iJRASET
International Journal For Research in
Applied Science and Engineering Technology

INTERNATIONAL JOURNAL FOR RESEARCH
IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 10 Issue: I Month of publication: January 2022

DOI: https://doi.org/10.22214/ijraset.2022.39772

International Journal for Research in Applied Science & Engineering Technology (IJRASET)
ISSN: 2321-9653; IC Value: 45.98; SJ Impact Factor: 7.538
Volume 10 Issue I Jan 2022- Available at www.ijraset.com

Eulers Totient Function (Number Theoretic Functions), Right Angle Triangle and Their Applications

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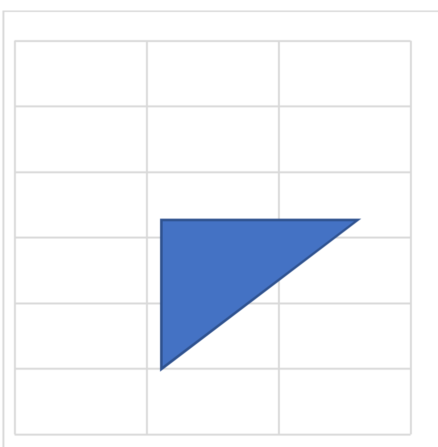
***Abstract:** In this paper, we introduce some new theorem and results(section II,III and VI) on Euler's Totient Function , Right angle triangle and their applications*
***Keywords:** Euler's totient function, right angle triangle, Golden ratio, Hardy – Ramanujan number, student's pencil compass and etc.*

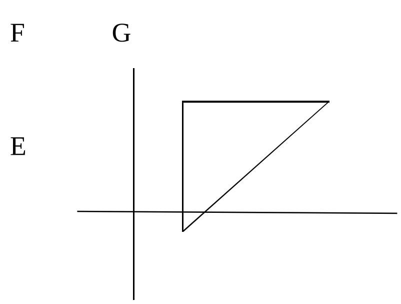
I. INTRODUCTION(PRELIMINARY)

1) *Definition 1.1 :* for $m \geq 1$, $\varphi(m)$ denote the number of positive integers not exceeding m that are relatively prime to m. Euler's totient function is also known as Euler's phi function.

If $m = p_1.p_2$ where p_1, p_2 are the primes $(p_2 > p_1)$, Let divisors of $m\ are\ 1, p_1, p_2\ and\ p_1.p_2$.
Then , $\varphi(m)$= (fourth divisors of m – second divisors of m) - (Third divisor of m – first divisor of m).

$$\text{Or, } \varphi(m) = (p_1.p_2 - p_1) - (p_2 - 1)$$
$$= p_1(p_2 - 1) - (p_2 - 1)$$
$$= (p_1 - 1)(p_2 - 1)$$
$$\varphi(m) = (p_1 - 1)(p_2 - 1)$$
$$If\ E\ [HCF(1, p_1), LCM(1, p_1)],$$
$$F\ [HCF(p_1, p_2), LCM\ (p_1, p_2)], and$$
$$G\ [HCF(p_2, p_1.p_2), LCM(p_2, p_1.p_2)]$$
$$Are\ the\ vertices\ of\ triangle\ EFG$$





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Then triangle EFG is inverted right angle triangle

$$And\ \varphi(m) = |A - B|$$

Where A,B are the Altitude and Base of triangle EFG

and $(i)\ If\ A > B$

then, $A = p_1.p_2 - p_1\ and\ B = p_2 - 1$

$(ii)\ If\ B > A$

then , $B = p_1.p_2 - p_1\ and\ A = p_2 - 1$

$$\varphi(m) = |A - B|$$

Squaring the both side,

$$(\varphi(m))^2 = A^2 + B^2 - 2.A.B$$

Since, $A^2 + B^2 = H^2$ where H is the hypotenuse

Then , $(\varphi(m))^2 = H^2 - 2.A.B$

divide both side of above equation by 4,

$$\left(\frac{\varphi(m)}{2}\right)^2 = \left(\frac{H}{2}\right)^2 - \frac{1}{2}.A.B$$

Or , $T = \left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(m)}{2}\right)^2$

Where T is the area of triangle EFG = $\frac{1}{2}.A.B$.

$H^n - (\varphi(m))^n$ is perfectly divisible by 2^n , $n \geq 1$.

If $m = p_1.p_2 \dots .p_z, Z \geq 2$ then total pair of primes = $\frac{z(z-1)}{2}$ Or $(Z-1)^{th}$ triangular number

Therefore, total number of inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(S)}{2}\right)^2$ is $(z-1)^{th}$ Triangular number.

