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# Connectedness Among the Pixels at the Product Digital Topology

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**Abstract:** The aim of this paper is to analyze at the concepts connectedness of the separated interior circles at the pixels at the product digital topology with the axioms C1, C2, C3 in the cartesian complex and also at the structure of the elements at the collections of the pixels and interior circles in that region.

**Keywords:** Cut point, classical axioms of the topological space, pixels, incidence, path, opponent, interior, closure and frontier, locally finite space.

## I. INTRODUCTION

Digital topology is to study at the topological properties of digital image arrays. These properties on cathode ray tubes are virtually important in a wide range of diverse applications, including computer graphics, computer tomography, pattern analysis and robotic design. A topological framework contains many pixels or 2-cell. A digital picture can be stored at them. These framework settings are in some of the devices for the focus purpose. In this case one can specify at the pixels on the simple closed curves which states that a simple closed curve separates at the plane into two connected sets. When a pixel is extended to be such a set of pixels possess connectivity and is called a region.

## II. PRELIMINARIES

- 1) *Definition 2.1[2]:* A point  $x$  in  $X$  is called a cut point (respectively endpoint) if  $X - \{x\}$  has two (one) components. (In the literature our cut-point is usually called a strong cut-point, but here it turns out that these two notions coincide.) The parts of  $X - \{x\}$  are its components if there are two, and  $X - \{x\}, \emptyset$  if there is only one.
- 2) *Definition 2.2[5]:* A nonempty set  $S$  is called a locally finite (LF) space if to each element  $e$  of  $S$  certain subsets of  $S$  are assigned as neighborhoods of  $e$  and some of them are finite.
- 3) *Definition 2.3 [5]:* Axiom 1. For each space element  $e$  of the space  $S$  there are certain subsets containing  $e$ , which are neighborhoods of  $e$ . The intersection of two neighborhoods of  $e$  is again a neighborhood of  $e$ . Since the space is locally finite, there exists the smallest neighborhood of  $e$  that is the intersection of all neighborhoods of  $e$ . Thus, each neighborhood of  $e$  contains its smallest neighborhood. We shall denote the smallest neighborhood of  $e$  by  $SN(e)$ .
- 4) *Definition 2.4[5]:* Axiom 2. There are space elements, which have in their  $SN$  more than one element.
- 5) *Definition 2.5[5]:* If  $b \in SN(a)$  or  $a \in SN(b)$ , then the elements  $a$  and  $b$  are called incident to each other.
- 6) *Definition 2.6[4]:* A path is a sequence  $(p_i / 0 \leq i \leq n)$ , and  $p_i$  is adjacent to  $p_{i+1}$ . In another way Let  $T$  be a subset of the space  $S$ . In another way [4] a sequence  $(a_1, a_2, \dots, a_k), a_i \in T, i = 1, 2, \dots, k$ ; in which each two subsequent elements are incident to each other, is called an incidence path in  $T$  from  $a_1$  to  $a_k$ .
- 7) *Definition 2.7 [4]:* A set of pixels is said to be connected if there is a path between any two pixels.
- 8) *Remark 2.8[5]:* In another way A subset  $T$  of the space  $S$  is connected iff for any two elements of  $T$  there exists an incidence path containing these two elements, which completely lies in  $T$
- 9) *Definition 2.9 [5]:* The topological boundary, also called the frontier, of a non-empty subset  $T$  of the space  $S$  is the set of all elements  $e$  of  $S$ , such that each neighborhood of  $e$  contains elements of both  $T$  and its complement  $S - T$ . It is denoted by the frontier of  $T \subseteq S$  by  $Fr(T, S)$ .
- 10) *Definition 2.10[5]:* A subset  $O \subset S$  is called open in  $S$  if it contains no elements of its frontier  $Fr(O, S)$ . A subset  $C \subset S$  is called closed in  $S$  if it contains all elements of  $Fr(C, S)$ .
- 11) *Definition 2.11[5]:* The neighbourhood relation  $N$  is a binary relation in the set of the elements of the space  $S$ . The ordered pair  $(a, b)$  is in  $N$  iff  $a \in SN(b)$ . We also write  $aNb$  for  $(a, b)$  in  $N$ .
- 12) *Definition 2.12 [5]:* A pair  $(a, b)$  of elements of the frontier  $Fr(T, S)$  of a subset  $T \subset S$  are opponents of each other, if  $a$  belongs to  $SN(b)$ ,  $b$  belongs to  $SN(a)$ , one of them belongs to  $T$  and the other one to its complement  $S - T$ .
- 13) *Definition 2.13[5]:* The smallest open subset of the ALF space  $S$  that contains the element  $a \in S$  is called the smallest open neighborhood of  $a$  in  $S$  and is denoted by  $SON(a, S)$ . It is denoted by  $SON(a, S) = SN(a)$

