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Jordan Canonical Form, Generalised Eigen Vectors and its Applications

Bratasee Bhunia[1], Prof. Diptiman Saha(Guide)[2]
Bachelor of Science, Department of Mathematics, St. Xavier's College (autonomous), Kolkata

I. INTRODUCTION

The advantage of a diagonalisable matrix lies in the simplicity of its description. We say a matrix is diagonalisable if it is similar to a diagonal matrix (like, A is diagonalisable if it similar to a diagonal matrix D i.e. ∃ a non-singular matrix P such that $P^{-1}AP=D$). A nxn matrix A is diagonalisable if and only if A has n linearly independent eigenvectors.

But we know that there exist non-diagonalisable matrices too (A nxn matrix A is non- diagonalisable if and only if A does not have n linearly independent eigenvectors i.e. for at least one eigenvalue of A, its geometric multiplicity is strictly less than its algebraic multiplicity).

However, we might still be curious to know the simplest form to which a non-diagonalisable matrixis similar?

Every matrix is similar to an upper triangular matrix over **C**. Therefore, we can atleast say that a non-diagonalisable matrix is similar to an upper triangular matrix, may be of a special structure.

The answer to the question posed above is the **Jordan Canonical Form** (JCF) of a matrix. JCF of a matrix is not only upper triangular but it is very close to being a diagonal matrix except for the few ones above the main diagonal.

In this project, we shall take a closer look at the Jordan Canonical Form of a given matrix A. In particular, we shall be interested in the following questions:

1) How to determine JCF of a matrix A
2) How to find a matrix P such that $P^{-1}AP=J$, where J is the JCF of A.

In addition, we shall look at some applications of Jordan Canonical Form. We shall see how the special structure of J allows us to do many of the nice computations we can do with the diagonal matrices.

II. JORDAN CANONICAL FORM OF A MATRIX

Before knowing Jordan Canonical Form (JCF) J of a matrix, we need to know what is called Jordan Block.

A. *Jordan Block*

The Jordan block of size n for eigen value λ(real) is the nxn upper triangular matrix having λ s on the principal diagonal,1s directly above the principal diagonal (super diagonal) and zeroes elsewhere.

Therefore, Jordan blocks of sizes 1,2,3,4,5 are $[\lambda]$, $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$,

$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$ respectively.

And, for complex eigen value λ=a+ib, the Jordan block is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Clearly, for complex eigen value Jordan block is only of size 2.

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B. Structure of Jordan Canonical Form

The Jordan Canonical form, J of a nxn matrix A is a "block diagonal" matrix

$$J=\begin{bmatrix} J_1(\lambda_1, m_1) & & & \mathbf{0} \\ & J_2(\lambda_2, m_2) & & \\ & & \ddots & \\ \mathbf{0} & & & J_k(\lambda_k, m_k) \end{bmatrix},$$

where each J_i is a Jordan block of size m_i corresponding to eigen value λ_i (λ_i's and m_i's may not be all distinct), $i = 1,2,\ldots,k$ and $m_1+m_2+m_3+\ldots+m_k=n$.

Indeed, any diagonal matrix is in Jordan Canonical form where each Jordan block is of size 1.

Example:

- $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is JCF of matrix $A=\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$
  (eigen values of A are $-2,-2,4$) with $J_1=[-2]$, $J_2=[-2]$, $J_3=[4]$.

- $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ is JCF of matrix $A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & -1 & 4 \end{bmatrix}$
  (eigen values of A are 3,3,3,3) with $J_1 = J_2=\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

- $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is JCF of matrix $A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ -1 & 3 & 2 & 1 & 0 \\ 1 & 4 & 5 & 2 & 1 \end{bmatrix}$
  (eigen values of A are 1,1,1,1,1) with $J_1=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

- $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ is JCF of matrix $A=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
  (eigen values of A are $1 \pm i, 2 \pm i$) with $J_1=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $J_2=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

- $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ is JCF of matrix $A=\begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$
  (eigen values of A are $-3, 2 \pm i$) with $J_1=[-3]$, $J_2=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

The Jordan Canonical Form of a given nxn matrix A is unique except for the order of the elementaryJordan blocks.

III. PROCEDURE TO FIND JORDAN CANONICAL FORM OF A MATRIX A

A. For Real Eigen Values of A

Algebraic multiplicity of eigen value λ equals the number of times λ is repeated along the diagonal of J. Geometric multiplicity of λ equals the number of Jordan blocks in J with eigen value λ.

The order of the Jordan blocks in the matrix is not unique. Although, it is conventional to group blocks for the same eigen value together, but no ordering is imposed among the eigen values, nor among the blocks for a given eigen value, but the blocks for instance be ordered in descending manner of size for a particular eigen value.

Basically, in order to find JCF of A, we need to know the number and sizes of Jordan blocks corresponding to eigen values of A.

We need to carry out the following procedure for each eigen value λ of A.

Suppose λ is an eigen value of A with multiplicity r.

Let δ_j=dim Ker$(A-\lambda I)^j$=n-rank $(A-\lambda I)^j$

First, find δ_1.If δ_1=r good, otherwise find δ_2.If δ_2=r good, otherwise find δ_3 and so on. We need to continue the process till the kth step when δ_k=r.

Eventually, we get $\delta_1 \le \delta_2 \le \ldots \le \delta_k = r$.

The number k is the size of the largest Jordan block associated to λ and δ_1 is the total number of Jordan blocks associated to λ.

If we define $s_1=\delta_1$, $s_2=\delta_2-\delta_1$, $s_3=\delta_3-\delta_2$,…, $s_k=\delta_k-\delta_{k-1}$, then s_i is the number of Jordan blocks of size at least i by i associated to λ.

Finally put, $v_1=s_1-s_2=2\delta_1-\delta_2$

$v_2=s_2-s_3=2\delta_2-\delta_3-\delta_1$

…

$v_j=s_j-s_{j+1}=2\delta_j-\delta_{j+1}-\delta_{j-1}$ for 1<j<k

$v_k=s_k$ where v_i is the number of ixi Jordan blocks associated to λ.

This procedure has been illustrated through examples in the later part.

B. For Complex Eigen Values of A

In general, if a matrix has complex eigenvalues, it is not diagonalisable. Complex eigenvalues appear in pairs. If λ =a+ib is a complex eigenvalue of A, so as its conjugate $\bar{\lambda}$=a-ib.

We know that Jordan block corresponding to complex eigen value is always of size 2.

For finding the Jordan block, we need to pick one complex eigen value, suppose we pick λ =a+ib. Then, we separate the real and imaginary part, Re(a+ib) =a, Im(a+ib) =b.

Then, Jordan block corresponding to λ =a+ib is of the form $=\begin{bmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{bmatrix}=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

Algebraic multiplicity of λ (say a+ib) equals the number of times the Jordan block corresponding to λ is repeated along the diagonal of J.

IV. GENERALISED EIGEN VECTORS

Every square matrix A_{nxn} can be put in Jordan Canonical Form J by a similarity transformation

i.e. ∃ an invertible matrix P such that $P^{-1}AP=J$,where $J=\begin{bmatrix} J_1 & & \mathbf{0} & \\ & J_2 & & \\ \mathbf{0} & & \ddots & \\ & & & J_k \end{bmatrix}$ where each

J_r is an elementary Jordan block of sizes n_r,r=1,2,…k and $\sum_{r=1}^{k} n_r=n$.

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A. Motivation behind the concept

For nxn diagonalisable matrix A, we get n linearly independent eigenvectors of A to fill the columns of P such that $P^{-1}AP=D$ where D is a diagonal matrix (diagonal entries are the eigen values of A). But for non-diagonalisable matrix A, there is at least one eigen value with geometric multiplicity which is strictly less than its algebraic multiplicity. Thus, while writing P we are lacking by some column entries of P. In order to make up for the deficiency of eigen vectors, the definition of eigenvectors is generalised and we get the concept of generalised eigenvectors.

B. Definition

Let λ be an eigen value of matrix $A_{n\times n}$ of multiplicity $m \le n$. Then for r =1, 2…, m any nonzero solution v of $(A-\lambda I)^r v=0$ is called a generalised eigenvector of A.

If $(A-\lambda I)^r v=0$ and $(A-\lambda I)^{r-1} v\neq 0$ $(v\neq 0)$, then v is a generalised eigenvector of rank r. We note that a generalised eigenvector of rank 1 is an ordinary eigenvector associated with λ.

Basically, the definition of ordinary eigenvector is generalised to get the concept of generalised eigenvectors.

Ordinary eigen vectors are elements in Ker(A-λI) whereas generalised eigen vectors are elements in the kernel of some positive power of (A-λI).

If λ is an eigen value of A with algebraic multiplicity k, there are k linearly independent generalised eigen vectors for λ.

V. JORDAN CHAINS

While filling the columns of P by generalised eigen vectors, we will find that any set of linearly independent generalised eigen vectors will not do, but the set of linearly independent generalised eigen vectors are to be related by Jordan chains so that $P^{-1}AP=J$.

Definition: Given an eigen value λ, we say that $v_1, v_2,\dots, v_r$ form a Jordan chain of generalised eigen vectors of length r if $v_1 \neq 0$ and

$v_{r-1}=(A-\lambda I)\, v_r$

$v_{r-2}=(A-\lambda I)\, v_{r-1}$

□

$v_1=(A-\lambda I)\, v_2$

$0=(A-\lambda I)\, v_1$

Using these relations, we get that $(A-\lambda I)^{i-1}v_i=v_1$. Thus, $(A-\lambda I)^{i-1}v_i \neq 0$ (since $v_1 \neq 0$) and $(A-\lambda I)^i v_i = (A-\lambda I)v_1 = 0$. Therefore, the element v_i is a generalised eigenvector of rank i.

Formation of these Jordan chains of various lengths corresponding to Jordan blocks of different sizes have been clearly illustrated through various examples in the later part.

VI. PROCEDURE FOR FINDING P SUCH THAT P-1AP=J OR EQUIVALENTLY AP=PJ

A. For real eigen values of A

Here we will just give a basic idea to find P and formation of Jordan chains.

Here we will just give a basic idea to find P and formation of Jordan chains.

Now, $P^{-1}AP = J = \begin{bmatrix} J_1 & & \mathbf{0} \\ & J_2 & \\ \mathbf{0} & & \ddots & \\ & & & J_k \end{bmatrix}$, where each J_r is an elementary Jordan block of sizes n_r, r =1,2,…k and $\sum_{r=1}^{k} n_r = n$.

Since $P^{-1}AP = J$, we write this equation as AP=PJ.

Expressing $P = [P_1 \quad P_2 \dots P_q]$ where $P_{i_{n\times n_i}}$ are the columns of P associated with the ith Jordan block J_i of size n_i corresponding to eigen value λ_i.

We have $AP_i = P_iJ_i$. Let $P_i = [v_{i1} \quad v_{i2} \dots v_{in_i}]$.

Thus, we have, $Av_{i1} = \lambda_i v_{i1}$

For $j=2,\dots, n_i$, $Av_{ij} = v_{ij-1} + \lambda_i v_{ij}$

So, $v_{i1}, v_{i2},\dots, v_{in_i}$ forms a Jordan chain of length n_i.

By solving these set of equations, we will find $v_{i1}, v_{i2},\dots, v_{in_i}$ and thus P_i. Following the same method we can find $P_1, P_2, \dots P_q$ and thus find P.

B. For Complex Eigen Values of A

Complex eigenvalues appear in pairs. If λ =a+ib is a complex eigenvalue of A, so as its conjugate $\bar{\lambda}$=a-ib. Now, we proceed to find the corresponding eigen vectors. If v is an eigen vector associated with λ, then $\bar{v}$,the conjugate of v is the eigen vector associated with $\bar{\lambda}$,the conjugate of λ.

Then, we need to pick the eigen vector in correspondence with the eigen value chosen for finding Jordan block. Suppose we chose λ =a+ib for finding Jordan block, then we will pick v, eigen vector corresponding to λ for finding P.

We then write P=$[\mathrm{Im}(\mathbf{v}) \quad \mathrm{Re}(\mathbf{v})]$.

VII. LINEAR INDEPENDENCE OF GENERALISED EIGEN VECTORS

Statement: The vectors in a chain of generalised eigenvectors are linearly independent.

Proof: We consider the linear combination $\sum_{i=1}^{r} a_i v_i=0$ … (1)

From the definition of Jordan chain of length r, we get that $v_i=(A-\lambda I)^{r-i}v_r$

Putting the value of v_i in (1), equation (1) reduces to $\sum_{i=1}^{r} a_i (A-\lambda I)^{r-i}v_r=0$…(2)

We want to prove that all the a_i are equal to zero in order to prove the linear independence. We aregoing to use the fact that $(A-\lambda I)^m u=0$ for all $m\geq r$… (3)

Indeed, $(A-\lambda I)^m v_r=(A-\lambda I)^{m-r}(A-\lambda I)^r v_r=(A-\lambda I)^{m-r}(A-\lambda I)\, v_1=0$

Applying $(A-\lambda I)^{r-1}$ to (2), we get $\sum_{i=1}^{r} a_i(A-\lambda I)^{2r-i-1}v_r=0$... (4)
Since $(A-\lambda I)^{2r-i-1}v_r=0$ for $i\leq r-1$(from (3)), the equation (4) simplifies and we get $a_r(A-\lambda I)^{r-1}v_r=a_r v_1=0$.
Hence, $a_r=0$ because $v_1\neq 0$. Now, we know that $a_r=0$ so that (2) reduces to $\sum_{i=1}^{r-1} a_i (A-\lambda I)^{r-i}v_r=0$ … (5)

Applying $(A-\lambda I)^{r-2}$ to (5), we get $\sum_{i=1}^{r-1} a_i(A-\lambda I)^{2r-i-2}v_r=a_{r-1}(A-\lambda I)^{r-1}v_r=a_{r-1}v_1=0$.
because $(A-\lambda I)^{2r-i-2}v_r=0$ for $i\leq r-2$. Therefore, $a_{r-1}=0$.

We proceed recursively with the same argument and prove that all the a_i are equal to zero so thatthe vectors v_i are linearly independent.