Theorem and Results

II. INEQUALITY RELATION

1) *Theorem 2.1:* If $= p_1.p_2 \dots .p_z$,
Where $p_1, p_2, \dots .p_z$ are the primes

$(p_1 < p_2 < \dots .. < p_z)$, $Z \geq 3$.

Let divisors of m are $d_1, d_2, \dots .. d_{\tau(m)}$, Consider m have ${}^{\tau(m)}C_4$ [${}^{\tau(m)}C_4 = \sum_{i=i}^{n} i[\{\tau(m)-2-i\}^{th}$ triangular number] , where n= $\tau(m) - 3$] groups , each group consists of four elements (at least one element of each group should be distinct) . let groups be (a', b', c', d') , (a'', b'', c'', d'') , ….. ($a^{''\dots e\ times}$, $b^{''\dots e\ times}$, $c^{''\dots .e\ times}$, $d^{''\dots .e\ times}$) where e = ${}^{\tau(m)}C_4$

And every four elements of each group belongs to the divisors of m that is $d_1, d_2, \dots \dots d_{\tau(m)}$. Let E_1[HCF(a', b'), LCM (a', b')],

F_1[HCF(b', c'), LCM(b', c')], G_1[HCF(c', d'), LCM (c', d')] , E_2[HCF(a'', b''), LCM (a'', b'')],

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F_2[HCF(b'', c''), LCM(b'', c'')], G_2[HCF(c'', d''), LCM (c'', d'')] , …….. E_e[HCF($a^{''....\text{e times}}$, $b^{''....\text{e times}}$), LCM ($a^{''....\text{e times}}$, $b^{''....\text{e times}}$)],

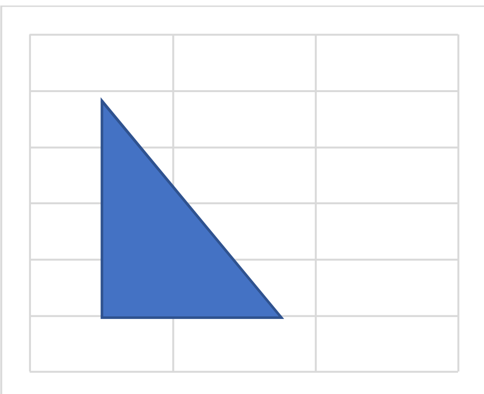
F_e[HCF($b^{''...\text{e times}}$,$c^{''...\text{e times}}$), LCM($b^{''...\text{e times}}$,$c^{''...\text{e times}}$)], G_e[HCF($c^{''...\text{e times}}$,$d^{''...\text{e times}}$), LCM ($c^{''..\text{e times}}$, $d^{''..\text{e times}}$)] are the vertices (or points) of $E_1F_1G_1$,$E_2F_2G_2$, ….$E_eF_eG_e$ and

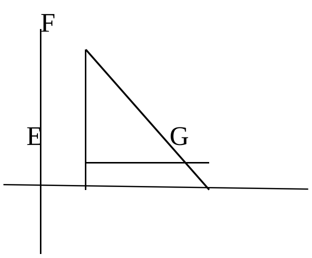
W_1 = Total number of groups for which E_Ψ,F_Ψ,G_Ψ form Collinear line.

Where Ψ belongs to [1,e]

W_2 = Total number of groups for which E_Ψ,F_Ψ,G_Ψ form triangles (other than right angle triangle).

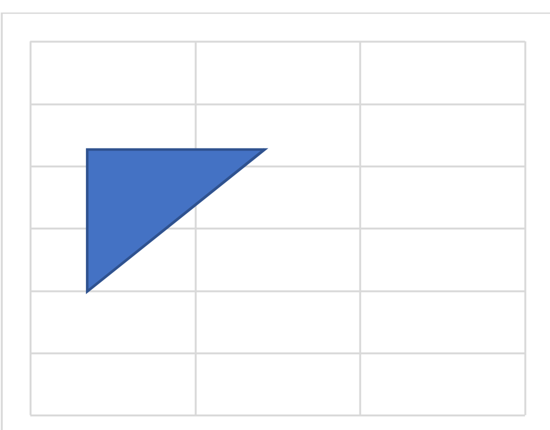
W_3 = Total number of groups for which E_Ψ,F_Ψ,G_Ψ form non-inverted right angle triangles





W_4 = Total number of groups for which E_Ψ,F_Ψ,G_Ψ form inverted right angle triangles whose area is not equal to $\left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(S)}{2}\right)^2$ and

W_5 = Total number of groups for which E_Ψ,F_Ψ,G_Ψ form inverted right angle triangles whose area is equal to $\left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(S)}{2}\right)^2$ where $S \neq m$.



