and Priest (1982) and are measured in GPa. The sample size for the 10 mm gauge length is 63, while the sample size for the 20 mm gauge length is 69.

Data set 1: X(10 mm gauge length, $m = 63$)

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Data set 2: Y(20 mm gauge length: $n = 69$)

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

The EED is fitted to data set 1 and EWD is fitted to data set 2 using Q-Q plots and Kolmogorov-Smirnov (K-S) test. The K-S statistic values for datasets 1 and 2 are 0.1187 and 0.1294 with corresponding p-values of 0.7127 and 0.3428, respectively. Fig. 1 displays the Q-Q plots for both data sets. Based on the p-values and visual inspection of the Q-Q plots, it can be concluded that EED provides a good fit to the data set 1 and EWD provides a good fit to the data set 2. The maximum likelihood and Bayes estimates of stress-strength reliability based on the parameters of $\hat{\alpha}_1 = 144.0232$, $\hat{\alpha}_2 = 50.53977$, $\hat{\beta}_1 = 1.824965$, $\hat{\beta}_2 = 30.289321$ and $k = 9.821343$ are $\hat{R}_{MLE} = 0.923607$.

Under informative prior,

$$\hat{R}_{SELF} = 0.925674,\ \hat{R}_{LLF} = 0.926138,\ \hat{R}_{LLF1} = 0.924321.$$

Under non-informative prior,

$$\hat{R}_{SELF} = 0.921931,\ \hat{R}_{LLF} = 0.922398,\ \hat{R}_{LLF1} = 0.920584.$$

The estimated value of the stress-strength reliability (R) is found to be greater than 0.5, indicating that the 10 mm carbon fibers have a higher strength compared to the 20 mm carbon fibers.

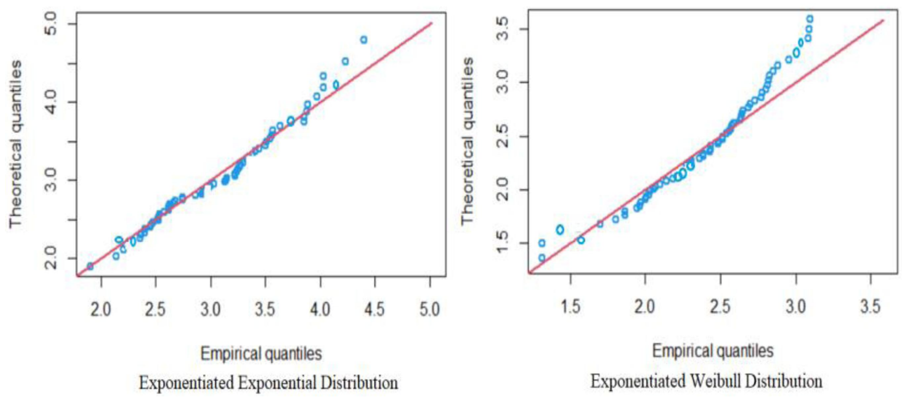

Fig. 1 Q-Q plots of datasets 1 and 2.

VI. SUMMARY AND CONCLUSIONS

The study focuses on estimating the stress-strength reliability using EED as the strength variable and EWD as the stress variable. The estimation is performed using MLE and Bayesian methods under LINEX loss function and SELF using Lindley's approximation. The performance of these estimators is compared based on the mean squared errors. The simulation study reveals the following findings.

1) Increasing the values of α_1 while keeping other parameters fixed leads to an increase in stress-strength reliability.
2) Decreasing the values of α_2, β_1 and β_2 while keeping the remaining parameters fixed results in an increase in stress-strength reliability.
3) The Bayes estimator with a gamma prior under LLF demonstrates better performance with smaller mean squared errors across all three sets of hyperparameters.

Hence, it can be concluded that the Bayes estimators for the gamma prior under the LINEX loss function with a positive loss parameter outperform other estimation methods. Additionally, the stress-strength reliability) of the EED with EWD is investigated using real data sets of breaking strength in carbon fibers with different gauge lengths. The findings reveal that the 10 mm length carbon fibers exhibit greater strength compared to the 20 mm length fibers.

VII. ACKNOWLEDGMENT

I would like to thank my supervisor for her continuous support and encouragement in preparing the manuscript.

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