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A Generalized Subclass of p-VALENT Analytic Functions

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***Abstract:** In this paper, a class $\Sigma_{p,\alpha}(\alpha,\beta,\mu,k)$ of functions F analytic in the unit disk U={z:|z|<1} of the form $F(z)=z^p-\sum_{t=n}^{\infty}a_{p+t}z^{p+t}$, (p, $n\in N$, $a_{p+t}\geq 0$ for $t\geq n$) is considered. Coefficient inequality, distortion theorem, extreme points, starlikeness and convexity for this class are obtained.*

***Keywords:** Analytic functions, p-valent functions, extreme points, radii of starlikeness and convexity.*
***2010 AMS Subject classification:** Primary 30C45.*

I. INTRODUCTION

Let $\Delta_{p,n}$ denotes the class of p-valent analytic functions in the unit disk U={z:|z|<1} which are of the form

(1.1) $$F(z)=z^p-\sum_{t=n}^{\infty}a_{p+t}z^{p+t},\ \text{p,n}\in \text{N},\ a_{p+t}\geq 0\,.$$

A function $P(z)=p+p_nz^n+.....,(n\geq 1)$ analytic in U is said to be in $\mathrm{M}_{p,n}(\alpha)$ if

(1.2) $$|P(z)-p|<(p-\alpha),\quad 0\leq\alpha<\text{p},\quad z\in U\,.$$

Alternatively, in terms of subordination, it is said that P(z) is in $\mathrm{M}_{p,n}(\alpha)$ if

(1.3) $$P(z)\prec p+(p-\alpha)z$$

where '$\prec$' stands for subordination which is defined by saying that F is subordinate to G written as $F\prec G$ if F(z) = G(ϕ(z)), $z\in U$ for some analytic functions ϕ (z) such that

ϕ (0) = 0 and | ϕ (z)| < 1 for $z\in U$.

Now, a class criterion $M_{p,n}(\alpha,\beta,\mu,k)$ whose members $P(z) = p + p_nz^n \ldots (n\geq 1)$, analytic in U satisfy the condition

(1.4) $$P(z)\prec\frac{k+(k-2\mu\alpha)\beta z}{1-(2\mu-1)\beta z}$$

with $1\leq 2\mu\leq 2$, $2\mu\alpha< k\leq p$, $0<\beta\leq 1$ and $0\leq\alpha<\dfrac{p}{2\mu}-\dfrac{(p-k)}{\mu}$ is considered.

Equivalently (1.4) can be written as:

(1.5) $$\left|\frac{P(z)-k}{(2\mu-1)P(z)+(k-2\mu\alpha)}\right|<\beta$$

Note that $M_{p,n}(\alpha,1,1/2,p)\equiv M_{p,n}(\alpha)$ and $M_{p,n}(\alpha,\beta,\mu,p)\equiv M_{p,n}(\alpha,\beta,\mu)$.

On putting P(z) = $\dfrac{F'(z)}{z^{p-1}}$ in (1.5) it follows that

(1.6) $$\left|\frac{\frac{F'(z)}{z^{p-1}}-k}{(2\mu-1)\frac{F'(z)}{z^{p-1}}+(k-2\mu\alpha)}\right|<\beta,\quad F(z)\in\Delta_{p,n}.$$

The class of such functions F(z) satisfying (1.6) is denoted by $\Sigma_{p,\alpha}(\alpha,\beta,\mu,k)$ and

$\Sigma_{p,n}(\alpha, \beta, \mu, p) \equiv \Sigma_{p,n}(\alpha, \beta, \mu)$.

Clearly, these classes are more general than the classes studied by Gupta and Jain [2,3], Aouf [1], Juneja–Mogra [4], Thirupathi Reddy [6] and Kulkarni, Aouf, Joshi [5] etc.

In this paper, coefficient inequalities, distortion theorem, closure theorem, radii of starlikeness and convexity for the class $\Sigma_{p,n}(\alpha, \beta, \mu, k)$ are obtained.

