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Homomorphism of Characteristic Fuzzy Subgroup and Abelian Fuzzy Subgroup

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Abstract: In this paper, we have established some independent proof of homomorphism on algebra of abelian and characteristic fuzzy subgroup. The characteristic of fuzzy subgroup [13] was first introduced by P. Bhattacharya and N. P. Mukharjee in 1986.
Keywords: Fuzzy subgroup, characteristic fuzzy subgroup, abelian fuzzy subgroup and normal fuzzy subgroup.

I. INTRODUCTION

The concept of fuzzy sets was introduced by L.A.Zadeh [15] in 1965.Study of algebraic structure was first introduced by A.Rosenfeld [1]. After that a series of researches have done in this direction P.Bhattacharya and N.P.Mukharjee[13] have defined fuzzy normal subgroup and characteristic fuzzy subgroup in 1986. In this paper we have tried to established some independent proof about the properties of fuzzy group homomorphism on algebra of characteristic fuzzy subgroup.

II. PRELIMINARIES

In this section, we recall and study some concepts associated with fuzzy sets and fuzzy group, which we need in the subsequent sections.

A. Fuzzy Set

Over the past three decades, a number of definitions of a fuzzy set and fuzzy group have appeared in the literature (cf., e.g., [15, 1, 3, 7, 10]). In [15], it has been shown that some of these are equivalent. We begin with the following basic concepts of fuzzy set, fuzzy point and fuzzy group.

Definition 2.1 [15] A fuzzy subset of D_1 be a function $f_1 : D_1 \rightarrow [0,1]$ the set of all fuzzy subset of D_1 is sad to be fuzzy power set of D_1 and designate by $P_1(D_1)$.

Definition 2.2 [15] **Support of fuzzy set**. Suppose $A_1 \in F_1\ P_1(D_1)$ then the set $\{ A_1\ (d_1) : d_1 \in D_1\}$ is said to be the image of A_1 is designate by $A_1\ (D_1)$. The set $\{d_1 : d_1 \in D_1\ , A_1\ (d_1) > 0\}$ is said to be the support of A_1 is designate by $A_1{}^*$.

Definition 2.3 [15] Let A_1, $C_1 \in F_1\ P_1(D_1)$ such that $A_1\ (d_1) \leq C_1\ (d_1), \forall\ d_1 \in D_1$ then A_1 is said to be contained in C_1 and it is designate by $A_1 \subseteq C_1$

Definition 2.4 [15] Let $B_1 \subseteq A_1$ and $d_1 \in [0\ ,1]$ we defined $d_{1_{B_1}} \in F_1\ P_1(D_1)$ as

$$d_{1_{C_1}}\ (\mathrm{a}) = \begin{cases} d_1, for\ a_1 \in B_1 \\ 0,\ for\ a_1 \in A_1 \end{cases}$$

If B_1 is a singleton $\{b_1\}$ then $D_{\{b_1\}}$ is called a fuzzy point.

For any collection $\{A_{i_1}, i_1 \in I_1\}$ of fuzzy subset of D_1, where I_1 is an index set the least upper bound (L.U.B.) $\cup_{i_1 \in I_1} A_{i_1}$ and greatest lower bound (G.L.B) $\cap_{i_1 \in I_1} A_{i_1}$ of A_{i_1} are given by

$$(\cup_{i_1 \in I_1} A_{i_1})\ (d_1) = \vee_{i_1 \in I_1} A_{i_1}\ (d_1),\ \forall\ d_1 \in D_1.$$

$$(\cap_{i_1 \in I_1} A_{i_1})\ (d_1) = \wedge_{\ i_1 \in I_1} A_{i_1}\ (d_1),\ \forall d_1 \in D_1$$

Fuzzy subgroup

In this section, we discuss the concept of a fuzzy subgroup in details (c.f.,[1]).

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Definition 2.5 Fuzzy subgroup (or F_1 (G_1)) Let G_1 be any group, we define the binary operation o' and unary operation $^{-1}$ on F_1P_1 (G_1) as follows, $\forall A_1, C_1 \in F_1P_1$ (G_1) and $\forall d_1 \in G_1$

$$(A_1 \text{ o } C_1)(d_1) = \vee\{A_1(y_1) \wedge C_1(z_1) : y_1 z_1 = d_1, \forall y_1, z_1 \in G_1\}$$
$$A_1^{-1}(d_1) = A_1(d_1^{-1})$$

Proposition 2.1 [3] If $A_1 \in F_1$ (G_1), then for all $d_1 \in G_1$

(i) $A_1(e_1) \geq A_1(d_1)$

(ii) $A_1(d_1) = A_1(d_1^{-1})$

Proof (i) Let $d_1 \in A_1$, then $d_1 d_1^{-1} = e_1$

$$\begin{aligned} A_1(e_1) &= A_1(d_1 d_1^{-1}) \\ &\geq A_1(d_1) \wedge A_1(d_1^{-1}) \\ &\geq A_1(d_1) \wedge A_1(d_1) = A_1(d_1) \end{aligned}$$

$\therefore$ $A_1(e_1) \geq A_1(d_1), \forall d_1 \in G_1$

(ii)
$$\begin{aligned} A_1(d_1) &= A_1((d_1^{-1})^{-1}) \\ &\geq A_1(d_1^{-1}) \\ &\geq A_1(d_1) \end{aligned}$$

Finally, $A_1(d_1) = A_1(d_1^{-1})$

Anti fuzzy subgroup

In this section we discuss the basic concepts of anti fuzzy subgroup of G_1,[5]