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If S=product of second and third elements of each group ($b^{'}.c^{'}$, $b^{''}.c^{''}$, ….. $b^{''...e\ times}.c^{''...e\ times}$) , then there exist at least Z-1 + ${}^{Z-1}C_2$ inverted right angle triangles whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

That is $w_5 \geq$ Z-1 + ${}^{Z-1}C_2$ or, $\sum_{i=1}^{n} i$ [{τ(m)-2-i}th triangular number] – $\sum_{j=1}^{4} wj \geq$ Z-1+${}^{Z-1}C_2$, where n=τ(m) –3 .

Where total number of groups = $W_1+W_2+W_3+W_4+W_5$.

Proof : If m= $P_1.P_2….P_z$, Z ≥ 3.

Total pair of primes = Z-1+${}^{Z-1}C_2$ and number of groups of the form (1,P',P'',P'.P'') is Z-1+${}^{Z-1}C_2$

Where P' and P'' are one of $P_1,P_2,….P_Z$ (P'<P''). If groups are of the form (1,P',P'',P'.P'') then E,F,G form inverted right angle triangles, and every Z-1+${}^{Z-1}C_2$ triangles follow the relation T= $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$. Where S= $p^{'}.p^{''}$. By considering the groups of four elements (elements belongs to the divisors of m) , there exist some other groups also which are not of the form (1,P',P'',P'.P'') but follow the relation T = $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$. If m=$P_1.P_2.P_3$, then total number of groups = ${}^{\tau(m)}C_4={}^{8}C_4$=70.

Out of these 70 groups there exist definitely Z−1+${}^{Z-1}C_2$, that is 3 groups for which the vertices form inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

Above 3 groups are of the form (1,P',P'',P'.P'') where P' and P'' are one of P_1,P_2,P_3 (P'<P''). But for some m there exist some other groups which are not of the form (1,P',P'',P'.P'') but follow the relation T = $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

If m =$2.P_2.P_3$, then the vertices (E,F,G) for the group ($2,2.p_2,2.p_3,2.p_2.p_3$) form inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

So, if m=$2.P_2.P_3$, then there exist 4 inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

If m=$3.P_2.P_3$, then the vertices (E,F,G) for the group ($3,3.P_2,3.P_3,3.P_2.P_3$) form inverted right angle triangle but area is not equal to $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

So, If m=$3.P_2.P_3$ then there exist 3 inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

Hence, there exist at least Z-1+${}^{Z-1}C_2$ inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$.

NOTE : (a) $a^{'}<b^{'}<c^{'}<d^{'}$, $a^{''}<b^{''}<c^{''}<d^{''}$, ….., $a^{''...e\ times}<b^{''...e\ times}<c^{''....e\ times}<d^{''....e\ times}$

(b) let m = 2.3.5 then , groups are (1,2,3,5) , (1,2,3,6) , (1,2,3,10) , (1,2,3,15) , (1,2,3,30) , (1,2,5,6) , (1,2,5,10) , (1,2,5,15) , (1,2,5,30) , (1,2,6,10) , (1,2,6,15) , (1,2,6,30) , (1,2,10,15) , (1,2,10,30) , (1,2,15,30) , (1,3,5,6) , (1,3,5,10) , (1,3,5,15) , ……………….. , (6,10,15,30) . If m=2.3.5 , then total number of groups = 70 .
(c) $d_1=a^{'}=1$.

Theorem 2.1 in another form :
A lot consists of groups for which E_Ψ,F_Ψ,G_Ψ form inverted right angle triangle, then probability (q_1) of getting the inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2-\left(\frac{\varphi(S)}{2}\right)^2$

is greater than or equal to $\frac{z-1+{}^{z-1}C_2}{w_4+w_5}$

Or, $q_1 \geq \frac{z-1+{}^{z-1}C_2}{w_4+w_5}$ and probability (q_2) of getting the inverted right angle triangle whose area is not equal to

$\left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(S)}{2}\right)^2$ is less than or equal to $\frac{w_4+w_5-{}^{z-1}C_2-z+1}{w_4+w_5}$

Or, $q_2 \leq \frac{w_4+w_5-{}^{z-1}C_2-z+1}{w_4+w_5}$.

q_1 *can also be written as* ,

$$q_1 \geq \frac{z-1+{}^{z-1}C_2}{\sum_{i=i}^{n} i\ [\{\tau(m)-2-i\}\text{th triangular number}]-\sum_{j=1}^{3} w_j}$$

Note : If $m = p_1.p_2.p_3$ then ,

$q_1 \geq 0.1364$ *and* $q_2 \leq 0.8637$.