II. COEFFICIENT INEQUALITIES:

Theorem 2.1: A function $F(z)\in T_{p,n}$ is in the class $\Sigma_{p,n}(\alpha, \beta, \mu, k)$ if and only if

(2.1) $$\sum_{t=n}^{\infty}(p+t)a_{p+t}\le\frac{\{2\mu\beta(p-\alpha)-(p-k)(1+\beta)\}}{1+(2\mu-1)\beta}.$$

The result is sharp, the extremal function being

(2.2) $$F(z)=z^p-\frac{\{2\mu\beta(p-\alpha)-(p-k)(1+\beta)\}}{(p+t)\{1+(2\mu-1)\beta\}}z^{p+t}\qquad \text{for } t\ge n.$$

Proof: Let $F(z)\in\Sigma_{p,n}(\alpha, \beta, \mu, k)$, then

$$\left|\frac{\frac{F'(z)}{z^{p-1}}-k}{(2\mu-1)\frac{F'(z)}{z^{p-1}}+(k-2\mu\alpha)}\right|$$

$$=\left|\frac{(p-k)-\sum_{t=n}^{\infty}(p+t)a_{p+t}z^t}{\{(2\mu-1)p+(k-2\mu\alpha)\}-(2\mu-1)\sum_{t=n}^{\infty}(p+t)z^t}\right|<\beta.$$

On letting $z\to 1^-$ through real values, it gives

$$(p-k)+\sum_{t=n}^{\infty}(p+t)a_{p+t}$$

$$\le\beta\{(2\mu-1)p+(k-2\mu\alpha)\}-\beta(2\mu-1)\sum_{t=n}^{\infty}(p+t)a_{p+t}$$

or,

$$\sum_{t=n}^{\infty}(p+t)a_{p+t}\le\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{1+(2\mu-1)\beta}.$$

Conversely, let (2.1) holds, then for $|z| = 1$

$$|F'(z)z^{1-p}-k|-\beta|(2\mu-1)F'(z)z^{1-p}+(k-2\mu\alpha)|$$

$$\leq (p-k)+\sum_{t=n}^{\infty}(p+t)a_{p+t}$$

$$-\beta\{2\mu-1)p+(k-2\mu\alpha)\}+\beta(2\mu-1)\sum_{t=n}^{\infty}(p+t)a_{p+t}$$

$$\leq \{1+\{(2\mu-1)\beta\}\sum_{t=n}^{\infty}(p+t)a_{p+t}-2\mu\beta(p-\alpha)+(p-k)(1+\beta)$$

$$\leq 0.$$

The result is sharp for function given by (2.2).

Corollary 2.2: For k = p, $F(z)\in \Sigma_{p,n}(\alpha, \beta, \mu, p)$ if and only if

$$\sum_{t=n}^{\infty}(p+t)a_{p+t}\leq \frac{2\mu\beta(p-\alpha)}{1+\beta(2\mu-1)}, \quad 0\leq\alpha<\frac{p}{2\mu},$$

$0 <\beta\leq 1$, $1 \leq 2\mu\leq 2$.

For n = 1, the result of Kulkarni et.al. [5] follows.

Corollary 2.3: For k = p and μ = 1, $F(z)\in \Sigma_{p,n}(\alpha, \beta, 1, p)$ if and only if

$$\sum_{t=n}^{\infty}(p+t)a_{p+t}\leq \frac{2\beta(p-\alpha)}{1+\beta}, \; 0\leq\alpha< p,\; 0<\beta\leq 1.$$

For n = 1, β = 1, the result of Juneja and Mogra [4] follows.

For n = 1, p = 1, the result of Gupta and Jain [3] follows.

Corollary 2.4: For k = p, $\mu=\frac{1+\upsilon}{2}, F(z)\in\Sigma_{p,n}\left(\alpha,\beta,\frac{1+\upsilon}{2},p\right)$, if and only if

$$\sum_{t=n}^{\infty}(p+t)a_{p+t}\leq \frac{(1+\upsilon)\beta(p-\alpha)}{1+\beta\upsilon}, \quad 0\leq\alpha<\frac{p}{2}\;.$$

For n = 1, the result of Thirupathi Reddy [6] follows.