14) *Definition 2.14 [5]:* The topology of a space  $S$  is defined if a collection of subsets of  $S$  is declared to be the collection of open subsets satisfying the following axioms:

Axiom C1. The entire set  $S$  and the empty subset  $\emptyset$  are open.

Axiom C2. The union of any number of open subsets is open.

Axiom C3. The intersection of a finite number of open subsets is open.

15) *Theorem KC (k-dimensional cell) 2.15 [5]:* The dimension of a cell  $c = (a_1, a_2, \dots, a_n)$  of an  $n$ -dimensional Cartesian complex  $S$  is equal to the number of its components  $a_i, i = 1, 2, \dots, n$ ; which are open in their axes.

16) *Definition 2.16[1]:* Let  $T$  be a subset in the space  $S$  with the axioms C1, C2, C3 then the interior,  $\text{int}(T, S) = \bigcup \{O : O \text{ is open, } O \subseteq T\}$  (i.e.,) the union of all open sets contained in  $T$ .

17) *Definition 2.17[1]:* Let  $T$  be a subset in the space  $S$  with the axioms C1, C2, C3 then the closure,  $\text{cl}(T, S) = \bigcap \{C : C \text{ is closed and } T \subseteq C\}$  (i.e.,) the intersection of all closed sets containing  $T$ .

18) *Definition 2.18[1]:* Let  $T$  be any subset of a space  $S$  with the axioms C1, C2, C3. Then  $\text{ext}(T, S) = S - \{\text{int}(T, S) \cup \text{Fr}(T, S)\}$  where  $\text{int}(T, S), \text{ext}(T, S)$  and  $\text{Fr}(T, S)$  are disjoint and also  $\text{Fr}(T, S)$  is a closed set.

19) *Theorem 2.19[1]:* Let  $S$  be a space with the axioms C1, C2, C3 and let  $T$  be a subset of  $S$ . Then  $\text{int}(T, S)$  is an open.

20) *Corollary 2.20[1]:* Let  $T$  be any subset of a space  $S$  with the axioms C1, C2, C3. Then  $S = \text{int}(T, S) \cup \text{Fr}(T, S) \cup \text{ext}(T, S)$  where  $\text{int}(T, S), \text{ext}(T, S)$  and  $\text{Fr}(T, S)$  are disjoint and also  $\text{Fr}(T, S)$  is a closed set.

21) *Remark 2.21[1]:* Let  $P_1$  be the one of the pixels in the Cartesian complex. Now the projections are  $\Pi_1$  and  $\Pi_2$  are onto. For if  $x \in X$  such that  $(x, y) \in X \times Y$  with the axioms C1, C2, C3 for which  $\Pi_1(x, y) = x$  and  $y \in Y$  such that  $(x, y) \in X \times Y$  with the axioms C1, C2, C3 for which  $\Pi_2(x, y) = y$  if one of  $X$  or  $Y$  is empty then  $X \times Y$  is also empty and there is no need to consider the function  $\Pi_1$  and  $\Pi_2$ . As  $\Pi_1$  and  $\Pi_2$  are surjective we say the  $\Pi_1$  and  $\Pi_2$  are projections of  $X \times Y$  onto its factors respectively.

### III. SEPARATION OF INTERIOR CIRCLES AT THE PIXELS

Let  $\wp$  be the collection of pixels at the product digital topology with the axioms C1, C2, C3 in the Cartesian complex and the set of all frontier points. Thus it is called rectangular region.

Now we take at the collection of all interior circular region  $\ell$  at a pixel in the product digital topology with the axioms C1, C2, C3 as a plane in the Cartesian complex.

The element of the structure  $\wp$  and  $\ell$  are  $P_1, P_2, P_3, \dots$  and  $C_1, C_2, C_3, \dots$  at the product digital topology with the axioms C1, C2, C3 in the Cartesian complex where  $P_1$  be a one of the pixel at  $\wp$  and  $C_1$  be the collection of interior circulars  $c_1, c_2, c_3, \dots$  at a pixel  $P_1$ .