This result confirms us with the fact that P is invertible i.e., P^{-1}exists. |

VIII. ILLUSTRATIONS THROUGH EXAMPLES FOR FINDING J AND P

A. Real Eigen Values

8.1.1 2X2 matrix

Example 1:

$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$

Eigen values of A are 1,1

- Finding the Jordan Canonical Form of A

$(A-I)=\begin{bmatrix} & \\ 0 & 0\end{bmatrix}$

Rank(A-I) =1

δ_1=dim Ker(A-I) =n-rank(A-I) =2-1=1(geometric multiplicity=1<algebraic multiplicity=2)

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$(A-I)^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\text{Rank}(A-I)^2=0$

$\delta_2=\dim \text{Ker}(A-I)^2=\text{n-rank}(A-I)^2=2\text{-}0=2$

$\nu_1=2\delta_1-\delta_2=2\text{-}2=0$

$\nu_2=\delta_2-\delta_1=2\text{-}1=1$

So, there will be 1 Jordan block of size 2 (say J_1) corresponding to $\lambda=1$

$\therefore$ JCF of A, $J=(J_1)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

- **Finding a matrix P such that $P^{-1}AP=J$ or equivalently $AP=PJ$**

Let $P=[v_1 \quad v_2]$ where $v_1,v_2 \in \mathbf{R}^2$

$AP=[Av_1 \quad Av_2]$, $PJ=[v_1 \quad v_2]\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=[v_1 \quad v_1+v_2]$

Since AP=PJ, we want to choose $v_1,v_2 \in \mathbf{R}^2$ such that $Av_1=v_1$, $Av_2=v_1+v_2$ where $v_1,v_2\neq 0$ and v_1,v_2 are linearly independent since P is invertible.

The equations can be written in the form: $(A\text{-}I)v_1=0$ ... (1)

$(A\text{-}I)v_2=v_1$... (2)

$\therefore$ v_1, v_2 form a Jordan chain of length 2.

Equation (1) implies $v_1 \in \text{Ker}(A\text{-}I)$

Equation (2) implies $v_2 \notin \text{Ker}(A\text{-}I)$, since $v_1 \neq 0$

Again, $(A-I)^2v_2=(A\text{-}I)v_1=0 \Rightarrow (A-I)^2v_2=0$

$\therefore v_2\in \text{Ker}(A-I)^2, v_2 \notin \text{Ker}(A\text{-}I)$

Finding $\text{Ker}(A-I)^2$

Since $(A-I)^2=\mathbf{0}$, therefore, $\text{Ker}(A-I)^2=\text{span}\left\{\begin{pmatrix}1\\ 0\end{pmatrix},\begin{pmatrix}0\\ 1\end{pmatrix}\right\}$

Finding Ker(A-I) Let

$v \in \text{ker}(A\text{-}I)$

$\therefore (A\text{-}I)v=0$ where $v\neq 0, v=\begin{pmatrix}a\\ b\end{pmatrix}$

$\Rightarrow \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow 2b=0 \Rightarrow b=0$

$\therefore v=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}a\\ 0\end{pmatrix}=a\begin{pmatrix}1\\ 0\end{pmatrix}$

So, $\text{Ker}(A\text{-}I)=\text{span}\left\{\begin{pmatrix}1\\ 0\end{pmatrix}\right\}$

Now, since $v_2 \in \text{Ker}(A-I)^2, v_2 \notin \text{Ker}(A\text{-}I)$, we consider $v_2=\begin{pmatrix}0\\ 1\end{pmatrix}$

Putting the value of v_2 in equation (2), $v_1=(A\text{-}I)v_2=\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}2\\ 0\end{pmatrix}$

Also, we find that $v_1 \in \text{Ker}(A\text{-}I)$ as per our condition.

$\therefore v_1=\begin{pmatrix}2\\ 0\end{pmatrix}$, $v_2=\begin{pmatrix}0\\ 1\end{pmatrix}$. v_1 and v_2 are linearly independent as required. v_1 is the ordinary eigen vector (generalised eigen vector of rank 1)and v_2 is the generalised eigen vector of rank 2 corresponding to $\lambda=1$. Thus, there are 2 linearly independent generalised eigen vectors corresponding to $\lambda=1$.

Thus, our desired matrix $P=[v_1 \quad v_2]=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.

8.1.2 3X3 matrix

Example 1:

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Eigen values of A are 1,1,3

- **Finding the Jordan Canonical Form of A**

$$(A-I)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Rank(A-I) =2

δ_1=dim Ker(A-I) =n-rank(A-I) =3-2 =1 (geometric multiplicity of λ=1 is 1 < algebraic multiplicity of λ=1 is 2)

$$(A-I)^2=\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

Rank$(A-I)^2$=1

δ_2=dim Ker$(A-I)^2$=n-rank$(A-I)^2$=3-1=2

$\nu_1=2\delta_1-\delta_2$=2-2=0
$\nu_2=\delta_2-\delta_1$=2-1=1

So, there will be 1 Jordan block of size 2 (say J_1)corresponding to λ=1 and there will be 1 Jordan block of size 1(say J_2) corresponding to λ=3(since algebraic multiplicity of λ=3 is 1, no need to go by computational procedure)

$\therefore$JCF of A, $J=\begin{bmatrix} J_1 & \mathbf{0} \\ & 2 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ where $J_1=\begin{bmatrix} \cdot & \cdot \\ 0 & 1 \end{bmatrix}$, $J_2=[3]$

- **Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ**

Let $P=[v_1 \quad v_2 \quad v_3]$ where $v_1,v_2,v_3 \in \mathbf{R}^3$

$$AP=[Av_1 \quad Av_2 \quad Av_3],\ PJ=[v_1 \quad v_2 \quad v_3]\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}=[v_1 \quad v_1+v_2 \quad 3v_3]$$

Since, AP=PJ, we want to choose $v_1,v_2,v_3 \in \mathbf{R}^3$, such that $Av_1=v_1$, $Av_2=v_1+v_2$, $Av_3=3v_3$ where $v_1,v_2,v_3 \neq 0$ and v_1,v_2,v_3 are linearly independent since P is invertible.

The equations can be written in the form: $(A-I)v_1=0$ ... (1)
$(A-I)v_2=v_1$ ... (2)
$(A-3I)v_3=0$ ... (3)

$\therefore$ v_1, v_2 form a Jordan chain of length 2 corresponding to block J_1 and v_3 form a Jordanchain of length corresponding to block J_2.

Equation (1) implies $v_1 \in Ker(A-I)$

Equation (2) implies $v_2 \notin Ker(A-I)$, since $v_1 \neq 0$Equation (3) implies $v_3 \in Ker(A-3I)$

Now, $(A-I)^2 v_2 = (A-I)v_1 = 0$

$\therefore v_2 \in Ker(A-I)^2, v_2 \notin Ker(A-I)$

<u>Finding Ker(A-I)</u>

Let $v \in Ker(A-I)$

$\therefore (A-I)v = 0$ where $v \neq 0, v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ 2c \\ 3c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b = 0, c = 0$$

$$\therefore v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So, $Ker(A-I) = span \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

<u>Finding $Ker(A-I)^2$</u>

Let $v \in Ker(A-I)^2$

$\therefore (A-I)^2 v = 0$ where $v \neq 0$, $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2c \\ 4c \\ 4c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c = 0$$

$$\therefore v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So, $Ker\ (A-I)^2 = span \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Now, since $v_2 \in Ker(A-I)^2, v_2 \notin Ker\ (A-I)$, we consider $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Putting the value of v_2 in equation(2), $v_1 = (A-I)v_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Also, we find that $v_1 \in Ker(A-I)$ as per our condition.

Finding Ker(A-3I)

Let $v \in$ Ker(A-3I)

$\therefore (A-3I)v=0$ where $v \neq 0, v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow -2a+b=0,\ -2b+2c=0 \Rightarrow 2a=b=c$

$\therefore v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} a \\ 2a \\ 2a \end{pmatrix}=a\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

So, Ker(A-3I) $=$ span $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Since $v_3 \in$ Ker (A-3I), we consider $v_3=\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\therefore v_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ v_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ v_3=\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. v_1, v_2, v_3 are linearly independent as required.

v_1 is the ordinary eigen vector(generalised eigen vector of rank 1) and v_2 is the generalisedeigen vector of rank 2 corresponding to $\lambda=1$. Thus, there are 2 linearly independent generalised eigen vectors corresponding to $\lambda=1$.

v_3 is the ordinary eigen vector corresponding to $\lambda=3$.

Thus, our desired matrix $P=[v_1 \quad v_2 \quad v_3]=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 1 & \frac{-3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}=J$

Example 2:

$A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

Eigen values of A are 2,2,2

- Finding the Jordan Canonical Form of A

$(A-2I)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

Rank(A-2I)=1

$\delta_1=$ dim Ker(A-2I) = n - rank(A-2I) = 3-1 = 2 (geometric multiplicity = 2 < algebraicmultiplicity = 3)

$\nu_1=2\delta_1-\delta_2=4-3=1$
$\nu_2=\delta_2-\delta_1=3-2=1$

So, there will be 1 Jordan block of size 2 (say J_1)and 1 Jordan block of size 1 (say J_2)corresponding to $\lambda=2$.

$\therefore$ JCF of A, $J=\begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & 2 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ where $J_1=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $J_2=[2]$

- **Finding a matrix P such that $P^{-1}AP=J$ or equivalently $AP=PJ$**

Let $P=[v_1 \quad v_2 \quad v_3]$ where $v_1, v_2, v_3 \in \mathbf{R}^3$

$AP=[v_1 \quad v_2 \quad v_3]$, $PJ=[v_1 \quad v_2 \quad v_3]\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}=[2v_1 \quad v_1+2v_2 \quad 2v_3]$

Since, AP=PJ, we want to choose $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $Av_1=2v_1$, $Av_2=v_1+2v_2$, $Av_3=2v_3$ where $v_1, v_2, v_3 \neq 0$ and v_1, v_2, v_3 are linearly independent since P is invertible.

The equations can be written in the form: $(A-2I)v_1=0$ ... (1)
$(A-2I)v_2=v_1$ ... (2)
$(A-2I)v_3=0$ ... (3)

$\therefore$ v_1, v_2 forms a Jordan chain of length 2 corresponding to block J_1 and v_3 form a Jordanchain of length 1 corresponding to block J_2.

Equation (1) implies $v_1 \in Ker(A-2I)$
Equation (2) implies $v_2 \notin Ker(A-2I)$, since $v_1 \neq 0$Equation
(3) implies $v_3 \in Ker(A-2I)$

Now, $(A-2I)^2v_2=(A-2I)v_1=0$
$\therefore v_2 \in Ker(A-2I)^2$, $v_2 \notin Ker(A-2I)$

Finding $Ker(A-2I)^2$

Since $(A-2I)^2=\mathbf{0}$, therefore, $Ker(A-I)^2=span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

Finding Ker(A-2I) Let
$v \in Ker(A-2I)$

$\therefore (A-2I)\ v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}b\\0\\-b\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow b=0$

$\therefore v=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\0\\c\end{pmatrix}=a\begin{pmatrix}1\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$

So, $Ker(A-2I)=span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

Now, since $v_2 \in Ker(A-2I)^2$, $v_2 \notin Ker\ (A-2I)$, we consider $v_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$

Putting the value of v_2 in equation(2), $v_1=(A-2I)v_2=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Also, we find that $v_1 \in Ker(A-2I)$ as per our condition.

Since, $v_3 \in Ker(A-2I)$, we consider $v_3=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\therefore v_1=\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $v_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $v_3=\begin{pmatrix}1\\0\\0\end{pmatrix}$. v_1, v_2, v_3 are linearly independent as required.

v_1, v_3 are the ordinary eigen vectors (generalised eigen vectors of rank 1) and v_2 is the generalised eigen vector of rank 2 corresponding to $\lambda=2$.Thus, there are 3 linearly independent generalised eigen vectors corresponding to $\lambda=2$

Thus, our desired matrix $P=[v_1 \quad v_2 \quad v_3]=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$=\begin{bmatrix} 0 & 1 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}=J$

Example 3:

$A=\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

Eigen values of A are 2,2,2

- Finding the Jordan Canonical form of A

$(A-2I)=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{bmatrix}$

Rank(A-2I)=2

δ_1=dim Ker(A-2I) = n - rank(A-2I) = 3-2 = 1 (geometric multiplicity = 1 < algebraic multiplicity=3)

$(A-2I)^2=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$

Rank$(A-2I)^2$= 1

δ_2=dim Ker$(A-2I)^2$=n-rank$(A-2I)^2$=3-1=2

$(A-2I)^3=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rank$(A-2I)^3$= 0

δ_3=dim Ker$(A-2I)^3$=n-rank$(A-2I)^3$=3-0=3

$\nu_1=2\delta_1-\delta_2=2-2=0$
$\nu_2=2\delta_2-\delta_3-\delta_1=4-3-1=0$
$\nu_3=\delta_3-\delta_2=3-2=1$

So, there will be 1 Jordan block of size 3 (say J_1) corresponding to λ=2.

$\therefore$ JCF of A, $J=[J_1]=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ

Let $P=[v_1 \quad v_2 \quad v_3]$ where $v_1, v_2, v_3 \in \mathbb{R}^3$

$$AP=[Av_1 \quad Av_2 \quad Av_3],\ PJ=[v_1 \quad v_2 \quad v_3]\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}=[2v_1 \quad v_1+2v_2 \quad v_3+2v_3]$$

Since, AP=PJ, we want to choose $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $Av_1=2v_1, Av_2=v_1+2v_2, Av_3=v_3+2v_3$ where $v_1, v_2, v_3 \neq 0$ and v_1, v_2, v_3 are linearly independent since P is invertible.