Definition 2.6 A fuzzy subset A_1 of G_1 is said to be anti fuzzy group of G_1, and is denoted as anti F_1 (G_1) if for all $d_1, c_1 \in G_1$

(i) $A_1(d_1 \,.\, c_1) \leq \max\{A_1(d_1), A(c_1)\}$

(ii) $A_1(d_1^{-1}) = A_1(d_1)$

Definition 2.7 Let G_1 be any group we define the binary operation 'o' and unary operation'$^{-1}$' on anti-fuzzy group of G_1 as follows ,$\forall A_1, B_1 \in$ anti F_1 (G_1) and $\forall d_1 \in G_1$

i. $(A_1B_1)(d_1) = \wedge\{A_1(c_1) \vee B_1(p_1) : c_1 p_1 = d_1, \forall c_1, p_1 \in G_1\}$

ii. $A_1(d_1^{-1}) = A_1^{-1}(d_1)$ $\forall d_1 \in G_1$

Proposition 2.2 [5] Suppose $A_1, B_1 \in$ anti $F_1 \forall P_1$ (G_1) also A_{1_i} anti F_1 P_1 (G_1) for each $i \in I$, the following holds

(i)
$$\begin{aligned} (A_1 \text{ o } B_1)(d_1) &= \wedge_{c_1 \in G_1}\{A_1(c_1) \vee B_1(c_1^{-1}d_1)\} \\ &= \wedge_{c_1 \in G_1}\{A_1(d_1 c_1^{-1}) \vee B_1(c_1)\} \end{aligned}$$

(ii)
$$(a_{c_1} \text{ o } A_1)(d_1) = A_1(c_1^{-1}d_1) \quad \forall d_1, c_1 \in G_1$$
$$(A_1 \text{ o } a_{c_1})(d_1) = A_1(d_1 c_1^{-1}) \quad d_1, c_1 \in G_1$$

PROOF:- (i) We have $d_1, c_1 \in G_1 \Rightarrow c_1^{-1} \in G_1$

$$(d_1 c_1^{-1}) c_1 = d_1 (c_1^{-1}c_1) = d_1 e = d_1$$

Also $c_1(c_1^{-1}d_1) = (c_1 c_1^{-1}) d_1 = e\, d_1 = d_1$

Thus,

$$\begin{aligned} \{A_1(d_1 c_1^{-1}) \vee B_1(c_1) &= \wedge_{c_1 \in G_1}\{(A_1(d_1) \vee A_1(c_1^{-1}) \vee B_1(c_1)\} \\ &= \wedge_{c_1 \in G_1}\{(A_1(d_1) \vee (\wedge\ A_1(c_1^{-1}) \vee B_1(c_1)\} \\ &= \wedge_{c_1 \in G_1}\{(A_1(d_1) \vee (A_1 \text{ o } B_1)(c_1^{-1}c_1)\} \\ &= \wedge_{c_1 \in G_1}\{A_1 \text{ o } (A_1 \text{ o } B_1)(d_1 e) \\ &= (A_1 \text{ o } B_1)\, d_1, \forall d_1 \in G_1 \end{aligned}$$

Similarly, we get

$$\wedge_{c_1 \in G_1}\{A_1(c_1) \vee B_1(c_1^{-1}d_1)\} = (A_1 \text{ o } B_1)(d_1) \ \ \forall d_1 \in G_1$$

(ii)
$$\begin{aligned} (a_{c_1} \text{o } A_1)(d_1) &= \wedge_{c_1 \in G_1}\{A_1(c_1^{-1}d_1) \vee A_1(d_1)\} \\ &= \wedge_{c_1 \in G_1}\{A_1(c_1^{-1}) \vee A_1(d_1) \vee A_1(d_1)\} \\ &= \wedge_{c_1 \in G_1}\{A_1(c_1^{-1}) \vee A_1(d_1)\} \end{aligned}$$

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$= A_1\ (c_1^{-1} d_1)\ \ \forall\ d_1, c_1 \in G_1$

Also,

$$(A_1 \text{ o } a_{c_1})\ (d_1) = \wedge_{c_1 \in G_1} \{ A_1\ (d_1) \vee A_1\ (d_1\ c_1^{-1})\}$$
$$= \wedge_{c_1 \in G_1} \{ A_1\ (d_1) \vee A_1\ (d_1) \vee A_1\ (c_1^{-1})\}$$
$$= \wedge_{c_1 \in G_1} \{ A_1\ (d_1) \vee A_1\ (c_1^{-1})\}$$
$$= A_1\ (d_1\ c_1^{-1})\ \ d_1, c_1 \in G_1$$

Fuzzy homomorphism

In this section author have extend the properties of fuzzy homomorphism in abelian fuzzy subgroup and anti-abelian fuzzy subgroup

III. ABELIAN FUZZY SUBGROUP [6]

Definition 2.8 If $A_1 \in F_1\ (G_1)$ and if $A_1\ (d_1\ \ c_1) = A_1\ (c_1\ d_1)$ for all d_1, $c_1 \in G_1$ then A_1 is called an abelian fuzzy subgroup of G_1

Proposition 3.1:- If $f_1 : G_1 \to G_2$ be a homomorphism of group G_1 into G_2. Let $A_1 \in F_1\ (G_1)$ is abelian fuzzy sub group then expression that $f_1\ (A_1) \in F_1\ (G_2)$ is also an abelian fuzzy subgroup.