III. DISTORTION THEOREM

Theorem 3.1: If $F(z)\in \Sigma_{p,n}(\alpha,\beta,\mu,k)$, then for $|z| = r < 1$

$$r^p-\left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}}\right]r^{p+n}<|F(z)| \tag{3.1}$$

$$<r^p+\left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}}\right]r^{p+n}.$$

and

$$pr^{p-1}-\left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{\{1+\beta(2\mu-1)\}}\right]r^{p+n-1}<|F'(z)| \tag{3.2}$$

$$<pr^{p-1}+\left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{\{1+\beta(2\mu-1)\}}\right]r^{p+n-1}.$$

Proof: From Theorem 2.1, it follows that

(3.3) $$\sum_{t=n}^{\infty} a_{p+t} < \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}}\right].$$

Hence

$$|F(z)| < r^p + \sum_{t=n}^{\infty} a_{p+t} r^{p+t}$$
$$< r^p + \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}}\right] r^{p+n}$$

and

$$|F(z)| > r^p - \sum_{t=n}^{\infty} a_{p+t} r^{p+t}$$
$$> r^p - \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}}\right] r^{p+n}.$$

Hence (3.1) follows.
In the same way, it follows that

$$|F'(z)| < pr^{p-1} + \sum_{t=n}^{\infty} (p+t) a_{p+t} r^{p+t-1}$$
$$< pr^{p-1} + \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{\{1+\beta(2\mu-1)\}}\right] r^{p+n-1}$$

and

$$|F'(z)| > pr^{p-1} - \sum_{t=n}^{\infty} (p+t) a_{p+t} r^{p+t-1}$$
$$> pr^{p-1} - \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{\{1+\beta(2\mu-1)\}}\right] r^{p+n-1}.$$

This completes the proof of the theorem.
The above bounds are sharp. Equalities can be attained for the function

(3.4) $$F(z) = z^p - \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+n)\{1+\beta(2\mu-1)\}}\right] z^{p+n}, \quad z = \pm r.$$

IV. CLOSURE THEOREMS

Theorem 4.1: If F(z)$\in \Sigma_{p,n}(\alpha, \beta, \mu, k)$ and $G(z) = z^p - \sum_{t=n}^{\infty} b_{p+t} z^{p+t}$ are also in $\Sigma_{p,n}(\alpha,\beta,\mu,k)$, then $H(z) = z^p - \frac{1}{2}\sum_{t=n}^{\infty}(a_{p+t} + b_{p+t}) z^{p+t}$ is also in $\Sigma_{p,n}(\alpha,\beta,\mu,k)$.

Proof: Since F(z) and G(z) both belong to $\Sigma_{p,n}(\alpha, \beta, \mu, k)$, then from Theorem 2.1

(4.1) $$\{1+\beta(2\mu-1)\}\sum_{t=n}^{\infty}(p+t)a_{p+t} \le 2\mu\beta(p-\alpha)-(p-k)(1+\beta).$$

and

(4.2) $$\{1+\beta(2\mu-1)\}\sum_{t=n}^{\infty}(p+t)b_{p+t} \le 2\mu\beta(p-\alpha)-(p-k)(1+\beta).$$

So for H(z), it follows that

$$\frac{1}{2}\{1+\beta(2\mu-1)\}\sum_{t=n}^{\infty}(p+t)(a_{p+t}+b_{p+t}) \le 2\mu\beta(p-\alpha)-(p-k)(1+\beta).$$

Using (4.1) and (4.2). Therefore, $H(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$.