The equations can be written in the form: $(A-2I)v_1=0$ ... (1)

$(A-2I)v_2=v_1$ ... (2)

$(A-2I)v_3=v_2$ ... (3)

$\therefore$ v_1, v_2, v_3 forms a Jordan chain of length 3 corresponding to $J_1(=J)$ Equation (1) implies $v_1 \in Ker(A-2I)$

Equation (2) implies $v_2 \notin Ker(A-2I)$, since $v_1 \neq 0$ Equation (3) implies $v_3 \notin Ker(A-2I)$, since $v_2 \neq 0$

Now, $(A-2I)^2 v_2=(A-2I)v_1=0$

$\therefore v_2 \in Ker(A-2I)^2, v_2 \notin Ker(A-2I)$

Again, $(A-2I)^2 v_3=(A-2I)v_2=v_1 \Rightarrow (A-2I)^3 v_3=(A-2I)v_1=0$

$\therefore v_3 \in Ker(A-2I)^3, v_3 \notin Ker(A-2I)^2,\ v_3 \notin Ker(A-2I)$

<u>Finding $Ker(A-2I)^3$</u>

Since $(A-2I)^3=\mathbf{0}$, therefore, $Ker(A-I)^3=span \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Finding $Ker(A-2I)^2$

Let $v \in Ker(A-2I)^2$

$\therefore (A-2I)^2 v=0$ where $v \neq 0$, $v=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0\\0\\5a\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow a=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\b\\c\end{pmatrix}=b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

So, $Ker(A-2I)^2=span \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

<u>Finding $Ker(A-2I)$</u> Let

$v \in Ker(A-2I)$

$\therefore (A-2I)v=0$ where $v \neq 0, v=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}0\\a\\-3a+5b\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow a=0, b=0$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}=c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \operatorname{Ker}(A-2I)=\operatorname{span}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, since $v_3 \in \operatorname{Ker}(A-2I)^3, v_3 \notin \operatorname{Ker}(A-2I)^2, v_3 \notin \operatorname{Ker}(A-2I)$, we consider

$$v_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Putting the value of v_3 in equation(3), $v_2=(A-2I)v_3=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$

Also, we find that $v_2 \in \operatorname{Ker}(A-2I)^2$, $v_2 \notin \operatorname{Ker}(A-2I)$ as per our condition.

Putting the value of v_2 in equation(2), $v_1=(A-2I)v_2=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

Also, we find that $v_1 \in \operatorname{Ker}(A-2I)$ as per our condition.

$\therefore v_1=\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$, $v_2=\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$, $v_3=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. v_1, v_2, v_3 are linearly independent as required.

v_1 is the ordinary eigen vector(generalised eigen vector of rank 1), v_2 is the generalised eigen vector of rank 2 and v_3 is the generalised eigen vector of rank 3 corresponding to $\lambda=2$. Thus, there are 3 linearly independent generalised eigen vectors corresponding to $\lambda=2$.

Thus, our desired matrix $P=[v_1 \quad v_2 \quad v_3]=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 0 \end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix} 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 0 \end{bmatrix}$

$$=\begin{bmatrix} 0 & \frac{11}{5} & \frac{2}{5} \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 0 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}=J$$

8.1.3 4X4 matrix

Example 1:

$$A=\begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & -1 & 4 \end{bmatrix}$$

Eigen values of A are 3,3,3,3

- Finding the Jordan Canonical Form of A

$$(A-3I) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Rank(A-3I)=2

δ_1 =dim Ker(A-3I) =n-rank(A-3I) =4-2 =2 (geometric multiplicity =2 < algebraic multiplicity=4)

$$(A-3I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(A-3I)^2$=0

δ_2=dim Ker$(A-2I)^2$=4-rank$(A-2I)^2$=4-0=4

$\nu_1=2\delta_1-\delta_2=4-4=0$

$\nu_2=\delta_2-\delta_1=4-2=2$

So, there will be 2 Jordan blocks of size 2(say $J_1$and J_2) corresponding to $\lambda=3$.

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ where } \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ Let P=[v_1 v_2 v_3 v_4] where $v_1,v_2,v_3,v_4 \in \mathbf{R}^4$

AP=[Av_1 Av_2 Av_3 Av_4]

$$PJ=[v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}=[v_1 \quad v_1+3v_2 \quad 3v_3 \quad v_3+3v_4]$$

Since, AP=PJ, we want to choose $v_1, v_2, v_3, v_4 \in \mathbf{R}^4$ such that $Av_1=3v_1$,$Av_2=v_1+3v_2$,$Av_3=3v_3$,$Av_4=v_3+3v_4$where $v_1, v_2, v_3, v_4 \neq 0$ and v_1, v_2, v_3, v_4 are linearly independent since P is invertible.

The equations can be written in the form: $(A-3I)v_1=0$ ... (1)

$(A-3I)v_2=v_1$ ... (2)

$(A-3I)v_3=0$ ... (3)

$(A-3I)v_4=v_3$ ... (4)

$\therefore$ v_1, v_2 forms a Jordan chain of length 2 corresponding to J_1 and also v_3, v_4 forms a Jordan chain of length 2 corresponding to J_2.

Equation (1) implies $v_1 \in$ Ker(A-3I)

Equation (2) implies $v_2 \notin$ Ker(A-3I), since $v_1 \neq 0$ Equation (3) implies $v_3 \in$ Ker(A-3I)

Equation (4) implies $v_4 \notin Ker(A-3I)$, since $v_3 \neq 0$ Now, $(A-3I)^2 v_2 = (A-3I)v_1 = 0$

$\therefore v_2 \in Ker(A-3I)^2, v_2 \notin Ker(A-3I)$

Similarly, $(A-3I)^2 v_4 = (A-3I)v_3 = 0$

$\therefore v_4 \in Ker(A-3I)^2, v_4 \notin Ker(A-3I)$

Finding $Ker(A-3I)^2$

Since $(A-3I)^2 = \mathbf{0}$, therefore, $Ker(A-3I)^2 = span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$

Finding Ker(A-3I)

Let $v \in Ker(A-3I)$

$\therefore (A-3I)v=0$ where $v \neq 0, v=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}-1 & 1 & 0 & 0\\-1 & 1 & 0 & 0\\-1 & 1 & -1 & 1\\-1 & 1 & -1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}-a+b\\-a+b\\-a+b-c+d\\-a+b-c+d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$\Rightarrow -a+b=0,\ -a+b-c+d=0 \Rightarrow a=b,\ c=d$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a\\a\\c\\c\end{pmatrix}=a\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

So, $Ker(A-3I) = span\left(\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right)$

Now, since $v_2, v_4 \in Ker(A-3I)^2$ and $v_2, v_4 \notin Ker(A-3I)$, we consider $v_2=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $v_4=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

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Putting the value of v_2 in equation(2), $v_1=(A-3I)v_2=$ $\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

Putting the value of v_4 in equation(4), $v_3=(A-3I)v_4=$ $\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}$

Also, we find that $v_1, v_3 \in Ker(A-3I)$ as per our condition.

$\therefore v_1=\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, v_2=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_3=\begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, v_4=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$

v_1, v_2, v_3, v_4 are linearly independent as required.

v_1, v_3 are the ordinary eigen vectors(generalised eigen vectors of rank 1), v_2, v_4 are the generalised eigen vectors of rank 2 corresponding to $\lambda=3$.Thus, there are 4 linearly independent generalised eigen vectors corresponding to $\lambda=3$.

Our desired matrix, $P=[v_1 \quad v_2 \quad v_3 \quad v_4]=\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ -1 & 1 & 2 & 1 \\ -1 & 1 & -1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$

$=\begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}=\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}=J$

Example 2:

$A=\begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Eigen values of A are 1,1,1,1.

- Finding the Jordan Canonical Form of A

$(A-I)=\begin{bmatrix} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank (A-3I) =2

δ_1=dimKer(A-3I) =n-rank(A-3I) =4-2=2 (geometric multiplicity=2<algebraic multiplicity=4)

$$(A-I)^2=\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(A-I)^2$=1

δ_2=dim Ker$(A-I)^2$=4-rank$(A-I)^2$=4-1=3

$$(A-I)^3=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(A-I)^3$=0

δ_3=dim Ker$(A-I)^3$=4-rank$(A-I)^3$=4-0=4

$\nu_1=2\delta_1-\delta_2$=4-3=1

$\nu_2=2\delta_2-\delta_3-\delta_1$=6-4-2=0

$\nu_3=\delta_3-\delta_2$=4-3=1

So, there will be 1 Jordan block of size 3 (say J_1) and 1 Jordan block of size 1 correspondingto λ=1(say J_2)

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ where } J_1=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } J_2=[1]$$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ

Let P=[v_1 v_2 v_3 v_4] where $v_1,v_2,v_3,v_4 \in \mathbf{R}^4$

AP=[Av_1 Av_2 Av_3 Av_4]

$$PJ=[v_1 \; v_2 \; v_3 \; v_4]\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=[v_1 \;\; v_1+v_2 \;\; v_2+v_3 \;\; v_4]$$

Since, AP=PJ, we want to choose $v_1, v_2, v_3, v_4 \in \mathbf{R}^4$ such that $Av_1=v_1$, $Av_2=v_1+v_2$, $Av_3=v_2+v_3$, $Av_4=v_4$ where $v_1, v_2, v_3, v_4 \neq 0$ and v_1, v_2, v_3, v_4 are linearly independent since P is invertible.

The equations can be written in the form: (A-I)v_1=0 ... (1)

(A-I)v_2=v_1 ... (2)

(A-I)v_3=v_2 ... (3)

(A-I)v_4=0 ... (4)

$\therefore$ v_1, v_2, v_3 form a Jordan chain of length 3 corresponding to block J_1 and v_4 forms a Jordanchain of length 1 corresponding to block J_2.

Equation (1) implies $v_1 \in$ Ker(A-I)

Equation (2) implies $v_2 \notin Ker(A-I)$, since $v_1 \neq 0$ Equation (3) implies $v_3 \notin Ker(A-I)$, since $v_2 \neq 0$ Equation (4) implies $v_4 \in Ker(A-I)$

Now, $(A - I)^2 v_2 = (A-I) v_1 = 0$
$\therefore v_2 \in Ker(A - I)^2, v_2 \notin Ker(A-I)$

Again, $(A - I)^2 v_3 = (A-I) v_2 = v_1 \Rightarrow (A - I)^3 v_3 = (A-I) v_1 = 0$
$\therefore v_3 \in Ker(A - I)^3, v_3 \notin Ker(A - I)^2, v_3 \notin Ker(A-I)$

<u>Finding $Ker(A - I)^3$</u>

Since $(A - I)^3 = 0$, $Ker(A - I)^3 = span\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$

<u>Finding $Ker(A - I)^2$</u>

Let $v \in Ker(A - I)^2$

$\therefore (A - I)^2 v = 0$ where $v \neq 0$, $v = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}-1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}-a-b-c-d\\ 0\\ a+b+c+d\\ 0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$\Rightarrow a+b+c+d = 0 \Rightarrow a = -b-c-d$

$$\therefore v = \begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}-b-c-d\\ b\\ c\\ d\end{pmatrix} = b\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} + c\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + d\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$$

So, $Ker(A - I)^2 = span\left(\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right)$

<u>Finding $Ker(A-I)$</u>

Let $v \in Ker(A-I)$

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$\therefore (A-I)v=0$ where $v\neq 0, v=\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -1 & -2 & -1 & -1\\ 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -a-2b-c-d\\ a+b+c+d\\ b\\ 0\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\end{pmatrix}$$

$\Rightarrow -a-2b-c-d=0,\ a+b+c+d=0,\ b=0 \Rightarrow b=0,\ a+c+d=0 \Rightarrow a=-c-d,\ b=0$

$$\therefore v=\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix} -c-d\\ 0\\ c\\ d\end{pmatrix}=c\begin{pmatrix} -1\\ 0\\ 1\\ 0\end{pmatrix}+d\begin{pmatrix} -1\\ 0\\ 0\\ 1\end{pmatrix}$$

$$Ker(A-I)=span\left\{\begin{pmatrix} -1\\ 0\\ 1\\ 0\end{pmatrix},\begin{pmatrix} -1\\ 0\\ 0\\ 1\end{pmatrix}\right\}$$

Now, since, $v_3 \in Ker(A-I)^3$, $v_3 \notin Ker(A-I)^2$, $v_3 \notin Ker(A-2I)$, we consider $v_3 = \begin{pmatrix} 0 \end{pmatrix}$

Putting the value of v_3 in equation (3),

$$v_2=(A-I)v_3=\begin{pmatrix} -1 & -2 & -1 & -1\\ 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} 1\\ 0\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} -1\\ 1\\ 0\\ 0\end{pmatrix}$$

Also, we find that $v_2 \in Ker(A-I)^2$, $v_2 \notin Ker(A-I)$ as per our condition.

Putting the value of v_2 in equation (3),

$$v_1=(A-I)v_2=\begin{pmatrix} -1 & -2 & -1 & -1\\ 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} -1\\ 1\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} -1\\ 0\\ 1\\ 0\end{pmatrix}$$

Also, we find that $v_1 \in Ker(A-I)$ as per our condition.

Again, since, $v_4 \in Ker(A-I)$, we consider $v_4=\begin{pmatrix} -1\\ 0\\ 0\\ 1\end{pmatrix}$

$\therefore v_1=\begin{pmatrix} -1\\ 0\\ 1\\ 0\end{pmatrix}, v_2=\begin{pmatrix} -1\\ 1\\ 0\\ 0\end{pmatrix}, v_3=\begin{pmatrix} 1\\ 0\\ 0\\ 0\end{pmatrix}, v_4=\begin{pmatrix} -1\\ 0\\ 0\\ 1\end{pmatrix}$. v_1, v_2, v_3, v_4 are linearly independent as required.

v_1, v_4 are the ordinary eigen vectors(generalised eigen vectors of rank 1), v_2 is the generalised eigen vector of rank 2 and v_3 is the generalised eigen vector of rank 3 corresponding to λ=1.Thus, there are 4 linearly independent generalised eigen vectors corresponding to λ=1.