III. RESULTS RELATED TO HARDY-RAMANUJAN NUMBER , AREA OF RECTANGLE AND TRAPEZIUM AND GOLDEN RATIO .

1) *Result 3.1:* If $m = 3.5.7$, out of the 70 groups if we take 27^{th} and 41^{th} groups that are $(1,7,15,35)$ and $(3,5,15,35)$ and

if $E_{27}[HCF(1,7), LCM\ (1,7)], F_{27}[HCF(7,15), LCM(7,15)]$, $G_{27}[HCF(15,35), LCM(15,35)]$

and $E_{41}[HCF(3,5), LCM(3,5)]$,

$F_{41}[HCF(5,15), LCM(5,15)]$,

$G_{41}[HCF(15,35), LCM(15,35)]$ are the vertices of triangles E_{27},F_{27},G_{27} and $E_{41}F_{41}G_{41}$, then the average of $\left(\frac{H_{27}}{2}\right)^2 - \left(\frac{\varphi(S_{27})}{2}\right)^2$

And $\left(\frac{H_{41}}{2}\right)^2 - \left(\frac{\varphi(S_{41})}{2}\right)^2$ is Hardy – Ramanujan number

Where $E_{27}F_{27}G_{27}$ is inverted right angle triangle , $E_{41}F_{41}G_{41}$ is non-inverted right angle triangle. H_{27},H_{41} are the length of hypotenuse of triangles $E_{27}F_{27}G_{27}$ and $E_{41}F_{41}G_{41}$

S_{27}=7x15 , S_{41}=5x15 and area of both triangle is not equal to $\left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(S)}{2}\right)^2$.

$\frac{\left(\frac{H_{27}}{2}\right)^2-\left(\frac{\varphi(S_{27})}{2}\right)^2+\left(\frac{H_{41}}{2}\right)^2-\left(\frac{\varphi(S_{41})}{2}\right)^2}{2} = 1729$.

2) *Result 3.2:* If m=$P_1.P_2$ ($P_1<P_2$), divisors of m are 1,P_1,P_2,$P_1.P_2$.
Let E [HCF(1,P_1), LCM(1,P_1)],

F [HCF(P_1,P_2), LCM(P_1,P_2)], and

G [HCF(P_2,$P_1.P_2$), LCM (P_2,$P_1.P_2$)] are the vertices of triangle EFG

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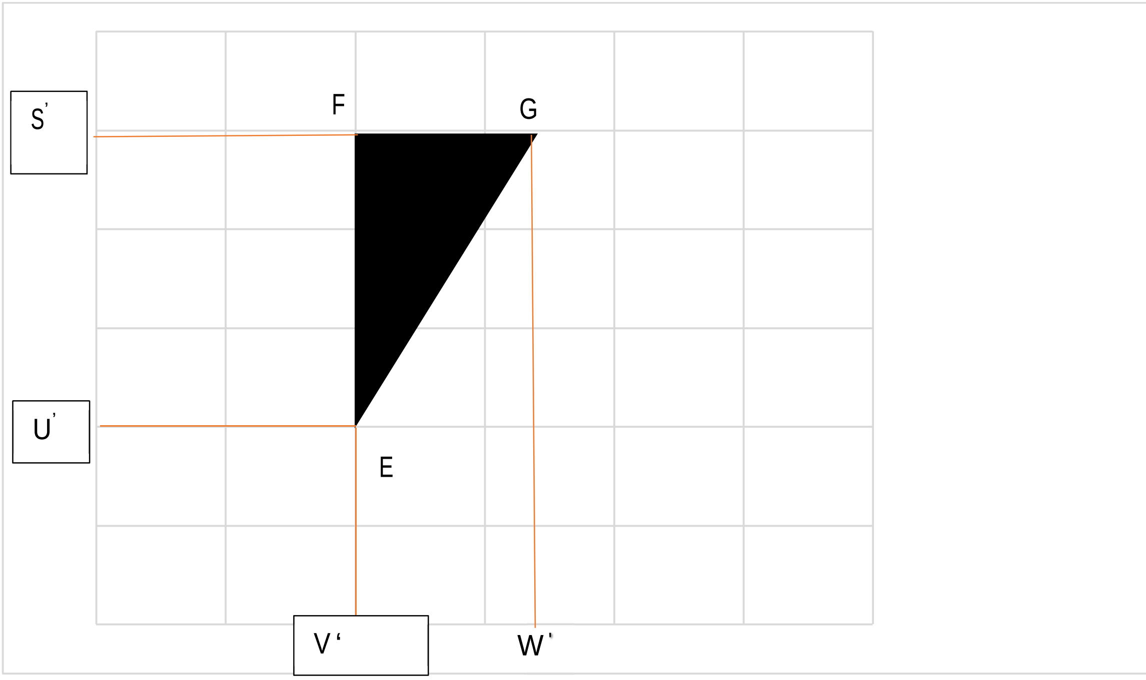