PROOF:- Let $m_1, n_1 \in G_2$ then

$$(f_1\ (A_1))\ (m_1\ n_1) = \vee\{A_1\ (p_1) : p_1 \in G_1, f_1\ (p_1) = m_1\ n_1\}$$
$$\geq \vee\{A_1\ (d_1\ c_1) : d_1, c_1 \in G_1, f_1\ (d_1) = m_1, f_1\ (c_1) = n_1\}$$
$$= \vee\{A_1\ (c_1\ d_1) : d_1, c_1 \in G_1, f_1\ (d_1) = m_1, f_1\ (c_1) = n_1\}$$
$$= \vee\{A_1\ (c_1) \wedge A_1\ (d_1) : d_1, c_1 \in G_1, f_1\ (d_1) = m_1, f_1\ (c_1) = n_1\}$$
$$= \vee\{ A_1\ (c_1) : c_1 \in G_1, f_1\ (c_1) = m_1\} \wedge \{\vee\{ A_1\ (d_1) : c_1 \in G_1, f_1\ (d_1) = n_1\}$$
$$= f_1\ (A_1)\ (m_1) \wedge f_1\ (A_1)\ \ (n_1)$$
$$= (f_1\ (A_1))\ (m_1\ n_1)\ \ \forall\, m_1, n_1 \in G_2$$

Hence, $f_1\ (A_1) \in F_1(G_2)$ is an abelian fuzzy subgroup (ABFSG) of G_2.

Proposition 3.2:- Let $f_1 : G_1 \to G_2$ is a homomorphism of group G_1 into a group G_2. If $A_1 \in F_1(G_2)$ is an abelian fuzzy subgroup of G_2 Then show that $f_1^{-1}(A_1) \in F_1\ (G_1)$ is also an abelian fuzzy subgroup of G_1.

PROOF:- Let $f_1 : G_1 \to G_2$ be homomorphism of group G_1 into group G_2. Let $A_1 \in F_1(G_2)$ be an abelian fuzzy subgroup of G_1. Then show $f_1^{-1}(A_1) \in F_1\ (G_1)$ is also an abelian fuzzy subgroup of G_1.

Suppose $d_1, c_1 \in G_1$ we have

$$(f_1^{-1}(A_1))\ (d_1\ c_1) = A_1\ (f_1\ (d_1\ c_1))$$
$$= A_1\ (f_1\ (d_1)\ f_1\ (c_1)), \qquad \text{since } f_1 \text{ is a homomorphism}$$
$$= A_1\ (f_1\ (c_1)\ f_1\ (d_1)), \qquad \text{since } G_2 \text{ is an abelian subgroup}$$
$$= A_1\ (f\ (c_1\ d_1))$$
$$= (f_1^{-1}(A_1))\ \ (c_1\ d_1)\ \forall\, d_1, c_1 \in G_1.$$

Hence, $f_1^{-1}(A_1) \in F_1\ (G_1)$ is an abelian fuzzy subgroup of G_1.

Proposition 3.3:- If $f_1 : G_1 \to G_1'$ is a homomorphism of group G_1 into G_1' and $g_1 : G_1' \to G_1''$ be a homomorphism of group G_1' into group G_1''. Let $A_1 \in F_1\ (G_1)$ then show that the composition $(g_1 \text{ o } f_1)\ (A_1) \in F_1\ (G_1'')$.

PROOF:- Let $\alpha_1, \beta_1 \in G_1''$. If possible, let $\alpha_1 \notin (g_1 \text{ o } f_1)\ (G_1)$ or $\beta_1 \notin (g_1 \text{ o } f_1)\ (G_1)$ then

$(g_1 \text{ o } f_1)\ A_1\ (\alpha_1) \wedge (g_1 \text{ o } f_1)\ A_1\ (\beta_1) = 0 \leq (g_1 \text{ o } f_1)\ A_1\ (\alpha_1\beta_1)$.

If we suppose $\alpha_1 \notin (g_1 \text{ o } f_1)\ (G_1)$ then $\alpha_1^{-1} \notin (g_1 \text{ o } f_1)\ (G_1)$

Implies that $(g_1 \text{ o } f_1)\ (A_1)\ \alpha_1 = 0 = (g_1 \text{ o } f_1)\ (A_1)\ \ \alpha_1^{-1}$

Again if we assume

$\alpha_1 = (g_1 \text{ o } f_1)\ (d_1)$ and $\beta_1 = (g_1 \text{ o } f_1)\ (c_1)$ for some $d_1, c_1 \in G_1$.

Also

$$(g_1 \text{ o } f_1)\ (A_1)\ (\alpha_1\beta_1) = \vee\{ A_1\ (p_1) : p_1 \in G_1, (g_1 \text{ o } f_1)\ p_1 = \alpha_1\beta_1\}$$
$$(g_1 \text{ o } f_1)\ (A_1)\ (\alpha_1\beta_1)$$
$$\geq \vee\ \{ A_1\ (d_1\ c_1) : d_1, c_1 \in G_1, (g_1 \text{ o } f_1)\ d_1 = \alpha_1, (g_1 \text{ o } f_1)\ c_1 = \beta_1\}$$
$$\geq \vee\ \{ A_1\ (d_1) \wedge A_1\ (c_1) : d_1, c_1 \in G_1, (g_1 \text{ o } f_1)\ d_1 = \alpha_1, (g_1 \text{ o } f_1)\ c_1 = \beta_1\}$$

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$= \vee \{ A_1 (d_1) : d_1 \in G_1, (g_1 \circ f_1)\, d_1 = \alpha_1\} \wedge \{\vee ((A_1 (c_1)) : c_1 \in G_1, (g_1 \circ f_1)\, c_1 \in \beta_1\}$