Our desired matrix, P= [v_1 v_2 v_3 v_4]=[

−11 −01 0 −01
0 0 0 0

Check: $P^{-1}AP$=[

$$\begin{matrix} 0 & 0 & 1 & 0 \\ & & & \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix} \quad \begin{matrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \quad \begin{matrix} -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = J$$

Example 3:

$$\begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Eigen values of A are 5,5,5,5

- Finding the Jordan Canonical Form of A

$$(A-5I) = \begin{bmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A-5I)=3

δ_1=dim Ker(A-5I) =n-rank(A-5I) =4-3=1(geometric multiplicity=1<algebraic multiplicity=4)

$$(A-5I)^2 = \begin{bmatrix} 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(A-5I)^2$=2

δ_2=dim Ker$(A - I)^2$=4-rank$(A-5I)^2$=4-2=2

$$(A-I)^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(A-I)^3$=1

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δ_3=dim Ker$(A-I)^3$=4-rank$(A-I)^3$=4-1=3

$$(A-I)^4=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank$(A-I)^4$=0
δ_4=dim Ker$(A-I)^3$=4-rank$(A-I)^4$=4-0=4

$\nu_1=2\delta_1-\delta_2$=2-2=0
$\nu_2=2\delta_2-\delta_3-\delta_1$=4-3-1=0
$\nu_3=2\delta_3-\delta_4-\delta_2$=6-4-2=0
$\nu_4=\delta_4-\delta_3$=4-3=1
So, there will be 1 Jordan block of size 4(say J_1)corresponding to λ=5.

$$\therefore \text{JCF of A, } J=[J_1]=\begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ Let P=[v_1 v_2 v_3 v_4] where $v_1,v_2,v_3,v_4 \in \mathbf{R}^4$

AP=[Av_1 Av_2 Av_3 Av_4]

$$PJ=[v_1 \quad v_2 \quad v_3 \quad v_4]\begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}=[5v_1 \quad v_1+5v_2 \quad v_2+5v_3 \quad v_3+5v_4]$$

Since, AP=PJ, we want to choose $v_1, v_2, v_3, v_4 \in \mathbf{R}^4$ such that $Av_1=5v_1$, $Av_2=v_1+5v_2$, $Av_3=v_2+5v_3$, $Av_4=v_3+5v_4$ where $v_1, v_2, v_3, v_4 \neq 0$ and v_1, v_2, v_3, v_4 are linearly independent since P is invertible.
The equations can be written in the form: $(A-5I)v_1=0$ ... (1)
$(A-5I)v_2=v_1$ ... (2)
$(A-5I)v_3=v_2$ ... (3)
$(A-5I)v_4=v_3$... (4)

$\therefore$ v_1, v_2, v_3, v_4 forms a Jordan chain of length 4 corresponding to $J_1(=J)$ Equation (1)

implies $v_1 \in$ Ker(A-5I)
Equation (2) implies $v_2 \notin$ Ker(A-5I), since $v_1 \neq 0$ Equation (3) implies $v_3 \notin$ Ker(A-5I), since $v_2 \neq 0$ Equation (4) implies $v_4 \notin$ Ker(A-5I), since $v_3 \neq 0$

Now, $(A-5I)^2v_2=(A-5I)v_1=0$
$\therefore v_2 \in \text{Ker}(A-5I)^2, v_2 \notin \text{Ker}(A-5I)$

Again, $(A-5I)^2v_3=(A-5I)v_2=v_1 \Rightarrow (A-5I)^3v_3=(A-5I)v_1=0$
$\therefore v_3 \in \text{Ker}(A-5I)^3, v_3 \notin \text{Ker}(A-5I)^2, v_3 \notin \text{Ker}(A-5I)$

Again, $(A-5I)^2v_4=(A-5I)v_3=v_2 \Rightarrow (A-5I)^3v_4=(A-5I)v_2=v_1 \Rightarrow (A-5I)^4v_3=(A-5I)v_1=0$

$\therefore v_4 \in Ker(A-5I)^4, v_4 \notin Ker(A-5I)^3, v_4 \notin Ker(A-5I)^2, v_4 \notin Ker(A-5I)$

Finding $Ker(A-5I)^4$

Since $(A-5I)^4=0$, $Ker(A-5I)^4=span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$

Finding $Ker(A-5I)^3$

Let $v \in Ker(A-5I)^3$

$\therefore (A-I)^3v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&0&6\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}6d\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow d=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a\\b\\c\\0\end{pmatrix}=a\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

So, $Ker(A-5I)^3=span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$

Finding $Ker(A-5I)^2$

Let $v \in Ker(A-5I)^2$

$\therefore (A-5I)^2v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&2&-4\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}2c-4d\\6d\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow 2c-4d=0, 6d=0 \Rightarrow c=0,\ d=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a\\b\\0\\0\end{pmatrix}=a\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

So, $Ker(A-5I)^2=span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$

Finding Ker(A-5I)

Let $v \in \text{Ker}(A-5I)$

$\therefore (A-5I)v=0$ where $v \neq 0, v=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b-2c+4d \\ 2c+2d \\ 3d \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow b=0,\ c=0,\ d=0$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $\text{Ker}(A-5I) = \text{span}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, since, $v_4 \in \text{Ker}(A-5I)^4, v_4 \notin \text{Ker}(A-5I)^3,\ v_4 \notin \text{Ker}(A-5I)^2,\ v_4 \notin \text{Ker}(A-5I)$,

we consider $v_4=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Putting the value of v_4 in equation(4), $v_3=(A-5I)v_4=\begin{pmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 4 \\ 2 \\ 3 \\ 0 \end{pmatrix}$

Also, we find that $v_3 \in \text{Ker}(A-5I)^3,\ v_3 \notin \text{Ker}(A-5I)^2,\ v_3 \notin \text{Ker}(A-5I)$ as per our condition.

Putting the value of v_3 in equation(3), $v_2=(A-5I)v_3=\begin{pmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \\ 3 \\ 0 \end{pmatrix}=\begin{pmatrix} -4 \\ 6 \\ 0 \\ 0 \end{pmatrix}$

Also, we find that $v_2 \in \text{Ker}(A-5I)^2, v_2 \notin \text{Ker}(A-5I)$ as per our condition.

Putting the value of v_2 in equation(2), $v_1=(A-5I)v_2=\begin{pmatrix} 0 & 1 & -2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -4 \\ 6 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Also, we find that $v_1 \in \text{Ker}(A-5I)$ as per our condition.

$\therefore v_1=\begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ v_2=\begin{pmatrix} -4 \\ 6 \\ 0 \\ 0 \end{pmatrix},\ v_3=\begin{pmatrix} 4 \\ 2 \\ 3 \\ 0 \end{pmatrix},\ v_4=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. v_1, v_2, v_3, v_4 are linearly independent as required.

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v_1 is the ordinary eigen vector(generalised eigen vector of rank 1), v_2 is the generalised eigen vector of rank 2 and v_3 is the generalised eigen vector of rank 3, v_4 is the generalised eigen vector of rank 4 corresponding to λ=5.Thus, there are 4 linearly independent generalised eigen vectors corresponding to λ=5.

Our desired matrix, $P=[v_1 \quad v_2 \quad v_3 \quad v_4]=\begin{bmatrix} 6 & -4 & 4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix} \frac{1}{6} & \frac{1}{9} & \frac{-8}{27} & 0 \\ 0 & \frac{1}{6} & \frac{-1}{9} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 6 & -4 & 4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$=\begin{bmatrix} \frac{5}{6} & \frac{13}{18} & \frac{-43}{27} & 0 \\ 0 & \frac{5}{6} & \frac{-2}{9} & 0 \\ 0 & 0 & \frac{5}{3} & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 6 & -4 & 4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}=J$

8.1.4 5 X 5 matrix

Example 1:

$$A=\begin{bmatrix} 2 & 5 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Eigen values of A are 2,2, -1, -1, -1

- Finding the Jordan Canonical Form of A

- Finding the Jordan block corresponding to λ=2(J_1)

$$(A-2I)=\begin{bmatrix} 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

Rank(A-2I) =4

δ_1=dim Ker(A-2I) =n-rank(A-2I) =5-4=1(geometric multiplicity=1<algebraic multiplicity=2)

$$(A-2I)^2=\begin{bmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Rank$(A-2I)^2$=3

δ_2=dim Ker$(A-2I)^2$=n-rank$(A-2I)^2$=5-3=2

$v_1=2\delta_1-\delta_2=2\text{-}2=0$
$v_2=\delta_2-\delta_1=2\text{-}1=1$
So,there will be 1 Jordan block of size 2 corresponding to $\lambda=2$.
$\therefore J_1=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

- Finding the Jordan block corresponding to λ= -1

$$(A+I)=\begin{bmatrix} 3 & 5 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(A+I) =2
δ_1=dim Ker(A+I) =n-rank(A+I) =5-2=3(geometric multiplicity=algebraic multiplicity)

So,there will be 3 Jordan blocks of size 1 corresponding to λ=-1(say J_2, J_3, J_4)
$J_2=J_3=J_4=[-1]$

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & J_3 & \\ 0 & & & J_4 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ where } J_1=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\ J_2=J_3=J_4=[-1]$$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ

Let $P=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5]$ where $v_1,v_2,v_3,v_4,v_5 \in \mathbf{R}^5$

$AP=[Av_1 \quad Av_2 \quad Av_3 \quad Av_4 \quad Av_5]$

$$PJ=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5]\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}=[2v_1 \quad v_1+2v_2 \quad -v_3 \quad -v_4 \quad -v_5]$$

Since, AP=PJ, we want to choose $v_1, v_2, v_3, v_4, v_5 \in \boldsymbol{R}^5$ such that $Av_1=2v_1$,$Av_2=v_1+2v_2$,$Av_3=-v_3$,$Av_4=-v_4$, $Av_5=-v_5$ where $v_1, v_2, v_3, v_4, v_5 \neq 0$ and v_1, v_2, v_3, v_4 ,v_5 are linearly independent since P is invertible.
The equations can be written in the form: $(A-2I)v_1=0$ ... . (1)
$(A-2I)v_2=v_1$ ...... (2)
$(A+I)v_3=0$ ....(3)
$(A+I)v_4=0$.............(4)
$(A+I)v_5=0$.............(5)

$\therefore$ v_1, v_2 form a Jordan chain of length 2 corresponding to J_1. v_3, v_4, v_5 form 3 Jordan chains each of length 1 corresponding to J_2, J_3, J_4 respectively.
Equation (1) implies $v_1 \in$ Ker(A-2I)
Equation (2) implies $v_2 \notin$ Ker(A-2I), since $v_1 \neq 0$
Equations (3), (4), (5) imply $v_3, v_4, v_5 \in$ Ker(A+I) respectively.

Finding Ker(A-2I)

Let v ∈ Ker(A-2I)

$\therefore$ (A-2I) v=0 where v≠0, $v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5b+e \\ 0 \\ -3c \\ -3d \\ -3e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b=0,\ c=0,\ d=0,\ e=0$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}=\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, Ker(A-2I) =span $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Finding Ker$(A-2I)^2$

Let v ∈ Ker$(A-2I)^2$

$\therefore (A-2I)^2 v=0$ where v≠0, $v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3e \\ 0 \\ 9c \\ 9d \\ 9e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c=0,\ d=0,\ e=0$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}=\begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}+b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, Ker $(A-2I)^2=\text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\right\}$

Finding Ker(A+I)

Let $v \in$ Ker(A+I)

$\because (A+I)\ v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}3&5&0&0&1\\0&3&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}3a+5b+e\\3b\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \Rightarrow b=0,\ e=-3a$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}a\\0\\c\\d\\-3a\end{pmatrix}=a\begin{pmatrix}1\\0\\0\\0\\-3\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+d\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$$

So, Ker(A+I) $=\text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\-3\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\right\}$

Now, since, $v_2 \in$ Ker$(A-2I)^2$, $v_2 \notin$ Ker(A-2I), we consider $v_2=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$

Putting the value of v_2 in equation(2), $v_1=(A-2I)v_2=\begin{pmatrix}0&5&0&0&1\\0&0&0&0&0\\0&0&-3&0&0\\0&0&0&-3&0\\0&0&0&0&-3\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}5\\0\\0\\0\\0\end{pmatrix}$

Also, we find that $v_1 \in$ Ker(A-2I) as per our condition.

Now, since, $v_3, v_4, v_5 \in$ Ker(A+I), we consider $v_3=\begin{pmatrix}1\\0\\0\\0\\-3\end{pmatrix}$, $v_4=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$, $v_5=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$

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$v_1=\begin{pmatrix}5\\0\\0\\0\\0\end{pmatrix}$, $v_2=\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}$, $v_3=\begin{pmatrix}1\\0\\0\\0\\-3\end{pmatrix}$, $v_4=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$, $v_5=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$. v_1, v_2, v_3, v_4, v_5 are linearly independent as required.

v_1 is the ordinary eigen vector(generalised eigen vector of rank 1) and v_2 is the generalised eigen vector of rank 2 corresponding to $\lambda=2$.Thus, there are 2 linearly independent generalised eigen vectors corresponding to $\lambda=2$.

v_3, v_4, v_5 are the ordinary eigen vectors(generalised eigen vectors of rank 1) corresponding to $\lambda=-1$.Thus,there are 3 linearly independent generalised eigen vectors corresponding to $\lambda=-1$.