V. EXTREME POINTS

Theorem 5.1: Let $F_{n-1}(z) = z^p$ and

$F_t(z) = z^p - \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+t)\{1+\beta(2\mu-1)\}}\right]z^{p+t}$, $t \ge n$ then $F(z) \in \Sigma_{p,n}(\alpha,\beta,\mu,k)$ if and only if it can be expressed in the form

$$F(z) = \sum_{t=n-1}^{\infty}\lambda_t F_t(z), \text{ where } \lambda_t \ge 0 \text{ and } \sum_{t=n-1}^{\infty}\lambda_t = 1.$$

Proof: Suppose

$$F(z) = \sum_{t=n-1}^{\infty}\lambda_t F_t(z) = z^p - \sum_{t=n}^{\infty}\left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+t)\{1+\beta(2\mu-1)\}}\right]\lambda_t z^{p+t},$$

then

$$\sum_{t=n}^{\infty}(p+t)\frac{\{2\mu\beta(p-\alpha)-(p-k)(1+\beta)\}}{(p+t)\{1+\beta(2\mu-1)\}}\lambda_t \le \frac{\{2\mu\beta(p-\alpha)-(p-k)(1+\beta)\}}{1+\beta(2\mu-1)}.$$

Thus, by Theorem 2.1, $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$.

Conversely, suppose $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$. Hence, by Theorem 2.1, it follows that

$$a_{p+t} \le \frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+t)\{1+\beta(2\mu-1)\}}, \quad t \ge n$$

Setting

$$\lambda_n = \frac{(p+n)\{1+(2\mu-1)\beta\}}{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}a_{p+n}, \quad n = 1,2,\ldots$$

and

$$\lambda_{n-1} = 1 - \sum_{t=n}^{\infty}\lambda_n.$$

It follows that

$$F(z) = \sum_{t=n-1}^{\infty} \lambda_t F_t(z).$$

This completes the proof.

The extreme points for the class $\Sigma_{p,n}(\alpha, \beta, \mu, k)$ are given by

$$F_{n-1}(z) = z^p$$

and

$$F_t(z) = z^p - \left[\frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{(p+t)\{1+\beta(2\mu-1)\}} \right] z^{p+t}, \ t \geq n.$$

VI. RADIUS OF STARLIKENESS:

Theorem 6.1: If $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$, then the function F(z) is startlike in the disk $0 < |z| < r = r(\alpha, \beta, \mu, k, n)$ where

$$r(\alpha,\beta,\mu,k,n) < \inf_{n\in N} \left\{ \frac{p\{1+(2\mu-1)\beta\}}{2\mu\beta(p-\alpha)-(p-k)(1+\beta)} \right\}^{1/n}, n \in N$$

Proof: It is enough to show that

$$\left| \frac{zF'(z)}{F(z)} - p \right| < p \text{ for } |z| < 1.$$

or,

$$\left| \frac{zF'(z)}{F(z)} - p \right| = \left| \frac{-\sum_{t=n}^{\infty} t\, a_{p+t} z^t}{1-\sum_{t=n}^{\infty} a_{p+t} z^t} \right| < p$$

or,

$$\sum_{t=n}^{\infty} t\, a_{p+t} |z|^t < p \left[1 - \sum_{t=n}^{\infty} a_{p+t} z^t \right]$$

or,

$$\sum_{t=n}^{\infty} \left(\frac{p+t}{p} \right) a_{p+t} |z|^t < 1.$$

But, Theorem 2.1 gives

$$\sum_{t=n}^{\infty} (p+t)\, a_{p+t} < \frac{2\mu\beta(p-\alpha)-(p-k)(1+\beta)}{1+\beta(2\mu-1)}.$$

Thus, F(z) is starlike if

$$|z| < \left\{ \frac{p[1+\beta(2\mu-1)]}{2\mu\beta(p-\alpha)-(p-k)(1+\beta)} \right\}^{1/n}, n = 1,2,\ldots .$$

VII. RADIUS OF CONVEXITY

Theorem 7.1: If $F(z) \in \Sigma_{p,n}(\alpha, \beta, \mu, k)$, then $F(z)$ is convex in the disk $0 < |z| < r = r(\alpha, \beta, \mu, k, n)$, where

$$r(\alpha,\beta,\mu,k,n) < \inf_{n\in N}\left\{\frac{p^2\{1+(2\mu-1)\beta\}}{(p+n)[2\mu\beta(p-\alpha)-(p-k)(1+\beta)]}\right\}^{1/n} \quad n = 1,2.. \; .$$

Proof: Putting $zF'(z)$ in place of $F(z)$ in Theorem 6.1, the result follows.