$= (g_1 \circ f_1)\, A_1 (\alpha_1) \wedge (g_1 \circ f_1)\, A_1 (\beta_1)$

Also,

$(g_1 \circ f_1)\, A_1\, {\alpha_1}^{-1}$

$= \vee \{ A_1 (p_1) : p_1 \in G, (g_1 \circ f_1)\, p_1 = {\alpha_1}^{-1} \}$

$= \vee \{ A_1 ({p_1}^{-1}) : p_1 \in G, (g_1 \circ f_1)\, {p_1}^{-1} = \alpha_1 \}$

$= (g_1 \circ f_1)\, A_1 (\alpha_1)$

Hence,

$(g_1 \circ f_1) (A_1) \in F_1 (G_1'')$

Proposition 3.4:- Suppose $f_1 : G_1 \rightarrow G_1'$ and $g_1 : G_1' \rightarrow G_1''$ where f_1 and g_1 are homomorphism of a group G_1 into group G_1' and from a group G_1' into a group G_1'' respectively then the composition homomorphism $(g_1 \circ f_1)$ from G_1 into G_1''. Let $A_1 \in F_1 (G_1)$ is an abelian group then prove that $(g_1 \circ f_1) (A_1) \in F_1 (G_1'')$ is also an abelian group.

PROOF :-Let $\alpha_1, \beta_1 \in G_1''$ then we have by extension principle

$(g_1 \circ f_1) (A_1) (\alpha_1, \beta_1)$

$= \vee \{ A_1 (p_1) : p_1\, G_1, (g_1 \circ f_1)\, p_1 = \alpha_1 \beta_1)\}$

$\geq \vee \{ A_1 (d_1\, c_1) : d_1, c_1 \in G_1, (g_1 \circ f_1)\; d_1 = \alpha_1, (g_1 \circ f_1)\, c_1 = \beta_1 \}$

$= \vee \{ A_1 (c_1\, d_1) : d_1, c_1 \in G_1, (g_1 \circ f_1)\; d_1 = \alpha_1, (g_1 \circ f_1)\, c_1 = \beta_1 \}$

Since $A_1 \in F_1(G_1)$ is an abelian group

$(g_1 \circ f_1) (A_1) (\alpha_1, \beta_1)$

$= \vee \{ A_1(c_1) \wedge A_1 (d_1) : d_1, c_1 \in G_1, (g_1 \circ f_1)\, d_1 = \alpha_1, (g_1 \circ f_1)\; c_1 = \beta_1 \}$

$= \vee [\{A_1 (c_1)\, c_1 \in G_1, (g_1 \circ f_1)\, c_1 = \beta_1 \}] \wedge [\vee A_1 \in (d_1) : d_1 \in G_1, (g_1 \circ f_1)\; d_1 = \alpha_1]$

$= (g_1 \circ f_1) (A_1) (\beta_1) \wedge (g_1 \circ f_1) (A_1) (\alpha_1)$

$= (g_1 \circ f_1) (A_1) (\beta_1\, \alpha_1)$

Hence,

$(g_1 \circ f_1)\, A_1 \in F_1 (G_1'')$ is an abelian fuzzy subgroup of G_1''.

Proposition on abelian anti fuzzy subgroup

Proposition 3.5 If $f_1 : G_1 \rightarrow G_2$ be a homomorphism of group G_1 into group G_2. Let $A_1 \in$ anti $F_1 (G_1)$ is abelian anti fuzzy subgroup of G_1, then show that $f_1\, A_1 \in F_1 (G_2)$ is also abelian anti fuzzy subgroup of G_2.

PROOF: Let $\alpha_1, \beta_1 \in G_2$

$(f_1\, A_1)) (\alpha_1\, \beta_1)$

$= \wedge \{A_1 (p_1) : p_1 \in G_1, f_1 (p_1) = \alpha_1\, \beta_1\}$

$= \wedge \{A_1 (d_1 c_1) : d_1, c_1 \in G_1, f_1 (d_1) = \alpha_1, f_1 (c_1) = \beta_1\}$

$= \wedge \{A_1 (c_1\, d_1) : d_1, c_1 \in G_1, f_1 (d_1) = \alpha_1, f_1 (c_1) = \beta_1\}$

$\leq \wedge \{A_1 (c_1) \vee A_1 (d_1) : d_1, c_1 \in G_1, f_1 (d_1) = \alpha_1, , f_1 (c_1) = \beta_1\}$

$= \wedge \{A_1 (c_1) : c_1 \in G_1, f_1 (c_1) = \beta_1 \vee (\wedge f_1 (d_1) : d_1 \in G_1, f_1 (d_1) = \alpha_1 \})$

$= \{f_1 (A_1) \vee f_1 (A_1)\} (\beta_1\, \alpha_1)$

$= (f_1 (A_1)) (\beta_1\, \alpha_1) \qquad \forall\, \alpha_1, \beta_1 \in G_2$

Hence $f_1 (A_1) \in$ anti $F_1 (G_2)$ is abelian anti-fuzzy subgroup of G_2

Proposition 3.6:- Let $f_1 : G_1$ □□G_2 is a homomorphism of a group G_1 into a group G_2. If A_1 □ anti $F_1 (G_2)$ is an abelian anti-fuzzy subgroup of G_2 then show that ${f_1}^{-1} (A_1)$ □ anti $F_1 (G_1)$ is also an abelian anti-fuzzy subgroup of G_1.

PROOF :- Suppose $f_1 : G_1$ □□G_2 is a homomorphism of a group G_1 into a group G_2. Let A_1 □ anti $F_1 (G_2)$ be abelian anti-fuzzy subgroup of G_2. Then show that ${f_1}^{-1}(A_1)$ □ anti $F_1 (G_1)$ is also an abelian anti-fuzzy subgroup G_1.