Our desired matrix, $P=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5] = \begin{bmatrix}5&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&-3&0&0\end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix}\frac{1}{5}&0&0&0&\frac{1}{15}\\0&1&0&0&0\\0&0&0&0&\frac{-1}{3}\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}\begin{bmatrix}2&5&0&0&0\\0&2&0&0&0\\0&0&-1&0&0\\0&0&0&-1&0\\0&0&0&0&-1\end{bmatrix}\begin{bmatrix}5&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&-3&0&0\end{bmatrix}$

$$=\begin{bmatrix}\frac{2}{5}&1&0&0&\frac{2}{15}\\0&2&0&0&0\\0&0&0&0&\frac{1}{3}\\0&0&-1&0&0\\0&0&0&-1&0\end{bmatrix}\begin{bmatrix}5&0&1&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&-3&0&0\end{bmatrix}=\begin{bmatrix}2&1&0&0&0\\0&2&0&0&0\\0&0&-1&0&0\\0&0&0&-1&0\\0&0&0&0&-1\end{bmatrix}=J$$

Example 2:

$$A=\begin{bmatrix}1&0&0&0&0\\2&1&0&0&0\\-1&3&1&0&0\\-1&3&2&1&0\\1&4&5&2&1\end{bmatrix}$$

Eigen values of A are 1,1,1,1,1

- **Finding the Jordan Canonical Form of A**

$$(A-I)=\begin{bmatrix}0&0&0&0&0\\2&0&0&0&0\\-1&3&0&0&0\\-1&3&2&0&0\\1&4&5&2&0\end{bmatrix}$$

Rank(A-I) =4

δ_1=dim Ker(A-I) =n - rank(A-I) = 5- 4 =1 (geometric multiplicity =1 < algebraic multiplicity =5)

$$(A-I)^2=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\6&0&0&0&0\\4&6&0&0&0\\1&21&4&0&0\end{bmatrix}$$

Rank$(A-I)^2$=3

δ_2=dim Ker$(A-I)^2$=n-rank$(A-I)^2$=5-3=2

$$(A-I)^3=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\12&0&0&0&0\\38&12&0&0&0\end{bmatrix}$$

Rank$(A-I)^3$=2

δ_3=dim Ker$(A-I)^3$=n-rank$(A-I)^3$=5-2=3

$$(A-I)^4=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\24&0&0&0&0\end{bmatrix}$$

Rank$(A-I)^4$=1

δ_4=dim Ker$(A-I)^4$=n-rank$(A-I)^4$=5-1=4

$$(A-I)^5=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Rank$(A-I)^5$=0

δ_5=dim Ker$(A-I)^5$=n-rank$(A-I)^5$=5-0=5

$v_1=2\delta_1-\delta_2$=2-2=0

$v_2=2\delta_2-\delta_3-\delta_1$=4-3-1=0

$v_3=2\delta_3-\delta_4-\delta_2$=6-4-2=0

$v_4=2\delta_4-\delta_5-\delta_3$=8-5-3=0

$v_5=\delta_5-\delta_4$=5-4=1

So,there will be 1 Jordan block of size 5(say J_1) corresponding to λ=1

$$\therefore \text{JCF of A, } J=[J_1]=\begin{bmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}$$

- <u>Finding a matrix P such that $P^{-1}AP=J$ or equivalently AP=PJ</u>

Let P=[v_1 v_2 v_3 v_4 v_5] where $v_1,v_2,v_3,v_4,v_5 \in \boldsymbol{R}^5$

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$AP=[Av_1 \quad Av_2 \quad Av_3 \quad Av_4 \quad Av_5]$

$$PJ=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5]\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$[v_1 \quad v_1+v_2 \quad v_2+v_3 \quad v_3+v_4 \quad v_4+v_5]$

Since, $AP=PJ$, we want to choose $v_1, v_2, v_3, v_4, v_5 \in \mathbf{R}^5$ such that $Av_1=v_1, Av_2=v_1+v_2$, $Av_3 = v_2+v_3, Av_4= v_3+v_4$, $Av_5 = v_4 + v_5$ where $v_1, v_2, v_3, v_4, v_5 \neq 0$ and v_1, v_2, v_3, v_4, v_5 are linearly independent since P is invertible.

The equations can be written in the form: $(A-I)v_1=0$ ... . (1)

$(A-I)v_2=v_1$ ...... (2)

$(A-I)v_3=v_2$ .... (3)

$(A-I)v_4=v_3$ .... (4)

$(A-I)v_5=v_4$ .... (5)

$\therefore$ v_1, v_2, v_3, v_4, v_5 forms a Jordan chain of length 5 corresponding to $J_1(=J)$

Equation (1) implies $v_1 \in Ker(A-I)$

Equation (2) implies $v_2 \notin Ker(A-I)$, since $v_1 \neq 0$

Equation (3) implies $v_3 \notin Ker(A-I)$, since $v_2 \neq 0$

Equation (4) implies $v_4 \notin Ker(A-I)$, since $v_3 \neq 0$

Equation (5) implies $v_5 \notin Ker(A-I)$, since $v_4 \neq 0$

Now, $(A-I)^2v_2=(A-I)v_1=0$

$\therefore v_2 \in Ker(A-I)^2, v_2 \notin Ker(A-I)$

Again, $(A-I)^2v_3=(A-I)v_2=v_1 \Rightarrow (A-I)^3v_3=(A-I)v_1=0$

$\therefore v_3 \in Ker(A-I)^3, v_3 \notin Ker(A-I)^2$, $v_3 \notin Ker(A-I)$

Again, $(A-I)^2v_4=(A-I)v_3=v_2 \Rightarrow (A-I)^3v_4=(A-I)v_2=v_1 \Rightarrow (A-I)^4v_4=(A-I)v_1=0$

$\therefore v_4 \in Ker(A-I)^4, v_4 \notin Ker(A-I)^3$, $v_4 \notin Ker(A-I)^2$, $v_4 \notin Ker(A-I)$

Again, $(A-I)^2v_5=(A-I)v_4=v_3 \Rightarrow (A-I)^3v_5=(A-I)v_3=v_2 \Rightarrow (A-I)^4v_5=(A-I)v_2=v_1$
$\Rightarrow (A-I)^5v_5=(A-I)v_1=0$

$\therefore v_5 \in Ker(A-I)^5, v_5 \notin Ker(A-I)^4$, $v_5 \notin Ker(A-I)^3$, $v_5 \notin Ker(A-I)^2$, $v_5 \notin Ker(A-I)$

<u>Finding $Ker(A-I)^5$</u>

Since $(A-I)^5=\mathbf{0}$, $Ker(A-I)^5=span\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$

<u>Finding $Ker(A-I)^4$</u>

Let $v \in Ker(A-I)^4$

$\therefore (A-I)^4 v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\24&0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \Rightarrow a=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\b\\c\\d\\e\end{pmatrix}=b\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+d\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}+e\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

So, $Ker(A-I)^4=span\left\{\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$

<u>Finding $Ker(A-I)^3$</u>

Let $v\in Ker(A-I)^3$

$\therefore (A-I)^4 v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\12&0&0&0&0\\38&12&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \Rightarrow a=0, b=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\c\\d\\e\end{pmatrix}=c\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+d\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}+e\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

So, $Ker(A-I)^3=span\left\{\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$

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Finding Ker$(A-I)^2$

Let $v \in$ Ker$(A-I)^2$

$\therefore (A-I)^2v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\6&0&0&0&0\\4&6&0&0&0\\1&21&4&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \Rightarrow a=0, b=0, c=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\d\\e\end{pmatrix}=d\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}+e\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

So, Ker$(A-I)^2$=span $\left\{\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$

Finding Ker(A-I)

Let $v \in$ Ker(A-I)

$\therefore (A-I)v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\-1&3&0&0&0\\-1&3&2&0&0\\1&4&5&2&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \Rightarrow a=0,\ b=0,\ c=0,\ d=0$$

$$\therefore v=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\e\end{pmatrix}=e\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

So, Ker(A-I) =span $\left\{\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$

Now, since, $v_5 \in Ker(A-I)^5, v_5 \notin Ker(A-I)^4, v_5 \notin Ker(A-I)^3, v_5 \notin Ker(A-I)^2, v_5 \notin Ker(A-I)$, we consider $v_5=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$

Putting the value of v_5 in equation(5), $v_4=(A-I)v_5=\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\-1&3&0&0&0\\-1&3&2&0&0\\1&4&5&2&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\2\\-1\\-1\\1\end{pmatrix}$

Also, we find that $v_4 \in Ker(A-I)^4, v_4 \notin Ker(A-I)^3, v_4 \notin Ker(A-I)^2, v_4 \notin Ker(A-I)$ as per our condition.

Putting the value of v_4 in equation(4), $v_3=(A-I)v_4=\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\-1&3&0&0&0\\-1&3&2&0&0\\1&4&5&2&0\end{pmatrix}\begin{pmatrix}0\\2\\-1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\6\\4\\1\end{pmatrix}$

Also, we find that $v_3 \in Ker(A-I)^3, v_3 \notin Ker(A-I)^2, v_3 \notin Ker(A-I)$

Putting the value of v_3 in equation(3), $v_2=(A-I)v_3=\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\-1&3&0&0&0\\-1&3&2&0&0\\1&4&5&2&0\end{pmatrix}\begin{pmatrix}0\\0\\6\\4\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\12\\38\end{pmatrix}$

Also, we find that $v_2 \in Ker(A-I)^2, v_2 \notin Ker(A-I)$ as per our condition.

Putting the value of v_2 in equation (2),

$v_1=(A-I)v_2=\begin{pmatrix}0&0&0&0&0\\2&0&0&0&0\\-1&3&0&0&0\\-1&3&2&0&0\\1&4&5&2&0\end{pmatrix}\begin{pmatrix}0\\0\\0\\12\\38\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\24\end{pmatrix}$

Also, we find that $v_1 \in Ker(A-I)$ as per our condition.

$v_1=\begin{pmatrix}0\\0\\0\\0\\24\end{pmatrix}$, $v_2=\begin{pmatrix}0\\0\\0\\12\\38\end{pmatrix}$, $v_3=\begin{pmatrix}0\\0\\6\\4\\1\end{pmatrix}$, $v_4=\begin{pmatrix}0\\2\\-1\\-1\\1\end{pmatrix}$, $v_5=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$. v_1, v_2, v_3, v_4, v_5 are linearly independent as required.

v_1 is the ordinary eigen vector, v_2 is the generalised eigen vector of rank 2 and v_3 is the generalised eigen vector of rank 3, v_4 is the generalised eigen vector of rank 4, v_5 is the generalised eigen vector of rank 5 corresponding to $\lambda=1$. Thus, there are 5 linearly independent generalised eigen vectors corresponding to $\lambda=1$.

Our desired matrix, $P=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5]=\begin{bmatrix}0&0&0&0&1\\0&0&0&2&0\\0&0&6&-1&0\\0&12&4&-1&0\\24&38&1&1&0\end{bmatrix}$

Check: $P^{-1}AP = \begin{bmatrix} 0 & \frac{-5}{108} & \frac{35}{432} & \frac{-19}{144} & \frac{1}{24} \\ 0 & \frac{1}{72} & \frac{-1}{18} & \frac{1}{12} & 0 \\ 0 & \frac{1}{12} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ [illegible] & 1 & 0 & 0 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ -1 & 3 & 2 & 1 & 0 \\ [illegible] & 4 & 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 12 & 4 & -1 & 0 \\ 24 & 38 & 1 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & \frac{-7}{216} & \frac{11}{432} & \frac{-7}{144} & \frac{1}{24} \\ 0 & \frac{7}{72} & \frac{1}{9} & \frac{1}{12} & 0 \\ 0 & \frac{7}{12} & \frac{1}{6} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 12 & 4 & -1 & 0 \\ 24 & 38 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = J$$

8.1.5 6X6 matrix

Example 1:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Eigen values of A are 2,2,2,2,2,2

- Finding the Jordan Canonical Form of A

$$(A-2I) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Rank(A-2I) =5

δ_1=dim Ker(A-2I) =n-rank(A-2I) =6-5=1(geometric multiplicity =1< algebraic multiplicity =6)

$$(A-2I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Rank$(A-2I)^2$=4

δ_2=dim Ker$(A-2I)^2$=n-rank$(A-2I)^2$=6-4=2

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$$(A-2I)^3=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\0&2&1&0&0&0\end{bmatrix}$$

Rank$(A-2I)^3=3$

δ_3=dim Ker$(A-2I)^3$=n-rank$(A-2I)^3$=6-3=3

$$(A-2I)^4=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\end{bmatrix}$$

Rank$(A-2I)^4=2$

δ_4=dim Ker$(A-2I)^4$=n-rank$(A-2I)^4$=6-2=4

$$(A-2I)^5=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\end{bmatrix}$$

Rank$(A-2I)^5=1$

δ_5=dim Ker$(A-2I)^5$=n-rank$(A-2I)^5$=6-1=5

$$(A-2I)^6=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

Rank$(A-2I)^6=0$

δ_6=dim Ker$(A-2I)^6$=n-rank$(A-2I)^6$=6-0=6

$v_1=2\delta_1-\delta_2$=2-2=0

$v_2=2\delta_2-\delta_3-\delta_1$=4-3-1=0

$v_3=2\delta_3-\delta_4-\delta_2$=6-4-2=0

$v_4=2\delta_4-\delta_5-\delta_3$=8-5-3=0

$v_5=2\delta_5-\delta_6-\delta_4$=10-6-4=0

$v_6=\delta_6-\delta_5$=6-5=1

So there will be 1 Jordan block of size 6 (say J_1) corresponding to $\lambda=2$

$$\therefore \text{JCF of A, } J=[J_1]=\begin{bmatrix}2&1&0&0&0&0\\0&2&1&0&0&0\\0&0&2&1&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{bmatrix}$$

- **Finding a matrix P such that $P^{-1}AP=J$ or equivalently $AP=PJ$**

Let $P=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6]$ where $v_1, v_2, v_3, v_4, v_5, v_6 \in \mathbf{R}^6$

$AP=[Av_1 \quad Av_2 \quad Av_3 \quad Av_4 \quad Av_5 \quad Av_6]$

$$PJ=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6]\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$=[2v_1 \quad v_1+2v_2 \quad v_2+2v_3 \quad v_3+2v_4 \quad v_4+2v_5 \quad v_5+2v_6]$

Since, $AP=PJ$, we want to choose $v_1, v_2, v_3, v_4, v_5, v_6 \in \mathbf{R}^6$ such that $Av_1=2v_1$, $Av_2=v_1+2v_2$, $Av_3=v_2+2v_3$, $Av_4=v_3+2v_4$, $Av_5=v_4+2v_5$, $Av_6=v_5+2v_6$ where $v_1, v_2, v_3, v_4, v_5, v_6 \neq 0$ and $v_1, v_2, v_3, v_4, v_5, v_6$ are linearly independent since P is invertible.