1) *Definition 3.1:* Let  $c_1$  and  $c_2$  be interior circulars at  $C_1$  in a pixel  $P_1$  in the Cartesian complex with the axioms C1, C2, C3 are separated at  $C_1$  in a pixel  $P_1$  if and only if  $c_1 \cap \text{cl}(c_2, P_1) = \text{cl}(c_1, P_1) \cap c_2 = \emptyset$ .

2) *Theorem 3.2:* If  $P_1$  and  $P_2$  are separated pixels of the product digital topology with the axioms C1, C2, C3 in the Cartesian complex and  $C_1 \subseteq P_1$  and  $C_2 \subseteq P_2$  then  $C_1$  and  $C_2$  are also separated.

*Proof:* Now  $P_1 \cap \text{cl}(P_2, \wp) = \emptyset$  and  $\text{cl}(P_1, \wp) \cap P_2 = \emptyset \dots \dots \dots (1)$ . Also  $C_1 \subseteq P_1 \Rightarrow \text{cl}(C_1, P_1) \subseteq \text{cl}(P_1, \wp)$  and  $C_2 \subseteq P_2 \Rightarrow \text{cl}(C_2, P_2) \subseteq \text{cl}(P_2, \wp) \dots \dots \dots (2)$ . It follows from (1) and (2) that  $C_1 \cap \text{cl}(C_2, P_2) = \emptyset$  and  $\text{cl}(C_1, P_1) \cap C_2 = \emptyset$ . Hence  $C_1$  and  $C_2$  are separated.

3) *Theorem 3.3:* Two closed (open) interior circles  $C_1$  and  $C_2$  at the pixels  $P_1$  and  $P_2$  at the product digital topology with the axioms C1, C2, C3 in the Cartesian complex are separated if and only if they are disjoint.

*Proof:* Since any two separated interior circles are disjoint. To prove that two disjoint closed (open) interior circles are separated. If  $C_1$  and  $C_2$  are both disjoint and closed, then  $C_1 \cap C_2 = \emptyset$ ,  $\text{cl}(C_1, P_1) = C_1$  and  $\text{cl}(C_2, P_2) = C_2$ . So that  $\text{cl}(C_1, P_1) \cap C_2 = \emptyset$  and  $C_1 \cap \text{cl}(C_2, P_2) = \emptyset$ . To show that  $C_1$  and  $C_2$  are separated. If  $C_1$  and  $C_2$  are both disjoint and open, then  $P_1 - C_1$  and  $P_2 - C_2$  are both closed interior circles so that  $P_1 - \text{cl}(C_1, P_1) = P_1 - C_1$  and  $P_2 - \text{cl}(C_2, P_2) = P_2 - C_2$ . Also  $C_1 \cap C_2 = \emptyset \Rightarrow C_1 \subseteq P_2 - C_2$  and  $C_2 \subseteq P_1 - C_1 \Rightarrow \text{cl}(C_1, P_1) \subseteq \text{cl}(P_2 - C_2, P_2) = P_2 - C_2$  and  $\text{cl}(C_2, P_2) \subseteq \text{cl}(P_1 - C_1, P_1) = P_1 - C_1 \Rightarrow \text{cl}(C_1, P_1) \cap \text{cl}(C_2, P_2) = \emptyset$  and  $\text{cl}(C_2, P_2) \cap C_1 = \emptyset \Rightarrow C_1$  and  $C_2$  are separated.



- 4) *Definition 3.4:* A pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex is disconnected if and only if there are disjoint non empty interior circular regions  $c_1$  and  $c_2$  at  $C_1$  in  $P_1$  such that  $P_1 = c_1 \sqcup c_2$ .
- 5) *Remark 3.5:* In another way of disconnected if and only if a pixel  $P_1$  is the union of two non empty separated interior circular regions  $c_1$  and  $c_2$  at  $C_1$  in  $P_1$  with the axioms  $C1, C2, C3$ .
- 6) *Definition 3.6:* A pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex is said to be connected if it cannot be expressed as union of two non empty separated interior circular regions  $c_1$  and  $c_2$  at  $C_1$  in  $P_1$  with the axioms  $C1, C2, C3$ . In another way, two non empty separated interior circular regions  $c_1$  and  $c_2$  at  $C_1$  in  $P_1$  with the axioms  $C1, C2, C3$  is said to be connected if and only if it is not disconnected.
- 7) *Theorem 3.7:* A pixel  $P_1$  is connected at the non empty separated interior circular regions  $c_1$  and  $c_2$  at  $C_1$  in  $P_1$  with the axioms  $C1, C2, C3$  that are both open and closed in a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex.