Let Area of Rectangle S'U'EF = R and Area of trapezium V'W' GE = K then,

(a) $\varphi(m) = R - \sqrt{H^2 - R^2}$

(b) $\varphi(m) = 2R - \sqrt{H^2 + 4(K - R)}$

where H is the length of hypotenuse of right angle triangle EFG .

Proof : $EG = \sqrt{(p_2 - 1)^2 + (p_1.p_2 - p_1)^2}$

$= (p_2 - 1).\sqrt{1 + {p_1}^2}$

$EG = H = (p_2 - 1).\sqrt{1 + {p_1}^2}$

Area of Rectangle = S'U'EF = $R = p_1(p_2 - 1)$

Area of Trapezium = V'W'GE =

$K = \frac{1}{2}p_1(p_2 - 1)\ (p_2 + 1)$

Put $R = p_1(p_2 - 1)$ and $H = (p_2 - 1).\sqrt{1 + {p_1}^2}$ in (a) that is $R - \sqrt{H^2 - R^2}$, We get

$$p_1(p_2 - 1) - \sqrt{(p_2 - 1)^2(1 + {p_1}^2) - {p_1}^2(p_2 - 1)^2}$$

$$= p_1(p_2 - 1) - (p_2 - 1)$$

$$= (p_2 - 1)(p_1 - 1)$$

$$= (p_1 - 1)(p_2 - 1)$$

$$= \varphi(m)$$

Hence (a) part is proved.

Now, put $R = p_1(p_2 - 1)$, $H = (p_2 - 1)\sqrt{1 + {p_1}^2}$

And $K = \frac{1}{2}p_1(p_2 - 1)\ (p_2 + 1)$ in (b) that is

$2R - \sqrt{H^2 + 4(K - R)}$, we get

$=2p_1(p_2-1)-[(p_2-1)^2(1+p_1^2)+2\{p_1(p_2-1)(p_2+1)-2p_1(p_2-1)\}]^{1/2}$

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$=2p_1(p_2-1) - [(p_2-1)^2(1+p_1^2)+2\,p_1(p_2-1)(p_2+1-2)]^{1/2}$

$=2p_1(p_2-1) - [(p_2-1)^2(1+p_1^2)+2\,p_1(p_2-1)^2]^{1/2}$

$=2p_1(p_2-1) - [(p_2-1)^2(1+p_1^2+2p_1)]^{1/2}$

$=2p_1(p_2-1) - (p_2-1)(p_1+1)$

$=(p_2-1)[2p_1-p_1-1]$

$=(p_2-1)(p_1-1)$

$=(p_1-1)(p_2-1)$

$=\varphi(m)$

Hence (b) part is proved.

3) *Result 3.3:* If $m=p_1.p_2$ where p_1,p_2 are the primes ($p_1<p_2$), divisors of m are $1,p_1,p_2,p_1.p_2$. let E [HCF(1,p_1), LCM(1,p_1)], F [HCF(p_1,p_2), LCM(p_1,p_2)] and G [HCF(p_2,$p_1.p_2$),LCM(p_2,$p_1.p_2$)] are the vertices of triangle EFG, then roots of the quadratic equation

$2.\,(\varphi(p_1))^2x^2 - 2(\varphi(p_1))^2x^1 - p_1 = 0$ are $\frac{\sqrt{H^2-4T}\pm H}{2\sqrt{H^2-4T}}$

Proof :- $2.\,(\varphi(p_1))^2x^2 - 2(\varphi(p_1))^2x^1 - p_1 = 0$

$x=\frac{(p_1-1)\pm\sqrt{1+p_1^2}}{2(p_1-1)}$

Since , H = $(p_2-1)\sqrt{1+p_1^2}$ and T = $\frac{1}{2}p_1(p_2-1)^2$

Where T is the area of triangle EFG

$$\frac{\sqrt{H^2-4T}\pm H}{2\sqrt{H^2-4T}} = \frac{(p_1-1)\pm\sqrt{1+p_1^2}}{2(p_1-1)}$$

Hence, Proved.