The equations can be written in the form: $(A-2I)v_1=0$ .... (1)

$(A-2I)v_2=v_1$ ....(2)

$(A-2I)v_3=v_2$ .... (3)

$(A-2I)v_4=v_3$ ....(4)

$(A-2I)v_5=v_4$ ....(5)

$(A-2I)v_6=v_5$ ....(6)

$\therefore v_1, v_2, v_3, v_4, v_5, v_6$ forms a Jordan chain of length 6 corresponding to $J_1(=J)$

Equation (1) implies $v_1 \in Ker(A-2I)$
Equation (2) implies $v_2 \notin Ker(A-2I)$, since $v_1 \neq 0$
Equation (3) implies $v_3 \notin Ker(A-2I)$, since $v_2 \neq 0$
Equation (4) implies $v_4 \notin Ker(A-2I)$, since $v_3 \neq 0$
Equation (5) implies $v_5 \notin Ker(A-2I)$, since $v_4 \neq 0$
Equation (6) implies $v_6 \notin Ker(A-2I)$, since $v_5 \neq 0$

Now, $(A-2I)^2v_2=(A-2I)v_1=0$
$\therefore v_2 \in Ker(A-2I)^2, v_2 \notin Ker(A-2I)$

Again, $(A-2I)^2v_3=(A-2I)v_2=v_1 \Rightarrow (A-2I)^3v_3=(A-2I)v_1=0$
$\therefore v_3 \in Ker(A-2I)^3, v_3 \notin Ker(A-2I)^2, v_3 \notin Ker(A-2I)$

Again, $(A-2I)^2v_4=(A-2I)v_3=v_2 \Rightarrow (A-2I)^3v_4=(A-2I)v_2=v_1 \Rightarrow (A-2I)^4v_4=(A-2I)v_1=0$
$\therefore v_4 \in Ker(A-2I)^4, v_4 \notin Ker(A-2I)^3, v_4 \notin Ker(A-2I)^2, v_4 \notin Ker(A-2I)$

Again, $(A-2I)^2v_5=(A-2I)v_4=v_3 \Rightarrow (A-2I)^3v_5=(A-2I)v_3=v_2 \Rightarrow (A-2I)^4v_5=(A-2I)v_2=v_1 \Rightarrow (A-2I)^5v_5=(A-2I)v_1=0$
$\therefore v_5 \in Ker(A-2I)^5, v_5 \notin Ker(A-2I)^4, v_5 \notin Ker(A-2I)^3, v_5 \notin Ker(A-2I)^2, v_5 \notin Ker(A-2I)$

Again, $(A-2I)^2v_6=(A-2I)v_5=v_4 \Rightarrow (A-2I)^3v_6=(A-2I)v_4=v_3 \Rightarrow (A-2I)^4v_6=(A-2I)v_3=v_2 \Rightarrow (A-2I)^5v_6=(A-2I)v_2=v_1 \Rightarrow (A-2I)^6v_6=(A-2I)v_1=0$

Finding Ker$(A-2I)^6$

Since $(A-2I)^6=0$, Ker$(A-2I)^6$=span $\left\{\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}\right\}$

Finding Ker$(A-2I)^5$

Let $v \in$ Ker$(A-2I)^5$

$\therefore (A-2I)^5 v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix} \Rightarrow a=0$$

$$v=\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix}=\begin{pmatrix}0\\b\\c\\d\\e\\f\end{pmatrix}=b\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}+d\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}+e\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}+f\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$$

Finding Ker$(A-2I)^4$

Let $v \in$ Ker$(A-2I)^4$

$\therefore (A-2I)^4 v=0$ where $v\neq 0$, $v=\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = b = 0$$

$$v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ c \\ d \\ e \\ f \end{pmatrix}=c\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}+d\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}+e\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}+f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So, $Ker(A-2I)^4$=span $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Finding $Ker(A-2I)^3$

Let $v \in Ker(A-2I)^3$

$\therefore (A-2I)^3v=0$ where $v\neq 0$, $v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$.

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a = b = c = 0$$

$$v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ d \\ e \\ f \end{pmatrix}=d\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}+e\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}+f\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So, $Ker(A-2I)^3$=span , $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Finding $Ker(A-2I)^2$

Let $v \in Ker(A-2I)^2$

$\therefore (A-2I)^2v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&2&1&0&0&0\\ 1&1&1&1&0&0 \end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \Rightarrow a=b=c=d=0$$

$$v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ e\\ f\end{pmatrix}=e\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 1\\ 0\end{pmatrix}+f\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix}$$

So, $Ker(A-2I)^2=span\left\{\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 1\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix}\right\}$

<u>Finding Ker(A-2I)</u>

Let $v \in Ker(A-2I)$

$\therefore (A-2I)\ v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0&0&0&0&0&0\\ 1&0&0&0&0&0\\ -1&1&0&0&0&0\\ 0&1&1&0&0&0\\ 1&1&1&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \Rightarrow a=b=c=d=e=0$$

$$v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ f\end{pmatrix}=f\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix}$$

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So, Ker(A-2I) =span $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Now, since, $v_6 \in Ker(A-2I)^6, v_6 \notin Ker(A-2I)^5$, $v_6 \notin Ker(A-2I)^4$, $v_6 \notin Ker(A-2I)^3$, $v_6 \notin Ker(A-2I)^2$, $v_6 \notin Ker(A\text{-}2I)$, we consider $v_6=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Putting the value of v_6 in equation (6),

$$v_5=(A-2I)v_6=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Also, we find that $v_5 \in Ker(A-2I)^5, v_5 \notin Ker(A-2I)^4$, $v_5 \notin Ker(A-2I)^3$, $v_5 \notin Ker(A-2I)^2$, $v_5 \notin Ker(A\text{-}2I)$ as per our condition.

Putting the value of v_5 in equation (5),

$$v_4=(A-2I)v_5=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Also, we find that $v_4 \in Ker(A-2I)^4$, $v_4 \notin Ker(A-2I)^3$, $v_4 \notin Ker(A-2I)^2$, $v_4 \notin Ker(A\text{-}2I)$

Putting the value of v_4 in equation(4), $v_3=(A-2I)v_4=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Also, we find that $v_3 \in Ker(A-2I)^3$, $v_3 \notin Ker(A-2I)^2$, $v_3 \notin Ker(A\text{-}2I)$

Putting the value of v_3 in equation(3), $v_2=(A-2I)v_3=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Also, we find that $v_2 \in Ker(A-2I)^2$, $v_2 \notin Ker(A\text{-}2I)$

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Putting the value of v_2 in equation(2), $v_1=(A-2I)v_2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Also, we find that $v_1 \in Ker(A-2I)$ as per our condition.

$$\therefore\ v_1=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ v_2=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},\ v_3=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ v_4=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ v_5=\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ v_6=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

v_1, v_2, v_3, v_4, v_5 , v_6 are linearly independent as required.

v_1 is the ordinary eigen vector(generalised eigen vector of rank 1), v_2 is the generalised eigen vector of rank 2 and v_3 is the generalised eigen vector of rank 3, v_4 is the generalised eigen vector of rank 4, v_5 is the generalised eigen vector of rank 5, v_6 is the generalised eigen vector of rank 6 corresponding to $\lambda=2$. Thus, there are 6 linearly independent generalised eigen vectors corresponding to $\lambda=2$.

Our desired matrix, $P=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$

Check:

$$P^{-1}AP=\begin{bmatrix} 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -1 & -2 & 1 & -1 & 2 \\ 0 & -2 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}=J$$

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B. Complex Eigen values

8.2.1 2 X 2 matrix

Example 1:

$$A=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

Eigen values of A are $1\pm i$

- Finding Jordan Canonical Form of A

We consider the eigen value $\lambda=1+i$

Separating the real and imaginary part of the complex eigen value $\lambda=1+i$, $Re(\lambda)=1$, $Im(\lambda)=1$

$\therefore$ JCF of A, $J=\begin{bmatrix}Re(\lambda) & -Im(\lambda)\\ Im(\lambda) & Re(\lambda)\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently $AP=PJ$

Finding eigen vector corresponding to $\lambda=1+i$

$(A-\lambda I)v=0$ where $\lambda=1+i, v\neq 0, v=\begin{pmatrix}a\\ b\end{pmatrix}$

$\Rightarrow \begin{pmatrix}1-1-i & 1\\ -1 & 1-1-i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \Rightarrow \begin{pmatrix}-ai+b\\ -a-ib\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow a=-ib$

$\therefore v=\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}-ib\\ b\end{pmatrix}=b\begin{pmatrix}-i\\ 1\end{pmatrix}$

$\therefore$ An eigen vector corresponding to $\lambda=i$ is $v=\begin{pmatrix}-i\\ 1\end{pmatrix}$

Separating the real and imaginary part of the eigen vector v, $v=\begin{pmatrix}-i\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}+i\begin{pmatrix}-1\\ 0\end{pmatrix}$

$\therefore Re(v)=\begin{pmatrix}0\\ 1\end{pmatrix}, Im(v)=\begin{pmatrix}-1\\ 0\end{pmatrix}$

Thus, our desired matrix $P=[Im(v) \quad Re(v)]=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=J$

Example 2:

$$A=\begin{bmatrix}0 & -4\\ 1 & 0\end{bmatrix}$$

Eigen values of A are $\pm 2i$.

- Finding Jordan Canonical Form of A

We consider the eigen value $\lambda=2i$

Separating the real and imaginary part of the complex eigen value $\lambda=2i$, $Re(\lambda)=0$, $Im(\lambda)=2$

$\therefore$ JCF of A, $J=\begin{bmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{bmatrix}=\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently $AP=PJ$

Finding eigen vector corresponding to $\lambda=2i$

$(A-\lambda I)v=0$ where $\lambda=2i, v\neq 0, v=\begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -2ai-4b \\ a-2ib \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a=2ib$

$\therefore v=\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 2ib \\ b \end{pmatrix}=b\begin{pmatrix} 2i \\ 1 \end{pmatrix}$

$\therefore$ An eigen vector corresponding to $\lambda=2i$ is $v=\begin{pmatrix} 2i \\ 1 \end{pmatrix}$

Separating the real and imaginary part of the eigen vector v, $v=\begin{pmatrix} 2i \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}+i\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$\therefore Re(v)=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Im(v)=\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Thus, our desired matrix $P=[Im(v) \quad Re(v)]=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Check: $P^{-1}AP=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}=J$

8.2.2 4 X 4 matrix

Example 1:

$A=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Eigen values of A are $1\pm i, 2\pm i$

- Finding the Jordan Canonical Form of A
- Finding the Jordan block corresponding to $\lambda_1=1+i(J_1)$

Considering the eigen value $\lambda_1=1+i$, we separate the real and imaginary part, $Re(\lambda_1)=1, Im(\lambda_1)=1$

$J_1=\begin{bmatrix} Re(\lambda_1) & -Im(\lambda_1) \\ Im(\lambda_1) & Re(\lambda_1) \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

- Finding the Jordan block corresponding to $\lambda_2=2+i(J_2)$

Considering the eigen value $\lambda_2=2+i$, we separate the real and imaginary part, $Re(\lambda_2)=2$, $Im(\lambda_2)=1$

$$J_2=\begin{bmatrix} Re(\lambda_2) & -Im(\lambda_2) \\ Im(\lambda_2) & Re(\lambda_2) \end{bmatrix}=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix}=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{ where } J_1=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, J_2=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

- Finding a matrix P such that $P^{-1}AP=J$ or equivalently $AP=PJ$

Finding eigen vector corresponding to $\lambda=1+i$

$(A-\lambda I)v=0$, where $v\neq 0, v=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, $\lambda=1+i$

$$\begin{pmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 2-i & -2 \\ 0 & 0 & 1 & -i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a=bi,\ c=0,\ d=0$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=\begin{pmatrix} bi \\ b \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ An eigen vector corresponding to $\lambda=1+i$ is $v_1=\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Separating the real and imaginary part of $v_1=\begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}+i\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Re(v_1)=\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $Im(v_1)=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Finding eigen vector corresponding to $\lambda=2+i$

$(A-\lambda I)\ v=0$, where $v\neq 0$, $v=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, $\lambda=2+i$

$$\Rightarrow \begin{pmatrix} -1-i & -1 & 0 & 0 \\ 1 & -1-i & 0 & 0 \\ 0 & 0 & 1-i & -2 \\ 0 & 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a=0,\ b=0,\ c=d(1+i)$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}=d\begin{pmatrix} 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix}$$

$\therefore$ An eigen vector corresponding to $\lambda=2+i$ is $v_2=\begin{pmatrix} 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix}$

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Separating the real and imaginary part of $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $Re(v_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$,

$Im(v_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

From J, it is clear that the 1st, 2nd columns of P will correspond to J_1 and the 3rd and 4th columns of P will correspond to J_2.
Thus, 1st and 2nd columns will be the imaginary and real part of eigen vector corresponding to $\lambda=1+i$ respectively and 3rd and 4th columns will be the imaginary and real part of eigen vector corresponding to $\lambda=2+i$ respectively.

Thus, our desired matrix $P=[Im(v_1) \quad Re(v_1) \quad Im(v_2) \quad Re(v_2)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Check: $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = J$

C. Real and Complex Eigen Values

8.3.1 3 X 3 matrix

Example 1:

$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

Eigen values of A are $-3, 2\pm i$

- Finding the Jordan Canonical Form of A
- Finding the Jordan Block corresponding to $\lambda= -3$ (J_1)

Since algebraic multiplicity of $\lambda=3$ is 1, clearly, there will be 1 Jordan block J_1 of size 1 corresponding to $\lambda=-3$.

$\therefore J_1=[-3]$

- Finding the Jordan block corresponding to complex eigen value (J_2)

Considering the eigen value, $\lambda=2+i$, we separate the real and imaginary part

Thus, $Re(\lambda)=2$, $Im(\lambda)=1$

$$\therefore J_2=\begin{bmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{bmatrix}=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}=\begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \text{ where } J_1=[-3],\ J_2=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

- Finding a matrix P such that $P^{-1}AP=J$

From J, it is clear that the 1st column of P will correspond to J_1 and the 2nd and 3rd columns will correspond to J_2.
Thus, 1st column will be the eigen vector corresponding to $\lambda=-3$. 2nd and 3rd column will be the imaginary and real part of eigen vector corresponding to $\lambda=2+i$ respectively.