Let d_1 □□c_1 □□G_1

We have $({f_1}^{-1} (A_1)) (d_1\, c_1) = A_1 (f_1 (d_1\, c_1))$

$= A_1 (f_1 (d_1)\, f_1 (c_1))$ since f_1 is a homomorphism

$= A_1 (f_1 (c_1)\, f_1 (d_1))$ since G_2 is an abelian subgroup

$= A_1 (f_1\, (c_1\, d_1))$

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$= f_1^{-1}(A_1)\,(c_1, d_1)$

Finally, $f_1^{-1}(A_1) \in$ anti F_1 (G_1) is an abelian anti-fuzzy subgroup.

Proposition 3.7: Suppose $f_1 : G_1 \to G_1'$ and $g_1 : G_1' \to G_1''$ where f_1 and g_1 are homomorphism of a group G_1 into group G_1' and from a group G_1' into a group G_1'' respectively. Let $A_1 \in$ anti $F_1(G_1)$ is an abelian anti fuzzy subgroup of G_1 then prove that the image of composition homo – morphism of fuzzy anti subgroup A_1 of G_1'' is also an abelian anti fuzzy subgroup of G_1''

PROOF: - Let $\alpha_1, \beta_1 \in G_1''$ then we have by extension principle

$(g_1 \circ f_1)\,(A_1)\,(\alpha_1 \beta_1)$

$= \wedge \{ A_1\,(p_1) : p_1 \in G_1, (g_1 \circ f_1)\, p_1 = \alpha_1\, \beta_1)\}$

$\leq \wedge \{ A_1\,(d_1\, c_1) : d_1, c_1 \in G_1, (g_1 \circ f_1)\;\; d_1 = \alpha_1, (g_1 \circ f_1)\, c_1 = \beta_1\}$

$= \wedge \{ A_1\,(c_1\, d_1) : d_1, c_1 \in G_1, (g_1 \circ f_1)\;\; d_1 = \alpha_1\, (g_1 \circ f_1)\, c_1\, \beta_1\}$

$\leq \wedge \{ A_1(c_1) \vee A_1\,(d_1) : d_1, c_1 \in G_1, (g_1 \circ f_1)\, d_1 = \alpha_1, (g_1 \circ f_1)\;\; c_1 = \beta_1\}$

$= \wedge\; [\{A_1\,(c_1)\, c_1 \in G_1, (g_1 \circ f_1)\, c_1 = \beta_1\}] \vee [\wedge A\,(d_1) : d_1 \in G_1, (g_1 \circ f_1)\;\; d_1]$

$= \alpha_1\, (g_1 \circ f_1)\,(A_1)\,(\beta_1) \vee (g_1 \circ f_1)\,(A_1)\,(\alpha_1)$

$= (g_1 \circ f_1)\,(A_1)\,(\beta_1\, \alpha_1)$

Finally,

$(g_1 \circ f_1)\, A_1$ F_1 (G_1'') is an abelian anti fuzzy subgroup of G_1''.

IV. CHARACTERISTIC FUZZY SUBGROUP [13]

DEFINITION: 4.1:- Let A_1 be a fuzzy subgroup of G_1 and ϕ be a function from G_1 into itself. Now define the fuzzy subset A_1^{ϕ} of G_1 by $A_1^{\phi}(d_1) = A_1\left(d_1^{\phi}\right)$, where $d_1^{\phi} = \phi(d_1)$ A_1 subgroup K of group G_1 is called a characteristic subgroup if $K^{\phi} = K$ for every automorphism ϕ of G_1, where K^{ϕ} denote $\phi(k)$.

Definition 4.2 Characteristic fuzzy subgroup: A fuzzy subgroup A_1 on a group K is called a fuzzy characteristic subgroup of G_1 if $A_1^{\phi}(d_1) = A_1(d_1)$ for every automorphism ϕ of G_1 and for all $d_1 \in G_1$

Proposition 4.1 :- Let A_1 is a fuzzy subgroup of a group G_1 if

a. If ϕ is a homomorphism of G_1 into itself, then A_1^{ϕ} is a fuzzy subgroup of G_1
b. If A_1 is a fuzzy characteristic subgroup of G_1 then A_1 is a normal.

PROOF : (i) $d_1, c_1 \in G_1$ then

$$A_1^{\phi}(d_1\; c_1) = A_1\,(d_1\; c_1)^{\phi}$$
$$= A_1\,(d_1^{\phi} c_1^{\phi})$$

Subsequently ϕ is a homomorphism and A_1 is a fuzzy subgroup of G_1.

$$A_1\left(d_1^{\phi} c_1^{\phi}\right) \geq A_1\left(d_1^{\phi}\right) \wedge A_1\,(c_1^{\phi})$$
$$A_1^{\phi}(d_1\; c_1) = A_1^{\phi}(d_1) \wedge A_1^{\phi}(c_1)$$

Also,
$$A_1^{\phi}\left(d_1^{-1}\right) = A_1\left(d_1^{-1}\right)^{\phi}$$
$$= A_1\left(d_1^{\phi}\right)^{-1}$$
$$= A_1\left(d_1^{\phi}\right)$$
$$= A_1^{\phi}(d_1)$$

Hence, A_1^{ϕ} is a fuzzy subgroup of G_1.