4) *Result 3.4:*

$$If\ m = p_1.p_2\ where\ p_1\ is\ either\ 2\ or\ 3\ and$$

$p_1 < p_2, Divisors\ of\ m\ are\ 1, p_1, p_2, p_1.p_2.$

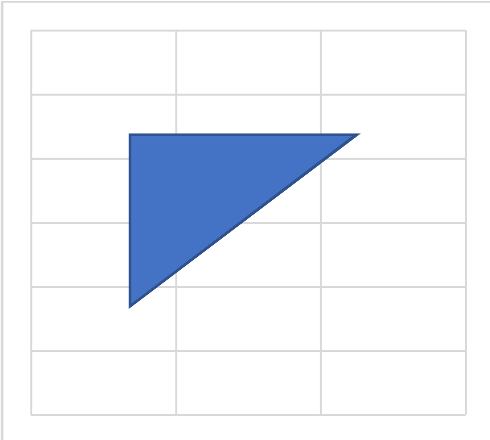
$$Let\ E[HCF(1,p_1), LCM(1,p_1)], F[HCF(p_1,p_2), LCM(p_1,p_2)$$

And $G[HCF(p_2,p_1.p_2), LCM(p_2,p_1.p_2)]$ are the vertices of triangle EFG , then

$2.\lim_{n\to\infty}\left(\frac{U_{n+1}}{U_n}\right) - 1 = \sqrt{\frac{4T+(\varphi(m))^2}{\varphi(p_2).\varphi(m)}} = \frac{H}{\varphi(p_2).\sqrt{\varphi(p_1)}}$

$$where\ T\ is\ the\ area\ of\ triangle\ EFG.$$

Proof : $If\ m = p_1.p_2, then\ \varphi(m) = (p_1-1).(p_2-1),$



Vertices of above triangle EFG are E(1,p_1) , F(1,$P_1.P_2$) and G(P_2,$P_1.P_2$) .

$$T = \frac{1}{2}.EF.FG$$

$$Or, T = \frac{1}{2}.p_1.(p_2 - 1)^2$$

$$Put\ T\ and\ \varphi(m)\ in\ R.H.S,$$

$$\sqrt{\frac{2p_1(p_2-1)^2 + (p_1-1)^2(p_2-1)^2}{(p_1-1)(p_2-1)^2}} = \sqrt{\frac{p_1^2+1}{p_1-1}}$$

(i) $If\ p_1 = 2, then \sqrt{\frac{p_1^2+1}{p_1-1}} = \sqrt{5}$

(ii) $If\ p_1 = 3, then \sqrt{\frac{p_1^2+1}{p_1-1}} = \sqrt{5}$

$$Since, for\ n \geq 1 \lim_{n\to\infty}\left(\frac{U_{n+1}}{U_n}\right) = \frac{1+\sqrt{5}}{2}$$

$$Or, 2.\lim_{n\to\infty}\left(\frac{U_{n+1}}{U_n}\right) - 1 = \sqrt{5}$$

Hence , $2.\lim_{n\to\infty}\left(\frac{U_{n+1}}{U_n}\right) - 1 = \sqrt{\frac{4T+(\varphi(m))^2}{\varphi(p_2).\varphi(m)}} = \frac{H}{\varphi(p_2).\sqrt{\varphi(p_1)}}$.

IV. ANGLES OF TRIANGLE EFG IN TERMS OF EULERS PHI FUNCTION

1) *Theorem 4.1:* If m=$p_1.p_2$ where p_1,p_2 are the primes (p_1<p_2), divisors of m are 1,p_1,p_2,$p_1.p_2$. let E [HCF(1,p_1), LCM(1,p_1)], F [HCF(p_1,p_2), LCM(p_1,p_2)] and G [HCF(p_2,$p_1.p_2$), LCM(p_2,$p_1.p_2$)] are the vertices of triangle EFG and angle E = α and angle G =β, then the angles of triangle EFG (in terms of euler's phi function) are $\alpha = \frac{1}{2}cot^{-1}\left[\frac{\varphi(m)\sqrt{2H^2-(\varphi(m))^2}}{H^2-(\varphi(m))^2}\right]$ and