Finding eigen vector corresponding to $\lambda=-3$

$(A-3I)\,v=0$, where $v\neq 0$, $v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow b=0,\ c=0$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ An eigen vector corresponding to $\lambda=-3$ is $v_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Finding eigen vector corresponding to $\lambda=2+i$

$(A-\lambda I)\,v=0$, where $v\neq 0$, $v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\lambda=2+i$

$$\Rightarrow \begin{pmatrix} -5-i & 0 & 0 \\ 0 & 1-i & -2 \\ 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a=0,\ b=c(1+i)$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \end{pmatrix}=c\begin{pmatrix} 0 \\ 1+i \\ 1 \end{pmatrix}$$

$\therefore$ An eigen vector corresponding to $\lambda=2+i$ is $v_2=\begin{pmatrix} 0 \\ 1+i \\ 1 \end{pmatrix}$

Separating the real and imaginary part of $v_2=\begin{pmatrix} 0 \\ 1+i \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}+i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\therefore Re(v_2)=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $Im(v_2)=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

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Thus, our desired matrix P=[v_1 Im(v_2) Re(v_2)]= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Check: $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} = J$$

8.3.2 5X5 matrix

Example 1:

$$A= \begin{bmatrix} -7 & -5 & -3 & 0 & 0 \\ 2 & -2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Eigen values of A are -3, -3, -3, 2±i

- Finding the Jordan Canonical Form of A
- Finding the Jordan Block corresponding to λ=-3(J_1)

$$(A+3I)= \begin{bmatrix} -4 & -5 & -3 & 0 & 0 \\ 2 & 1 & -3 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 6 & -2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Rank(A+3I) =4

δ_1=dim Ker(A+3I) =n-rank(A+3I) =5-4 =1

$$(A+3I)^2= \begin{bmatrix} 6 & 12 & 18 & 0 & 0 \\ -6 & -12 & -18 & 0 & 0 \\ 2 & 4 & 6 & 0 & 0 \\ 0 & 0 & 0 & 34 & -20 \\ 0 & 0 & 0 & 10 & 14 \end{bmatrix}$$

Rank$(A+3I)^2$=3

δ_2=dim Ker$(A+3I)^2$=n-rank$(A+3I)^2$=5-3=2

$$(A-I)^3= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 184 & -148 \\ 0 & 0 & 0 & 74 & 36 \end{bmatrix}$$

Rank$(A-I)^3$=2

δ_3=dim Ker$(A-I)^3$=n-rank$(A-I)^3$=5-2=3

$v_1=2\delta_1-\delta_2$=2-2=0

$v_2=2\delta_2-\delta_3-\delta_1$=4-3-1=0

$v_3=\delta_3-\delta_2$=3-2=1

So, there will be 1 Jordan block of size 3 corresponding to λ=-3.

$$\therefore J_1= \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

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- Finding the Jordan Block corresponding to complex eigen value (J_2)

Considering the eigen value, $\lambda=2+i$, we separate the real and imaginary part

Thus, $Re(\lambda)=2$, $Im(\lambda)=1$

$$J_2 = \begin{bmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}=\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ where } J_1=\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}, J_2=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

- Finding a matrix P such that $P^{-1}AP=J$

From J, it is clear that the 1st, 2nd, 3rd columns of P will correspond to J_1 and the 4th and 5th columns will correspond to J_2.
Thus, ordinary eigen vectors and generalised eigen vectors corresponding to $\lambda= -3$ will fill the 1st, 2nd, 3rd columns of P. 4th and 5th columns will be the imaginary and real part of eigen vector corresponding to $\lambda=2+i$ respectively.

Finding ordinary eigen vector and generalised eigen vectors corresponding to $\lambda= -3$
Since geometric multiplicity of $\lambda=-3$ is 1, there will be one ordinary eigen vector (say v_1) and two generalised eigen vectors (say v_2, v_3) corresponding to $\lambda=-3$.
By careful observation (Example 3 of 8.1.2), it is clearly understood that for a 3 X 3 Jordan Block, (here J_1) generalised eigen vectors corresponding to $\lambda=-3$ are related by Jordan chains in the following manner:

$(A+3I)v_1=0$ ... (1)

$(A+3I)v_2=v_1$ … (2)

$(A+3I)v_3=v_2$ ...(3)

v_1, v_2, v_3 form a Jordan chain of length 3 corresponding to J_1.
Equation (1) implies $v_1 \in Ker(A+3I)$
Equation (2) implies $v_2 \notin Ker(A+3I)$, since $v_1 \neq 0$
Equation (3) implies $v_3 \notin Ker(A+3I)$, since $v_2 \neq 0$

Now, $(A + 3I)^2v_2=(A+3I)v_1 =0$
$\therefore v_2 \in Ker(A + 3I)^2, v_2 \notin Ker(A+3I)$

Again, $(A + 3I)^2v_3=(A+3I)v_2 =v_1 \Rightarrow (A + 3I)^3v_3=(A+3I)v_1 =0$
$\therefore v_3 \in Ker(A + 3I)^3, v_3 \notin Ker(A + 3I)^2, v_3 \notin Ker(A+3I)$
Finding $Ker(A + 3I)^3$
Let $v \in Ker(A + 3I)^3$

$\therefore (A+3I)^3 v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&184&-148\\ 0&0&0&74&-36\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix} 0\\ 0\\ 0\\ 184d-148e\\ 74d-36e\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \Rightarrow d=0,\ e=0$$

$$\therefore v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} a\\ b\\ c\\ 0\\ 0\end{pmatrix}=a\begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix}+b\begin{pmatrix} 0\\ 1\\ 0\\ 0\\ 0\end{pmatrix}+c\begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix}$$

So, $\operatorname{Ker}(A+3I)^3=\operatorname{span}\left\{\begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix},\begin{pmatrix} 0\\ 1\\ 0\\ 0\\ 0\end{pmatrix},\begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix}\right\}$

<u>Finding $\operatorname{Ker}(A+3I)^2$</u>

Let $v\in \operatorname{Ker}(A+3I)^2$

$\therefore (A+3I)^2 v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6&12&18&0&0\\ -6&-12&-18&0&0\\ 2&4&6&0&0\\ 0&0&0&34&-20\\ 0&0&0&10&14\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \Rightarrow a=-2b-3c,\ d=0,\ e=0$$

$$\therefore v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} -2b-3c\\ b\\ c\\ 0\\ 0\end{pmatrix}=b\begin{pmatrix} -2\\ 1\\ 0\\ 0\\ 0\end{pmatrix}+c\begin{pmatrix} -3\\ 0\\ 1\\ 0\\ 0\end{pmatrix}$$

So, $\operatorname{Ker}(A+3I)^2=\operatorname{span}\left\{\begin{pmatrix} -2\\ 1\\ 0\\ 0\\ 0\end{pmatrix},\begin{pmatrix} -3\\ 0\\ 1\\ 0\\ 0\end{pmatrix}\right\}$

<u>Finding $\operatorname{Ker}(A+3I)$</u>

Let $v\in \operatorname{Ker}(A+3I)$

$\therefore (A+3I)\ v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -4 & -5 & -3 & 0 & 0\\ 2 & 1 & -3 & 0 & 0\\ 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 0 & 6 & -2\\ 0 & 0 & 0 & 1 & 4\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \Rightarrow a=3c,\ b=-3c,\ d=0,\ e=0$$

$$\therefore v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} 3c\\ -3c\\ c\\ 0\\ 0\end{pmatrix}=b\begin{pmatrix} 3\\ -3\\ 1\\ 0\\ 0\end{pmatrix}$$

So, $\operatorname{Ker}(A+3I)=\operatorname{span}\begin{pmatrix} 3\\ -3\\ 1\\ 0\\ 0\end{pmatrix}$

Now, since $v_3 \in \operatorname{Ker}(A+3I)^3$, $v_3 \notin \operatorname{Ker}(A+3I)^2$, $v_3 \notin \operatorname{Ker}(A+3I)$, we consider

$$v_3=\begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

Putting the value of v_3 in equation (3),

$$v_2=(A+3I)v_3=\begin{pmatrix} -4 & -5 & -3 & 0 & 0\\ 2 & 1 & -3 & 0 & 0\\ 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 0 & 6 & -2\\ 0 & 0 & 0 & 1 & 4\end{pmatrix}\begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} -4\\ 2\\ 0\\ 0\\ 0\end{pmatrix}$$

Also, we find that $v_2 \in \operatorname{Ker}(A+3I)^2$, $v_2 \notin \operatorname{Ker}(A+3I)$ as per our condition.

Putting the value of v_2 in equation (2),

$$v_1=(A+3I)v_2=\begin{pmatrix} -4 & -5 & -3 & 0 & 0\\ 2 & 1 & -3 & 0 & 0\\ 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 0 & 6 & -2\\ 0 & 0 & 0 & 1 & 4\end{pmatrix}\begin{pmatrix} -4\\ 2\\ 0\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} 6\\ -6\\ 2\\ 0\\ 0\end{pmatrix}$$

Also, we find that $v_1 \in \operatorname{Ker}(A+3I)$ as per our condition.

$v_1=\begin{pmatrix} 6\\ -6\\ 2\\ 0\\ 0\end{pmatrix}$, $v_2=\begin{pmatrix} -4\\ 2\\ 0\\ 0\\ 0\end{pmatrix}$, $v_3=\begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$. v_1, v_2, v_3 are linearly independent as required.

v_1 is the ordinary eigen vector, v_2 is the generalised eigen vector of rank 2 and v_3 is the generalised eigen vector of rank 3 corresponding to $\lambda=-3$. Thus, there are 3 linearly independent generalised eigen vectors corresponding to $\lambda=-3$.

Finding eigen vector corresponding to $\lambda=2+i$

$(A-\lambda I)v=0$, where $v\neq 0, v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$, $\lambda=2+i$

$$\Rightarrow \begin{pmatrix} -9-i & -5 & -3 & 0 & 0 \\ 2 & -4-i & -3 & 0 & 0 \\ 0 & 1 & -2-i & 0 & 0 \\ 0 & 0 & 0 & 1-i & -2 \\ 0 & 0 & 0 & 1 & -1-i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a=0,\ b=0,\ c=0,\ d=e(1+i)$

$$\Rightarrow v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = e\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix}$$

$\therefore$ An eigen vector corresponding to $\lambda=2+i$ is $v_4=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix}$

Separating the real and imaginary part of $v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$Re(v_4)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $Im(v_4)=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Thus, our desired matrix $P=[v_1 \quad v_2 \quad v_3 \quad Im(v_4) \quad Re(v_4)] = \begin{bmatrix} 6 & -4 & 1 & 0 & 0 \\ -6 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Check: $P^{-1}AP = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} -7 & -5 & -3 & 0 & 0 \\ 2 & -2 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -4 & 1 & 0 & 0 \\ -6 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & -\frac{3}{2} & 0 & 0 \\ -3 & -6 & -9 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 6 & -4 & 1 & 0 & 0 \\ -6 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = J$$

8.3.3 6 X 6 matrix

Example 1:

$$A=\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Eigen values of A are 3,3,2,2,2±i

- Finding the Jordan Canonical Form of A
- Finding the Jordan Block corresponding to λ=3(J_1)

$$(A-3I)=\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Rank(A-3I) =5

δ_1=dim Ker(A-3I) =n-rank(A-3I) =6-5 =1

$$(A-3I)^2=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

Rank$(A-3I)^2$=4

δ_2=dim Ker$(A-3I)^2$=n-rank$(A-3I)^2$=6-4=2

$v_1=2\delta_1-\delta_2$=2-2=0

$v_2=\delta_2-\delta_1$=2-1=1

So, there will be 1 Jordan block of size 2 corresponding to $\lambda = 3$

$$\therefore J_1=\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

- Finding the Jordan Block corresponding to λ=2(J_2)

$$(A-2I)=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Rank(A-2I) =5

δ_1=dim Ker(A-2I) =n-rank(A-2I) =6-5 =1

$$(A-2I)^2=\begin{bmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$\mathrm{Rank}(A-2I)^2=4$

$\delta_2=\dim \mathrm{Ker}(A-2I)^2=n\text{-rank}(A-2I)^2=6\text{-}4=2$

$v_1=2\delta_1-\delta_2=2\text{-}2=0$

$v_2=\delta_2-\delta_1=2\text{-}1=1$

So, there will be 1 Jordan block of size 2 corresponding to $\lambda = 2$

$\therefore J_2=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

- Finding the Jordan Block corresponding to λ=2+i(J₃)

Considering the eigen value, λ=2+i, we separate the real and imaginary part

Thus, Re(λ)=2, Im(λ)=1

$J_3=\begin{bmatrix} \mathrm{Re}(\lambda) & -\mathrm{Im}(\lambda) \\ \mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix}=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$$\therefore \text{JCF of A, } J=\begin{bmatrix} J_1 & & \mathbf{0} \\ & J_2 & \\ \mathbf{0} & & J_3 \end{bmatrix}=\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

- Finding a matrix P such that $P^{-1}AP=J$

From J, it is clear that the 1st ,2nd columns of P will correspond to J_1 and the 3rd and 4th columns of P will correspond to J_2,5th and 6th columns of P will correspond to J_3.

Thus, eigen vector and generalised eigen vectors corresponding to λ=3 will fill the 1stand 2ndcolumns of P, eigen vector and generalised eigen vectors corresponding to λ=2 will fill the 3rd and 4th columns of P ,5th and 6th columns will be the imaginary and real part of eigen vector corresponding to λ=2+i respectively.

Finding eigen vector and generalised eigen vectors corresponding to λ=3

Since geometric multiplicity of λ=3 is 2, there will be one ordinary eigen vector (say v_1) and one generalised eigen vectors (say v_2) corresponding to λ=3.
By careful observation (Example 1 of 8.1.1) it is clearly understood that for a 2X2 Jordan Block (here J_1) ,the eigen vectors and generalised eigen vectors corresponding to λ=3 are related by Jordan chains in the following manner: $(A\text{-}3I)v_1=0$ … (1)

$(A\text{-}3I)v_2=v_1$… (2)

$\therefore$ v_1, v_2 form a Jordan chain of length 2 corresponding to J_1.