(ii) Let $d_1, c_1 \in G_1$ to prove that A_1 is normal we have to show

$$A_1\,(d_1\; c_1) = A_1\,(c_1\; d_1)$$

Let ϕ be function from G_1 into itself definition by

$\phi(z) = d_1^{-1}\, z\, d_1$, $\forall$ $z \in G_1$

Since A_1 is a fuzzy characteristic subgroup of G_1,

$$\therefore A_1^{\phi} = A_1$$

Thus
$$A_1\,(d_1\; c_1) = A_1^{\phi}(d_1\; c_1)$$
$$= A_1\,(d_1\; c_1)^{\phi}$$
$$= A_1\,(\phi\,(d_1\; c_1))$$

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$$= A_1\,(d_1^{\,-1}(d_1\ c_1)d_1)$$
$$= A_1\,(c_1\ \ d_1)$$

Hence A_1 is normal subgroup of G_1.

V. MAIN RESULT

Proposition 5.1 : Let A_1, C_1 be the fuzzy subgroup of G_1 if

(i) If ϕ is a homomorphism of G_1 into itself, then $A_1^{\,\phi}$ is a fuzzy subgroup of G_1

(ii) If A_1 is a fuzzy characteristic subgroup of G_1 then A_1 is a normal.

PROOF : (i) $d_1, c_1 \in G_1$ then

$$A_1^{\,\phi}(d_1\ c_1) = A_1\,(d_1\ c_1)^{\phi}$$
$$= A_1\,(d_1^{\,\phi} c_1^{\,\phi})$$

Subsequently ϕ is a homomorphism and A_1 is a fuzzy subgroup of G_1.

$$A_1\left(d_1^{\,\phi} c_1^{\,\phi}\right) \geq A_1\left(d_1^{\,\phi}\right) \wedge A_1\,(c_1^{\,\phi})$$
$$A_1^{\,\phi}(d_1\ c_1) = A_1^{\,\phi}(d_1) \wedge A_1^{\,\phi}(c_1)$$

Also, $$A_1^{\,\phi}\left(d_1^{\,-1}\right) = A_1\left(d_1^{\,-1}\right)^{\phi}$$
$$= A_1\left(d_1^{\,\phi}\right)^{-1}$$
$$= A_1\left(d_1^{\,\phi}\right)$$
$$= A_1^{\,\phi}(d_1)$$

Hence, $A_1^{\,\phi}$ is a fuzzy subgroup of G_1.

Proposition 5.2 : Let A_1, C_1 be the fuzzy subgroups of a group G_1. Then the following statement hold

(i) If ϕ is a homomorphism of G_1 into itself. Then $A_1^{\,\phi}$ & $C_1^{\,\phi}$ are fuzzy subgroup of G_1. Then show that **(a)** $(A_1 \cup C_1)^{\phi}$ and **(b)** $(A_1 \cap C_1)^{\phi}$ are fuzzy subgroup of G_1.

(ii) If A_1, C_1 are fuzzy characteristic subgroup of G_1, so A_1 and C_1 are normal then we have to show that $A_1 \cup C_1$ and $A_1 \cap C_1$ are also normal.

Proof:(i) Let $A_1, C_1 \in F_1P_1\,(G_1)$ and ϕ is a homomorphism of G_1 into itself. Let $d_1\ c_1 \in G_1$, we have

$$(A_1 \cup C_1)^{\phi}\left((d_1\ c_1)\right) = (A_1 \cup C_1)\,((d_1\ c_1)^{\phi})$$
$$= (A_1 \cup C_1)\left(d_1^{\,\phi} c_1^{\,\phi}\right)$$
$$= A_1\left(d_1^{\,\phi} c_1^{\,\phi}\right) \vee C_1\left(d_1^{\,\phi} c_1^{\,\phi}\right)$$
$$\geq \left(A_1\left(d_1^{\,\phi}\right) \wedge A_1\left(c_1^{\,\phi}\right)\right) \vee \left(C_1\left(d_1^{\,\phi}\right) \wedge C_1\left(c_1^{\,\phi}\right)\right)$$
$$= \left(A_1\left(d_1^{\,\phi}\right) \vee C_1\left(d_1^{\,\phi}\right)\right) \wedge \left(A_1\left(c_1^{\,\phi}\right) \vee C_1\left(c_1^{\,\phi}\right)\right)$$
$$= (A_1 \cup C_1)\,d_1^{\,\phi} \wedge (A_1 \cup C_1)\,c_1^{\,\phi}$$
$$(A_1 \cup C_1)^{\phi}(d_1\ c_1) \geq (A_1 \cup C_1)^{\phi}(d_1) \wedge (A_1 \cup C_1)^{\phi}(c_1^{\,\phi})$$
$$(A_1 \cup C_1)^{\phi}\left(d_1^{\,-1}\right) = (A_1 \cup C_1)^{\phi}\left(d_1^{\,-1}\right)^{\phi}$$
$$= (A_1 \cup C_1)\left(\left(d_1^{\,\phi}\right)^{-1}\right)$$
$$= A_1\left(d_1^{\,\phi}\right)^{-1} \wedge C_1\left(d_1^{\,\phi}\right)^{-1} \quad \text{since } A_1, C_1 \in F_1\,(G_1)$$
$$= A_1\left(d_1^{\,\phi}\right) \wedge C_1\left(d_1^{\,\phi}\right)$$
$$= (A_1 \cup C_1)\left(d_1^{\,\phi}\right)$$
$$= (A_1 \cup C_1)^{\phi}(d_1)$$