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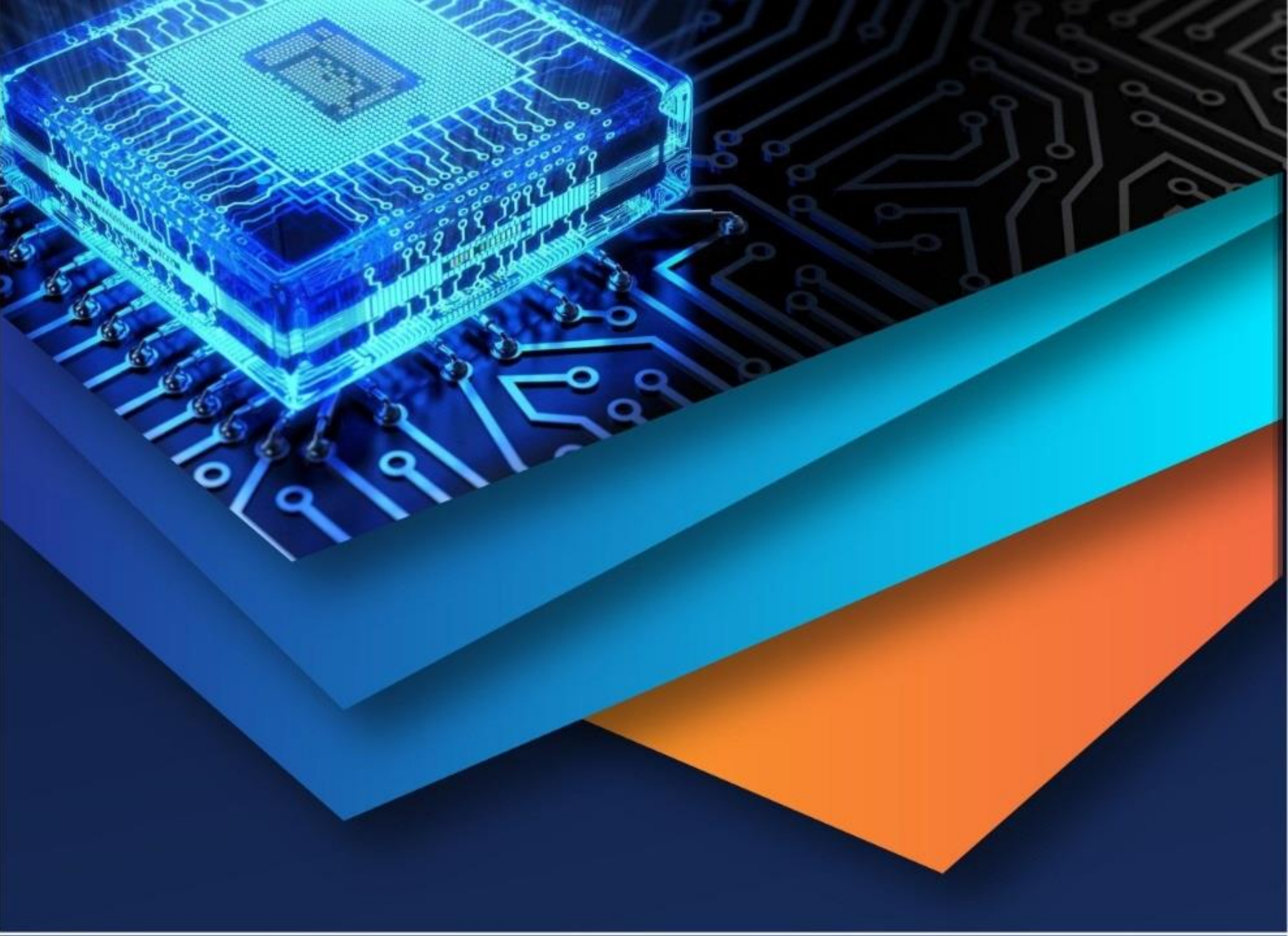

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