*Proof:* If  $c_1$  is non empty proper circular region of  $P_1$  which is both open and closed at a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex then  $u=c_1$  and  $v=P_1-c_1$ . Consistute a separation of  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex for they are open disjoint and non-empty whose union is  $P_1$ .

Conversely if  $u$  and  $v$  are separation of a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex then  $u=P_1-v$  that implies  $u$  is closed. Therefore  $v$  is open. Similarly  $v=P_1-u$  that implies  $v$  is closed therefore  $u$  is open. Thus  $u$  and  $v$  are both open and closed at a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex.

- 8) *Theorem 3.8:* A pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex is disconnected if and only if there exists a non-empty proper interior circles of  $P_1$  which is both open and closed at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex.

*Proof:* Let  $c_1$  be a non-empty proper interior circles of  $P_1$  which is both open and closed. To show that  $P_1$  is disconnected. Let  $c_2=P_1-c_1$ . Then  $c_2$  is non-empty since  $c_1$  is a proper interior circle of a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex. Moreover,  $c_1 \sqcup c_2=P_1$  and  $c_1 \cap c_2=\emptyset$ . Since  $c_1$  is both closed and open interior circle,  $c_2$  is also both closed and open interior circle. Hence  $cl(c_1, P_1)=c_1$  and  $cl(c_2, P_2)=c_2$ . It follows that  $cl(c_1, P_1) \cap c_2=\emptyset$  and  $c_1 \cap cl(c_2, P_2)=\emptyset$ . Thus  $P_1$  has been expressed as a union of two separated interior circles and so  $P_1$  is disconnected at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex.

Conversely, let  $P_1$  be disconnected. Then there exist non-empty interior circles  $c_1$  and  $c_2$  of a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex such that  $c_1 \cap cl(c_2, P_1)=\emptyset$ ,  $cl(c_1, P_1) \cap c_2=\emptyset$  and  $c_1 \sqcup c_2=P_1$ . Since  $c_1 \sqcup cl(c_1, P_1), cl(c_1, P_1) \cap c_2=\emptyset \Rightarrow c_1 \cap c_2=\emptyset$ . Hence  $c_1=P_1-c_2$ . Since  $c_2$  is non-empty interior circle, and  $c_2 \sqcup P_1-c_2=P_1$ , it follows that  $c_2=P_1-c_1$  is a proper interior circle of  $P_1$ . Now  $c_1 \sqcup cl(c_2, P_1)=P_1$ .

Also  $c_1 \cap cl(c_2, P_1)=\emptyset \Rightarrow c_1=P_1-cl(c_2, P_1)$  and similarly  $cl(c_1, P_1) \cap c_2=\emptyset \Rightarrow c_2=P_1-cl(c_1, P_1)$ . Since  $cl(c_2, P_1)$  and  $cl(c_1, P_1)$  are closed interior circles, it follows that  $c_1$  and  $c_2$  are open interior circles. Since  $c_1=P_1-c_2$ ,  $c_1$  is also closed interior circle. Thus  $c_1$  is non-empty proper sub interior circle of a pixel  $P_1$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex which is both open and closed interior circles.

- 9) *Theorem 3.9:* A pixel of  $\emptyset$  at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex is connected if and only if every non-empty proper interior circles of pixels has a non-empty frontier.

*Proof:* Let every non-empty proper interior circle of pixels have a non-empty frontier. To show that a pixel  $P_1$  is connected. Suppose, if possible,  $P_1$  is disconnected. Then there exist non-empty disjoint interior circles  $c_1$  and  $c_2$  both open and closed interior circles at a pixel  $P_1$  such that  $P_1 = c_1 \sqcup c_2$ . Therefore  $c_1 = \text{int}(c_1, P_1) = cl(c_1, P_1)$ . But  $\text{Fr}(c_1, P_1) = cl(c_1, P_1) - \text{int}(c_1, P_1)$ . Hence  $\text{Fr}(c_1, P_1) = cl(c_1, P_1) - \text{int}(c_1, P_1) = \emptyset$ , which is contrary to our hypothesis. Hence  $P_1$  must be connected.