Hence, $(A_1 \cup C_1) \in F_1\,(G_1)$

Similarly,

i (b) we have

$$(A_1 \cap C_1)^{\phi}(d_1\ c_1) = (A_1 \cap C_1)\,((d_1\ c_1)^{\phi})$$
$$= (A_1 \cap C_1)\left(d_1^{\,\phi} c_1^{\,\phi}\right)$$
$$= A_1\left(d_1^{\,\phi} c_1^{\,\phi}\right) \wedge C_1\left(d_1^{\,\phi} c_1^{\,\phi}\right)$$

$$\geq \left(A_1(d_1{}^{\phi}) \wedge A_1(c_1{}^{\phi})\right) \wedge \left(C_1(d_1{}^{\phi}) \wedge C_1(c_1{}^{\phi})\right)$$
$$= \left(A_1(d_1{}^{\phi}) \wedge C_1(d_1{}^{\phi})\right) \wedge \left(A_1(c_1{}^{\phi}) \wedge C_1(c_1{}^{\phi})\right)$$
$$= (A_1 \cap C_1)\, d_1{}^{\phi} \wedge (A_1 \cap C_1)\, c_1{}^{\phi}$$
$$= (A_1 \cap C_1)^{\phi}(d_1) \wedge (A_1 \cap C_1)^{\phi} c_1$$

i.e., $(A_1 \cap C_1)^{\phi}(d_1\ c_1) \geq (A_1 \cap C_1)^{\phi}(d_1) \wedge (A_1 \cap C_1)^{\phi}(c_1)$

Also, $(A_1 \cap C_1)^{\phi}(d_1{}^{-1}) = (A_1 \cap C_1)^{\phi}(d_1{}^{-1})^{\phi}$
$$= (A_1 \cap C_1)\left((d_1{}^{\phi})^{-1}\right)$$
$$= A_1\,(d_1{}^{\phi})^{-1} \wedge C_1(d_1{}^{\phi})^{-1} \quad \text{since } A_1, C_1 \in F_1(G_1)$$
$$= A_1\,(d_1{}^{\phi}) \wedge C_1(d_1{}^{\phi})$$
$$= (A_1 \cap C_1)\,(d_1{}^{\phi})$$
$$= (A_1 \cap C_1)^{\phi}(d_1)$$

Hence, $(A_1 \cap C_1) \in F_1(G_1)$

(ii) Let $d_1, c_1 \in G_1$ to prove that A_1 is normal we have to show
$$A_1\,(d_1\ c_1) = A_1\,(c_1\ d_1)$$
Let ϕ be function from G_1 into itself definition by
$\phi(z) = d_1{}^{-1} z\, d_1$, $\forall\ z \in G_1$
Since A_1 is a fuzzy characteristic subgroup of G_1,

$$\therefore A_1{}^{\phi} = A_1$$

Thus $A_1\,(d_1\ c_1) = A_1{}^{\phi}(d_1\ c_1)$
$$= A_1\,(d_1\ c_1)^{\phi}$$
$$= A_1\,(\phi\,(d_1\ c_1))$$
$$= A_1\,(d_1{}^{-1}(d_1\ c_1) d_1)$$
$$= A_1\,(c_1\ d_1)$$

Hence A_1 is normal subgroup of G_1.

Again, Suppose $d_1, c_1 \in F_1(G_1)$ to prove that $(A_1 \cap C_1)$ is a normal fuzzy subgroup of G_1 it is necessary to show
$$(A_1 \cap C_1)(d_1 c_1) = (A_1 \cap C_1)(c_1 d_1)$$
Let ϕ be the function of group G_1 into itself defined by
$$\phi\,(z) = d_1{}^{-1} z\, d_1 \quad \forall\ d_1 \in G_1$$
Since A_1 and C_1 are fuzzy characteristic subgroup of G_1, hence be normal as we prove
$(A_1 \cap C_1)^{\phi} = (A_1 \cap C_1)$
$$(A_1 \cap C_1)(d_1 c_1) = (A_1 \cap C_1)^{\phi}(d_1 c_1)$$
$$= (A_1 \cap C_1)\,(d_1 c_1)^{\phi}$$
$$= (A_1 \cap C_1)\,(d_1{}^{-1}(d_1 c_1) d_1)$$
$$= (A_1 \cap C_1)\left((d_1{}^{-1} d_1)\,(c_1 d_1)\right)$$
$$= (A_1 \cap C_1)\,(c_1 d_1)$$

Hence $(A_1 \cap C_1) \in F_1(G_1)$ is normal.
Similarly,
$$(A_1 \cup C_1)^{\phi} = (A_1 \cup C_1)$$
$$(A_1 \cup C_1)\,(c_1 d_1) = (A_1 \cup C_1)^{\phi}(c_1 d_1)$$
$$= (A_1 \cup C_1)(c_1 d_1)^{\phi}$$
$$= (A_1 \cup C_1)(d_1{}^{-1}(c_1 d_1) d_1)$$
$$= (A_1 \cup C_1)\left(d_1{}^{-1} d_1)\,(c_1 d_1)\right)$$
$$= (A_1 \cup C_1)(c_1 d_1)$$

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Hence $(A_1 \cup C_1) \in F_1(G_1)$ is also normal.