$\beta = \frac{1}{2}cot^{-1}\left[\frac{\varphi(m)\sqrt{2H^2-(\varphi(m))^2}}{(\varphi(m))^2-H^2}\right]$

Proof : Put $\varphi(m) = (p_1 - 1)(p_2 - 1)$ and

$H = (p_2 - 1)\sqrt{1 + {p_1}^2}$ in

$\alpha = \frac{1}{2}cot^{-1}\left[\frac{\varphi(m)\sqrt{2H^2-(\varphi(m))^2}}{H^2-(\varphi(m))^2}\right]$,

$= \frac{1}{2}cot^{-1}\left[\frac{(p_1-1)(p_2-1)\{2(p_2-1)^2(1+{p_1}^2)-(p_1-1)^2(p_2-1)^2\}^{1/2}}{(p_2-1)^2(1+{p_1}^2)-(p_1-1)^2(p_2-1)^2}\right]$

$= \frac{1}{2}cot^{-1}\left[\frac{(p_1-1)\{2(1+{p_1}^2)-(p_1-1)^2\}^{1/2}}{\{(1+{p_1}^2)-(p_1-1)^2\}}\right]$

$= \frac{1}{2}cot^{-1}\left[\frac{(p_1-1)\{2+2{p_1}^2-{p_1}^2-1+2p_1\}^{1/2}}{1+{p_1}^2-{p_1}^2-1+2p_1}\right]$

$= \frac{1}{2}cot^{-1}\left[\frac{(p_1-1)\{{p_1}^2+2p_1+1\}^{1/2}}{2p_1}\right]$

$= \frac{1}{2}cot^{-1}\left[\frac{(p_1-1)(p_1+1)}{2p_1}\right]$

$= \frac{1}{2}cot^{-1}\left[\frac{{p_1}^2-1}{2p_1}\right] = \frac{1}{2}tan^{-1}\left[\frac{2p_1}{{p_1}^2-1}\right]$ --------(*)

Since, tanα = $\frac{GF}{EF}$

Or, tanα = $\frac{p_2-1}{p_1(p_2-1)} = \frac{1}{p_1}$

From (*) equation,

$$\frac{1}{2}tan^{-1}\left[\frac{2p_1}{{p_1}^2-1}\right] = \alpha$$

Or, $tan(2\alpha) = \frac{2p_1}{{p_1}^2-1}$

Or, $\frac{2tan\alpha}{1-tan^2\alpha} = \frac{2p_1}{{p_1}^2-1}$

Put $\alpha = \frac{1}{p_1}$,

$$\frac{2\text{x}\frac{1}{p_1}}{1-\frac{1}{{p_1}^2}} = \frac{2p_1}{{p_1}^2-1}$$

L.H.S = R.H.S

Hence, Proved.

Since, $\alpha + \beta = \frac{\pi}{2}$

$$\frac{1}{2}tan^{-1}\left[\frac{2p_1}{{p_1}^2-1}\right] + \beta = \frac{\pi}{2}$$

$$\beta = \frac{\pi}{2} - \frac{1}{2}tan^{-1}\left[\frac{2p_1}{{p_1}^2-1}\right]$$

$$\beta = \frac{1}{2}\left[\pi - tan^{-1}\left\{\frac{2p_1}{{p_1}^2 - 1}\right\}\right]$$

Or, $\beta = \frac{1}{2}\left[\pi - cot^{-1}\left\{\frac{{p_1}^2-1}{2p_1}\right\}\right]$

$$\beta = \frac{1}{2}\left[cot^{-1}\left\{\frac{1-{p_1}^2}{2p_1}\right\}\right]$$

Hence, $\beta = \frac{1}{2}cot^{-1}\left[\frac{\varphi(m)\sqrt{2H^2-(\varphi(m))^2}}{(\varphi(m))^2-H^2}\right]$

V. OBSERVATIONS

(5.1) If $m = 2.3.5$, then $w_4 = 18$, $w_5 = 4$, and $w_4 + w_5 = 22$.
If m= 3.5.7 , then w_4=19, w_5=3, and w_4+w_5=22
So, $w_4 + w_5$ is constant but w_4, w_5 are not constant.
If m=$p_1.p_2.p_3$, for different m w_1, w_2, w_3 and w_4+w_5 are constants.
$w_1 = 10, w_2 = 36, w_3 = 2$ and $w_4 + w_5 = 22$.

$$w_2 > w_4 + w_5 > w_1 > w_3.$$

Total number of right angle triangles = $w_3 + w_4 + w_5$

$= \varphi(\sum_{k=1}^{5} w_k) = \varphi(8_{C_4}) =$

$\varphi[\sum_{i=i}^{5} i\ [\{\tau(m) - 2 - i\}th$ triangular number]] = 24 .