Conversely, let  $P_1$  be connected and suppose, if possible, there exists a non-empty proper interior circle  $c_1$  of  $P_1$  such that  $\text{Fr}(c_1, P_1) = \emptyset$ . Now  $cl(c_1, P_1) = \text{int}(c_1, P_1) \sqcup \text{Fr}(c_1, P_1) = c_1 \sqcup \text{Fr}(cl(c_1, P_1))$ . Hence  $cl(c_1, P_1) = \text{int}(c_1, P_1) = c_1$  showing that  $c_1$  is both open and closed interior circle in  $P_1$  and therefore  $P_1$  is disconnected. But this is a contradiction. Hence every non-empty proper interior circles of  $P_1$  must have a non-empty frontier.

10) *Theorem 3.10:* Let  $P_1$  be a pixel at the product digital topology with the axioms  $C1, C2, C3$  in the Cartesian complex and let  $c_1$  be separated interior circle of  $P_1$  such that  $c_1 \sqcap C_1 \sqcap C_2$  where  $C_1$  and  $C_2$  are separated interior circles. Then  $c_1 \sqcap C_1$  or  $c_1 \sqcap C_2$ , that is,  $c_1$  cannot intersect  $C_1$  and  $C_2$ .

*Proof:* Since  $C_1$  and  $C_2$  are separated interior circles,  $C_1 \cap cl(C_2, P_1) = \emptyset$ ,  $cl(C_1, P_1) \cap C_2 = \emptyset$ . Let  $c_1 \sqcap C_1 \sqcap C_2 \Rightarrow c_1 \cap (C_1 \sqcap C_2) = (c_1 \cap C_1) \sqcap (c_1 \cap C_2) \dots \dots \dots (1)$ .

Claim that at least one of the interior circle  $c_1 \cap C_1$  and  $c_1 \cap C_2$  is empty. If possible, suppose none of these interior circles is empty, that is, suppose that  $c_1 \cap C_1 \neq \emptyset$  and  $c_1 \cap C_2 \neq \emptyset$ . Then  $(c_1 \cap C_1) \cap (c_1 \cap C_2) \sqcap (c_1 \cap C_1) \cap (cl(C_1, P_1) \cap cl(C_2, P_1)) = (c_1 \cap (cl(C_1, P_1))) \cap (C_1 \cap cl(C_2, P_1)) = (c_1 \cap (cl(C_1, P_1))) \cap \emptyset = \emptyset$ . Clearly  $(c_1 \cap C_1) \cap (c_1 \cap C_2) = \emptyset$ . Since  $c_1 \cap C_1$  and  $c_1 \cap C_2$  are separated interior circles. Thus  $c_1$  has been expressed at the union of two non-empty separated interior circles and consequently  $c_1$  is disconnected. But this is a contraction. Hence at one of the interior circles  $c_1 \cap C_1$  and  $c_1 \cap C_2$  is empty. If  $c_1 \cap C_1 = \emptyset \dots \dots (1)$  gives  $c_1 = c_1 \cap C_2$  which implies that  $c_1 \sqcap C_2$ . Similarly if  $c_1 \cap C_2 = \emptyset$ , then  $c_1 \sqcap C_1$ . Hence either  $c_1 \sqcap C_1$  or  $c_1 \sqcap C_2$ .

11) *Corollary 3.11:* If  $c_1$  is a connected interior circle of a pixel  $P_1$  such that  $c_1 \sqcap C_1 \sqcap C_2$  where  $C_1, C_2$  are disjoint open (closed) interior circle of a pixel  $P_1$ , then  $C_1$  and  $C_2$  are separated interior circles.

12) *Remark 3.12:* Let  $c_1$  be a connected interior circle of a pixel  $P_1$  such that  $c_1 \sqcap c_1 \sqcap cl(c_1, P_1)$ . Then  $c_1$  is connected. In particular  $cl(c_1, P_1)$  is connected.

#### IV. CONCLUSIONS

Further work, at the connectedness among the pixels and interior circles will be extended upto the concepts compactness at the product digital topology with the axioms  $C1, C2, C3$  in the cartesian complex.

#### V. ACKNOWLEDGMENT

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