PROPOSITION 5.3: Let A_1 is a normal fuzzy subgroup of G_1 and let ϕ be a homomorphism of G_1 into itself. Then ϕ induces a homomorphism $\overline{\phi}$ of $\frac{G_1}{A_1}$ into itself defined by

$$\overline{\phi}(d_1 A_1) = \phi(d_1)A_1 \qquad \text{For all } d_1 \in (G_1)$$

Proof : Let $d_1, c_1 \in G_1$ we have

$$d_1 A_1 = c_1 A_1$$

Then we have to show that

$$\phi(d_1)A_1 = \phi(c_1)A_1$$

Since

$$d_1 A_1 = c_1 A_1$$

we have

$$d_1 A_1(d_1) = c_1 A_1(d_1)$$
$$\Rightarrow A_1(e) = A_1(c_1^{-1}d_1)$$
$$d_1 A_1(c_1) = c_1 A_1(c_1)$$
$$\Rightarrow A_1(d_1^{-1}c_1) = A_1(e)$$
$$A_1(c_1^{-1}d_1) = A_1(d_1^{-1}c_1) = A_1(e)$$

Implies that

$$(c_1^{-1}d_1), (d_1^{-1}c_1) \in A_{1_*}$$

Since we have

$$\phi(A_{1_*}) = A_{1_*}$$

Therefore $\phi(c_1^{-1}d_1)$ and $\phi(d_1^{-1}c_1)$ also belong to A_{1_*}

Which implies that

$$A_1(\phi(c_1^{-1}d_1)) = A_1(\phi(d_1^{-1}c_1)) = A_1(e)$$

Let g ∈ G, Then

$$\phi(d_1)A_1(g_1) = A_1(\phi(d_1^{-1})g_1)$$
$$= A_1(\phi(d_1^{-1})\phi(c_1)\phi(c_1^{-1})g_1)$$
$$\geq A_1(\phi(d_1^{-1})\phi(c_1) \wedge A_1(\phi(c_1^{-1})g_1)$$
$$= A_1(\phi(d_1^{-1}c_1)) \wedge)\phi(c_1)A_1(g_1)$$
$$= A_1(e) \wedge \phi(c_1) \wedge A_1(g_1)$$
$$= \phi(c_1)A_1(g_1)$$

Finally,

$$\phi(d_1)A_1(g_1) \geq \phi(c_1)A_1(g_1) \quad \ldots\ldots\ldots\ldots (i)$$

Similarly, we can prove that

$$\phi(d_1)A_1(g_1) \leq \phi(c_1)A_1(g_1) \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Since $g_1 \in G_1$ is arbitrary

Hence,

$$\phi(d_1)A_1 = \phi(c_1)A_1$$

Therefore,

we find that $\overline{\phi}$ is well defined

Now we have only to show that $\overline{\phi}$ is a homomorphism

Let $d_1, c_1 \in G_1$.

Since ϕ is homomorphism

$$\phi(d_1 c_1) = \phi(d_1)\phi(c_1)$$
$$\phi(d_1 c_1)A_1 = \phi(d_1)\phi(c_1)A_1.$$
$$\overline{\phi}(d_1 c_1)A_1 = \phi(d_1)A_1 . \phi(c_1)A_1.$$
$$= \overline{\phi}(d_1 A_1 . c_1 A_1)$$
$$= \overline{\phi}(d_1 A_1) . \overline{\phi}(c_1 A_1).$$

Hence $\overline{\phi}$ is a homomorphism.

REFERENCES

[1] A.Rosenfeld.: Fuzzy group. J.Math.Anal.Appl.35,512-517 (1971).
[2] S.Sebastian and S.Babunder. Fuzzy groups and group homomorphism.Fuzzy sets and systems.8,397-401 (1996).
[3] S. Abou-zaid, On fuzzy subgroup, Fuzzy sets and systems,55. 1993, pp. 237–240.
[4] N.Ajmal and A.S.Prajapati,Fuzzy cosets and fuzzy normal subgroup, Inform.Sci, 64 (1992) 17-25.
[5] R.Biswas,Fuzzy subgroup and anti fuzzy subgroup.Fuzzy sets and systems.35,121-124 (1990).
[6] D.S.Malik,and J.N.Mordeson,Fuzzy subgroup and abelian group.Chinese.J.Math(Taipei).19,129-145 (1991).
[7] J.M.Anthony,and H.Sherwood,Fuzzy subgroup redefined,J.Math.Anal.Appl.69,124-130 (1979).
[8] J.M.Anthony,and H.Sherwood,A characterization of fuzzy subgroup, J.Math.Anal.Appl,69 (1979) 297-305.
[9] P.S.Das,Fuzzy groups and level subgroups J.Math.Anal.Appl.,84 (1981) 264-269.
[10] V.N.Dixit,R.kumar,and N.Ajmal,Level subgroups and union of fuzzy subgroups. Fuzzy Sets and Systems, **37** (1990) 359-371. and Technology, 1994, Vol. 2, pp. 87-98.
[11] M.S.Eroglu, The homomorphic image of a fuzzy subgroup is always a fuzzy subgroup, Fuzzy Sets and Systems, **33** (1989) 255-256.
[12] I.J.Kumar,P.K.Saxena,and P.Yadav,Fuzzy normal subgroups and fuzzy quotients, Fuzzy sets and systems, 46 (1992) 121-132.
[13] N.P.Mukherjee,and P.Bhattacharya,Fuzzy normal subgroups and fuzzy cosets: Information Science, 34 (1984) 225-239.
[14] M.T.A.Osman, On the direct product of fuzzy subgroups, Fuzzy sets and sysems, 12 (1984) 87-91.
[15] L.A.Zadeh. Fuzzy sets, Inform.Control, **8** (1965) 338-353.
[16] S.M.A.Zaidi, and Q.A.Ansari,: Some results of categories of L-fuzzy subgroups.Fuzzy sets and systems, **64** (1994) 249-256. Information Technology, 2012, Vol. 2, pp. 527-531. automata. Soft Computing, 2013 (Communicated).
[17] Y.Yu, A theory of isomorphism of fuzzy groups, Fuzzy system and Math.2,(1988),57-68.