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# Integral Solutions of the Ternary Cubic Equation

$$3(x^2 + y^2) - 4(xy) + 2(x + y + 1) = 522z^3$$

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**Abstract:** The non-homogenous cubic equation with three unknowns represented by the Diophantine equation  $3(x^2 + y^2) - 4(xy) + 2(x + y + 1) = 522z^3$  is analyzed for its patterns of stringfy integral solutions. A few interesting properties among the solutions and some special polygonal numbers are presented.

**Keywords:** Cubic equation, integral solutions, polygonal number, square number, special number.

## I. INTRODUCTION

Number theory is a vast and fascinating field of mathematics. Concerned with the properties of numbers in general and integers in particular as well as the wider classes of problems that arises from their study. The study of number theory is very important because of all other branches depends on this branch for their final results. Solving equations in integers is the central problem of Number theory. Number theory may be subdivided into several fields, according to the method use and the type of questions investigated. The term “arithmetic” is also used to refer to Number theory. The word Diophantine refers to the Hellenistic Mathematician of 3<sup>rd</sup> century, Diophantus of Alexandria who made of such equations and was one of the first Mathematician to introduce symbolism into algebra. In this communication, the non-homogeneous cubic equation with three unknowns represented by the equation  $3(x^2 + y^2) - 4(xy) + 2(x + y + 1) = 522z^3$  is considered and in particular a few interesting relations among the solutions are presented.

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$T_{m,n}$  = Polygonal number

$O_n$  = Octrahedral number

$CS_n$  = Centered square number

$CC_n$  = Centered cube number

$Gno_n$  = Gnomonic number

$SO_n$  = Stella Octangula number

$CH_n$  = Centered Hexagonal number

$TT_n$  = Truncated tetrahedral number

## II. METHOD OF ANALYSIS

The ternary cubic equation to be considered for its quasi integral solution is

$$3(x^2 + y^2) - 4(xy) + 2(x + y + 1) = 522z^3 \tag{1}$$

After using the transformations,

$$x = r + s, y = r - s \tag{2}$$

in (1) leads to  $(r + 1)^2 + 5s^2 = 261z^3$  (3)

We demonstrate five different patterns of quasi distinct integer solutions of (1) as below:

**A. Pattern: 1**

Assume

$$z = z(a, b) = a^2 + 5b^2 \tag{4}$$

$$\text{Write } 261 = (16 + i\sqrt{5})(16 - i\sqrt{5}) \tag{5}$$

Applying the factorization method and substituting (4) and (5) in (3),

$$((r + 1) + i\sqrt{5}s)((r + 1) - i\sqrt{5}s) = (16 + i\sqrt{5})(16 - 6i\sqrt{5})((a + i\sqrt{5}b)^3(a - i\sqrt{5}b)^3) \tag{6}$$

Equating real and imaginary parts, the values of r and s are

$$\begin{aligned} r = r(a, b) &= 16a^3 - 240ab^2 - 15a^2b + 25b^3 - 1 \\ s = s(a, b) &= a^3 - 15ab^2 + 48a^2b - 80b^3 \end{aligned}$$

Substituting the values of *u* & *v* in equation (2), the non zero distinct integral solutions of (1) are executed as

$$\begin{aligned} x = x(a, b) &= 17a^3 - 255ab^2 - 33a^2b - 55b^3 - 1 \\ y = y(a, b) &= 15a^3 - 225ab^2 - 63a^2b + 105b^3 - 1 \\ z = z(a, b) &= a^2 + 5b^2 \end{aligned}$$

Properties

1.  $6z(a, a)$  is a Perfect square
2.  $42z(1, 1)$  is a palindrom number
3.  $-x(a, 1) + y(a, 1) - z(a, 1) + SO_a + 6TT_n + T_{25,a} + CH_a + T_{6,a} + 11Gno_a + 2T_{3,a} \equiv 0(mod 177)$
4.  $y(1, 1) - x(1, 1) - 28z(1, 1)$  is a Nasty number
5.  $22z(1, 1) - y(1, 1)$  is a happy couple number

**B. Pattern: 2**

$$\text{Instead of (5), Write } 261 = (4 + i7\sqrt{5})(4 - i7\sqrt{5}) \tag{7}$$

Equating real and imaginary parts, we get

$$\begin{aligned} r = r(a, b) &= 4a^3 - 60ab^2 - 105a^2b + 175b^3 - 1 \\ s = s(a, b) &= 7a^3 - 105ab^2 + 12a^2b - 20b^3 \end{aligned}$$

substituting (7) and (4) in (3) and the factorization method, as shown in pattern1, the corresponding integral solutions of (1) are represented by

$$\begin{aligned} x = x(a, b) &= 11a^3 - 165ab^2 - 93a^2b + 155b^3 - 1 \\ y = y(a, b) &= -3a^3 + 45ab^2 - 117a^2b + 195b^3 - 1 \\ z = z(a, b) &= a^2 + 5b^2 \end{aligned}$$

Properties

1.  $6y(1, 1)$  is a Ruth-Aaron number
2.  $y(a, 1) - x(a, 1) + 14az(a, 1) + 12CC_a - 128Gno_a \equiv 0(mod 180)$
3.  $y(a, a) + z(a, a)$  is a cubical integer
4.  $y(1, 1) - x(1, 1)$  is a palindrom number
5.  $17z(1, 1)$  is a duck number