Or, $w_3 + w_4 + w_5 = \varphi(w_1 + w_2 + w_3 + w_4 + w_5)$

Or, $\sum_{k=1}^{5} w_k - \varphi(\sum_{k=1}^{5} w_k) = w_1 + w_2$

That is $70 - 24 = 10 + 36$

From Theorem 2.1 : $w_5 \geq z - 1 +^{z-1}c_2$

If $m = p_1.p_2.p_3$, then,

$\varphi(\sum_{k=1}^{5} w_k) \geq z - 1 +^{z-1}c_2 + w_3 + w_4$, Where $\sum_{k=1}^{5} w_k$ = Total No. of groups.

And $\varphi(\sum_{k=1}^{5} w_k) - z + 1 -^{z-1}c_2 - w_3 \geq w_4 \geq \frac{z-1+^{z-1}c_2}{q1} - w_5$. where q_1 is the probability of getting the inverted right angle triangle whose area is $\left(\frac{H}{2}\right)^2 - \left(\frac{\varphi(S)}{2}\right)^2$

VI. APPLICATION OF EULER'S TOTIENT FUNCTION AND NUMBER THEORETIC FUNCTIONS IN STUDENT'S PENCIL COMPASS

Mathematical instruments are used to understand mathematical constructions and concepts . If we talk about mathematical instruments then there are many mathematical instruments like protractor , ruler , set-square , divider , pencil compass and etc . There are two types of mathematical instruments , one is used by student and other are used by teachers . Mathematical instruments used by teachers are very large as compare to mathematical instruments used by student because they are used in black boards or white boards . The mathematical instruments used by student are smaller because they are usable in their books only . If we talk about the student's pencil compass , then one can form the circles with a radius of about 1 to 15 centimeters .

Here we have applied the concepts of number theory .

If it is assumed that a circle of radius up to 20 centimeter is being formed by student's pencil compass then also the following result will be correct . Normally circles of radius up to 14 or 15 centimeters are formed by the student's pencil compass .

VII. RESULT

If we draw all the possible circles by student's pencil compass having radius = r (r = 1,2,3,4,……,maximum possible radius to be made by student's pencil compass) , then 6 is the only number whose every positive proper divisors as a radius follow the relation

$$\varphi(2r) + \tau(2r) + \sigma(2r) = \text{greatest integer function of P}$$

and the number 6 (itself) as a radius follow the relation

$$\varphi(2r) + \tau(2r) + \sigma(2r) = \text{lowest integer function of P} .$$

$\varphi(2r) + \tau(2r) + \sigma(2r)$ can also be written as $\varphi(d) + \tau(d) + \sigma(d)$,

where P = π x d , π = 3.1415 (four digits after the decimal point) , P is circumference(perimeter) of circle and d is diameter of circle .

NOTE: (a) Draw all the possible circles by student's pencil compass means to make circles of each radius which is made by student's pencil compass .

Where $\varphi(d)$ is the is the euler's phi function which denote the number of positive integers not exceeding d that are relatively prime to d .

$\tau(d)$ denote the number of positive divisors of d and $\sigma(d)$ denote the sum of these divisors .
In mathematical form : If r = 1 , $\varphi(2) + \tau(2) + \sigma(2)$ = greatest integer function of 2 x 3.1415 x 1 = 6 ,
If r = 2 , $\varphi(4) + \tau(4) + \sigma(4)$ = greatest integer function of 2 x 3.1415 x 2 = 12 ,
If r = 3 , $\varphi(6) + \tau(6) + \sigma(6)$ = greatest integer function of 2 x 3.1415 x 3 = 18 ,
If r = 6 , $\varphi(12) + \tau(12) + \sigma(12)$ = lowest integer function of 2 x 3.1415 x 6 = 38 ,
Where 1,2 and 3 are the positive proper divisors of 6 .

VIII. ACKNOWLEDGEMENT

I would like to thanks Dr. Pravin Hudge , Dr. V. Patil , Dr. Vikas Deshmane , and Mandar Khasnis sir for useful comments .

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