**C. Pattern: 3**

$$\text{Instead of(5), Write } 261 = (9 + i6\sqrt{5})(9 - i6\sqrt{5}) \tag{8}$$

Equating real and imaginary parts, we get

$$\begin{aligned} r = r(a, b) &= 9a^3 - 135ab^2 - 90a^2b + 150b^3 - 1 \\ s = s(a, b) &= 6a^3 - 90ab^2 + 27a^2b - 45b^3 \end{aligned}$$

substituting (8) and (4) are in (3) and the factorization method, as shown in pattern 1, the corresponding integral solutions of (1) are represented by

$$\begin{aligned} x = x(a, b) &= 15a^3 - 255ab^2 - 63a^2b - 105b^3 - 1 \\ y = y(a, b) &= 3a^3 - 45ab^2 - 117a^2b + 195b^3 - 1 \end{aligned}$$

$$z = z(a, b) = a^2 + 5b^2$$

**Properties**

1.  $24z(1,1)$  is a perfect square
2.  $y(a, 1) - x(a, 1) + 54z(a, 1) + 6SO_a - 87Gno_a \equiv 0(mod 447)$
3.  $z(a, a) - x(a, a)$  is a palindrom number
4.  $6y(1,1)$  is a duck number
5.  $2y(1,1) - z(1,1)$  is a cubical integer

**D. Pattern: 4**

instead of (5), Write  $261 = \frac{(96+i6\sqrt{5})(96-i6\sqrt{5})}{36}$  (9)

Equating real and imaginary parts, we get

$$r = u(a, b) = \frac{1}{6}(96a^3 - 1440ab^2 - 90a^2b + 150b^3) - 1$$

$$s = v(a, b) = \frac{1}{6}(6a^3 - 90ab^2 + 288a^2b - 480b^3)$$

$$z = z(a, b) = a^2 + 5b^2$$

Since finding integral solutions is the topic we are interested in, we have choose a and b in such a way that r, s, and z are integers.

Let us take  $a = 6A$  and  $b = 6B$ , we have

$$r = u(A, B) = 3456A^3 - 3240AB^2 - 51840A^2B + 5400B^3 - 1$$

$$s = v(A, B) = 216A^3 - 3240AB^2 + 10368A^2B - 17280B^3$$

$$z = z(A, B) = 36A^2 + 180B^2$$

In terms of (2), the integral solutions of (1) are given by

$$x = x(A, B) = 3672A^3 - 6480AB^2 - 41472A^2B + 11880B^3 - 1$$

$$y = y(A, B) = 3240A^3 - 62208A^2B + 22680B^3 - 1$$

$$z = z(A, B) = 36A^2 + 180B^2$$

**Properties**

1.  $6z(1,1)$  is a Perfect square
2.  $z(1,1)$  is a cubical integer
3.  $5z(1,1)$  is a duck number
4.  $12z(1,1)$  is a palindrom number
5.  $x(a, 1) + y(a, 1) - 192az(a, 1) + 103680C_{2,a} + 72360Gno_a \equiv 0(mod 42118)$

**E. Pattern: 5**

instead of (5), Write  $261 = \frac{(48+i3\sqrt{5})(48-i3\sqrt{5})}{9}$  (10)

Equating real and imaginary parts, we get

$$r = r(a, b) = \frac{1}{3}(48a^3 - 720ab^2 - 45a^2b + 75b^3) - 1$$

$$s = s(a, b) = \frac{1}{3}(3a^3 - 45ab^2 + 144a^2b - 240b^3)$$

$$z = z(a, b) = a^2 + 5b^2$$

Since finding integral solutions is the topic we are interested in, we have choose a and b in such a way that r, s, and z are integers.

Let us take  $a = 3A$  and  $b = 3B$ , we have

$$u = u(A, B) = 432A^3 - 6480AB^2 - 405A^2B + 675B^3 - 1$$

$$v = v(A, B) = 27A^3 - 405AB^2 + 1296A^2B - 2160B^3$$

$$z = z(A, B) = 9A^2 + 45B^2$$

In terms of (2), the integral solutions of (1) are given by

$$x = x(A, B) = 459A^3 - 6885AB^2 - 891A^2B + 1485B^3 - 1$$

$$y = y(A, B) = 405A^3 - 6075AB^2 - 1701A^2B + 2835B^3 - 1$$

$$z = z(A, B) = 36A^2 + 180B^2$$

Properties

1.  $\frac{z(1,1)}{6}$  is a perfect square
2.  $y(a, 1) - x(a, 1) + 6az(a, 1) + 654CH_a + 441Gno_a \equiv 0(mod 4533)$
3.  $y(a, 1) - x(a, 1) + a6z(a, 1) + 327 star a + 441Gno_a \equiv 0(mod 4206)$
4.  $z(1,1)+10$  is a cubical integer
5.  $z(1,1)+9$  is a woodall number

**III. CONCLUSION**

In this paper, we have presented five different patterns of non-zero distinct integer solutions of the non-homogeneous cone given by

$$3(x^2 + y^2) - 4(xy) + 2(x + y + 1) = 522z^3$$

To conclude, one may search for other patterns of non-zero integer distinct solutions and their corresponding properties for other choices of cubic Diophantine equations.

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