Equation (1) implies $v_1 \in \mathrm{Ker}(A\text{-}3I)$

Equation (2) implies $v_2 \notin \mathrm{Ker}(A\text{-}3I)$, since $v_1 \neq 0$

Again, $(A-3I)^2v_2=(A\text{-}3I)v_1=0 \Rightarrow (A-3I)^2v_2=0$

$\therefore v_2 \epsilon\ \mathrm{Ker}(A-3I)^2, v_2 \notin \mathrm{Ker}\ (A\text{-}3I)$

Finding $\mathrm{Ker}(A-3I)^2$

Let $v \in \mathrm{Ker}(A-3I)^2$

$\therefore (A-3I)^2 v=0$ where $v\neq 0$, $v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ -c-2d \\ 2c+3d \\ -2e+4f \\ -2e+2f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c=0,\ d=0,\ e=0,\ f=0$$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}=a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}+b\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So. $\mathrm{Ker}(A-3I)^2=\mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

<u>Finding Ker(A-3I)</u>

Let $v \in \mathrm{Ker}(A-3I)$

$(A-3I)\,v=0$ where $v\neq 0$, $v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -a-b \\ a+b \\ d \\ -c-2d \\ -2f \\ e-2f \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a=-b,\ c=0,\ d=0,\ e=0,\ f=0$

$$\therefore v=\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}=\begin{pmatrix} -b \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}=b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $\mathrm{Ker}(A-3I)=\mathrm{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

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Now, since, $v_2 \in \text{Ker}(A-3I)^2, v_2 \notin \text{Ker}(A-3I)$, we consider $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Putting the value of v_2 in equation (2),

$$v_1 = (A-3I)v_2 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Also, we find that $v_1 \in \text{Ker}(A-3I)$ as per our condition.

$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. v_1 and v_2 are linearly independent as required.

v_1 is the ordinary eigen vector and v_2 is the generalised eigen vector of rank 2 corresponding to $\lambda=3$. Thus, there are 2 linearly independent generalised eigen vectors corresponding to $\lambda=3$.

<u>Finding eigen vector and generalised eigen vectors corresponding to $\lambda=2$</u>

Since geometric multiplicity of $\lambda=2$ is 2, there will be one ordinary eigen vector (say v_3) and one generalised eigen vectors (say v_4) corresponding to $\lambda=2$.

Similarly, for J_2, the eigen vectors and generalised eigen vectors for $\lambda=2$ is related by Jordan chains in the following manner: $(A-2I)v_3=0$ … (1)

$(A-2I)v_4=v_3$… (2)

$\therefore$ v_3, v_4 form a Jordan chain of length 2 corresponding to J_2.

Equation (1) implies $v_3 \in \text{Ker}(A-2I)$

Equation (2) implies $v_4 \notin \text{Ker}(A-2I)$, since $v_3 \neq 0$

Again, $(A-2I)^2 v_4 = (A-2I)v_3 = 0 \Rightarrow (A-2I)^2 v_4 = 0$

$\therefore v_4 \epsilon \text{ Ker}(A-2I)^2, v_4 \notin \text{Ker}(A-2I)$

<u>Finding $\text{Ker}(A-2I)^2$</u>

Let $v \in \text{Ker}(A-2I)^2$

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$\therefore (A-2I)^2 v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0\\ 2 & 3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\Rightarrow\begin{pmatrix} -a-2b\\ 2a+3b\\ 0\\ 0\\ -e\\ -f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

$\Rightarrow a=0,\ b=0,\ e=0,\ f=0$

$$\therefore v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ c\\ d\\ 0\\ 0\end{pmatrix}=c\begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\\ 0\end{pmatrix}+d\begin{pmatrix} 0\\ 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix}$$

So. $\mathrm{Ker}(A-2I)^2=\mathrm{span}\left\{\begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\\ 0\end{pmatrix},\begin{pmatrix} 0\\ 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix}\right\}$

<u>Finding Ker(A-2I)</u>

Let $v \in \mathrm{Ker}(A-2I)$

$(A-2I)\,v=0$ where $v\neq 0$, $v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 1 & -1\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\Rightarrow\begin{pmatrix} -b\\ a+2b\\ c+d\\ -c-d\\ e-2f\\ e-f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$$

$\Rightarrow a=0,\ b=0,\ c=-d,\ e=0,\ f=0$

$$v=\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ -d\\ d\\ 0\\ 0\end{pmatrix}=d\begin{pmatrix} 0\\ 0\\ -1\\ 1\\ 0\\ 0\end{pmatrix}$$

So, $\mathrm{Ker}(A-2I)=\mathrm{span}\left\{\begin{pmatrix} 0\\ 0\\ -1\\ 1\\ 0\\ 0\end{pmatrix}\right\}$

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Now, since, $v_4 \in Ker(A-2I)^2, v_4 \notin Ker(A-2I)$, we consider $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Putting the value of v_4 in equation (2),

$$v_3 = (A-2I)v_4 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Also, we find that $v_3 \in Ker(A-2I)$ as per our condition.

$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. v_3 and v_4 are linearly independent as required.

v_3 is the ordinary eigen vector and v_4 is the generalised eigen vector of rank 2 corresponding to $\lambda=2$. Thus, there are 2 linearly independent generalised eigen vectors corresponding to $\lambda=2$.

<u>Finding eigen vector corresponding to $\lambda=2+i$</u>

$(A-\lambda I)\ v=0$, where $v \neq 0$, $v = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}$, $\lambda=2+i$

$$\rightarrow \begin{pmatrix} -i & -1 & 0 & 0 & 0 & 0 \\ 1 & 2-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-i & 1 & 0 & 0 \\ 0 & 0 & -1 & -1-i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-i & -2 \\ 0 & 0 & 0 & 0 & 1 & -1-i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a=0, b=0, c=0, d=0, e=(1+i)\ f$

$$\Rightarrow v = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = f \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1+i \\ 1 \end{pmatrix}$$

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$\therefore$ An eigen vector corresponding to $\lambda=2+i$ is $v_5=\begin{pmatrix}0\\0\\0\\0\\1+i\\1\end{pmatrix}$

Separating the real and imaginary part of $v_5=\begin{pmatrix}0\\0\\0\\0\\1+i\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}+i\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$

$\therefore Re(v_5)=\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}$, $Im(v_5)=\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}$

Thus, our desired matrix P

$$=[v_1 \quad v_2 \quad v_3 \quad v_4 \quad Im(v_5) \quad Re(v_5)]=\begin{bmatrix}-1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Check: $P^{-1}AP$

$$=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & -1 & 0 & 0 & 0 & 0\\ 1 & 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 3 & -2\\ 0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 4 & 0 & 0 & 0 & 0\\ 3 & 3 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 2 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & -3\\ 0 & 0 & 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}3 & 1 & 0 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & -1\\ 0 & 0 & 0 & 0 & 1 & 2\end{bmatrix}=$$

IX. APPLICATIONS OF JORDAN CANONICAL FORM

Finally, we look at the applications of Jordan Canonical Form of a matrix. The Jordan Canonical form of a matrix gives some insight into the form of the solution of a linear system of differential equations.

A. *Solution of a Linear System of Differential Equations*

We know that the linear system $\dot{\mathbf{x}}=A\mathbf{x}$ where $\mathbf{x} \in \mathbf{R}^n$, A is a nxn matrix and $\dot{\mathbf{x}}=\frac{dx}{dt}=\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$ with the initial condition $\mathbf{x}(0)=\mathbf{x}_0$ has a unique solution given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

So, we wish to calculate e^{At}. But, how to calculate e^{At}?

The most obvious procedure is to take the power series which defines the exponential,

$$e^X=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots+\frac{1}{k!}x^k+\ldots$$

and just formally plug-in x=At. (e^{At} is a nxn matrix, so we have to think of the term "1" as the identity matrix)

Thus, we define $e^{At}=I+At+\frac{1}{2!}(At)^2+\frac{1}{3!}(At)^3+\ldots+\frac{1}{k!}(At)^k+\ldots=\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}$

Now, this calculation can be a bit cumbersome. So, we will make use of JCF to make the computation easier.

The key concept for simplifying the computation of matrix exponentials is that of matrix similarity. And we know that every square matrix A can be put in Jordan Canonical Form J by a similarity transformation i.e., ∃ an invertible matrix P such that $P^{-1}AP=J$.

Now, we will prove some results that will gradually lead us to the solution of linear system.

Result 1: $e^{At}=Pe^{Jt}P^{-1}$ when $P^{-1}AP=J$.

Proof: Since $P^{-1}AP=J \Rightarrow A=PJP^{-1}$

Now, $e^{At}=I+At+\frac{1}{2!}(At)^2+\frac{1}{3!}(At)^3+\ldots+\frac{1}{k!}(At)^k+\ldots$

$=PP^{-1}+P(Jt)P^{-1}+\frac{1}{2!}P(Jt)^2P^{-1}+\frac{1}{3!}P(Jt)^3P^{-1}+\ldots+\frac{1}{k!}P(Jt)^kP^{-1}+\ldots$

$=P\left(I+Jt+\frac{1}{2!}(Jt)^2+\frac{1}{3!}(Jt)^3+\ldots+\frac{1}{k!}(Jt)^k+\ldots\right)P^{-1}$

$=Pe^{Jt}P^{-1}$

When A is a diagonalisable matrix, then $A=PDP^{-1}$ where D is a diagonal matrix (diagonal entries are the eigen values of A).

Thus,in that case, $e^{At}=Pe^{Dt}P^{-1}$

Result 2: $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$ when $D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

Proof: $e^{Dt} = I + Dt + \frac{1}{2!}(Dt)^2 + \frac{1}{3!}(Dt)^3 + \dots + \frac{1}{k!}(Dt)^k + \dots$

$$= \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & & 0 \\ & \lambda_2 t & & \\ & & \ddots & \\ 0 & & & \lambda_n t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (\lambda_1 t)^2 & & & 0 \\ & (\lambda_2 t)^2 & & \\ & & \ddots & \\ 0 & & & (\lambda_n t)^2 \end{bmatrix} +$$

$$\frac{1}{3!}\begin{bmatrix} (\lambda_1 t)^3 & & & 0 \\ & (\lambda_2 t)^3 & & \\ & & \ddots & \\ 0 & & & (\lambda_n t)^3 \end{bmatrix} + \dots + \frac{1}{k!}\begin{bmatrix} (\lambda_1 t)^k & & & 0 \\ & (\lambda_2 t)^k & & \\ & & \ddots & \\ 0 & & & (\lambda_n t)^k \end{bmatrix} + \dots$$

$$= \operatorname{diag}\Big(1 + \lambda_1 t + \frac{1}{2!}(\lambda_1 t)^2 + \frac{1}{3!}(\lambda_1 t)^3 + \dots + \frac{1}{k!}(\lambda_1 t)^k + \dots, 1 + \lambda_2 t + \frac{1}{2!}(\lambda_2 t)^2 + \frac{1}{3!}(\lambda_2 t)^3 + \dots + \frac{1}{k!}(\lambda_2 t)^k + \dots, 1 + \lambda_n t + \frac{1}{2!}(\lambda_n t)^2 + \frac{1}{3!}(\lambda_n t)^3 + \dots + \frac{1}{k!}(\lambda_n t)^k + \dots\Big)$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

But if A is not a diagonalizable matrix, then $A = PJP^{-1}$, where J is the JCF of A with at least one elementary Jordan block of size ≥ 2

It is not very difficult to find exponentials of upper triangular matrices. But the Jordan Canonical Form is not only upper triangular but has even more special structure.

Nilpotent matrix: A nxn matrix matrix N is said to be nilpotent of order k if $N^{k-1} \neq 0$ and $N^k = 0$.

Result 3: If DN=ND, then $e^{D+N} = e^D e^N$

Proof: If DN=ND, then by binomial theorem $(D+N)^n = n!\sum_{j+k=n} \frac{D^j N^k}{j!k!}$

Therefore, $e^{D+N} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{D^j N^k}{j!k!} = \sum_{j=0}^{\infty} \frac{D^j}{j!} \sum_{k=0}^{\infty} \frac{N^k}{k!} = e^D e^N$

9.1.1 Deduction of e^{Jt}

- **When J is Jordan block (of size ≥2) corresponding to real eigen value**

Let J be a mxm Jordan block corresponding to eigen value λ. Then J=D+N where D= $\lambda I_{m \times m}$ and N is a nilpotent matrix of order m and DN=ND

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & & \\ 0 & \dots & & 0 & 1 \\ 0 & \dots & & & 0 \end{bmatrix}, N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & \dots & & & \\ 0 & \dots & & 0 & & 0 \\ 0 & \dots & & & & 0 \end{bmatrix}, \dots, N^{m-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ & & \dots & & & \\ 0 & \dots & & 0 & & 0 \\ 0 & \dots & & & & 0 \end{bmatrix}$$

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Now, since $Jt=Dt+Nt$ and $(Dt)(Nt)=(Nt)(Dt)$, then $e^{Jt}=e^{Dt+Nt}=e^{Dt}e^{Nt}$

Now, $e^{Dt}=\begin{bmatrix} e^{\lambda t} & & & \mathbf{0} \\ & e^{\lambda t} & & \\ & & \ddots & \\ \mathbf{0} & & & e^{\lambda t} \end{bmatrix}=e^{\lambda t}I_{m\times m}$

$e^{Nt}= I+Nt+\frac{1}{2!}(Nt)^2+\frac{1}{3!}(Nt)^3+\ldots+\frac{1}{(m-1)!}(Nt)^{m-1}$

$=\begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 1 \end{bmatrix}+\begin{bmatrix} 0 & t & 0 & \ldots & 0 \\ 0 & 0 & t & \ldots & 0 \\ & & & \ldots & \\ 0 & \ldots & & & 0 \end{bmatrix}+\frac{1}{2!}\begin{bmatrix} 0 & 0 & t^2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & t^2 & \ldots & 0 \\ & & & \ldots & & \\ 0 & \ldots & & & & 0 \end{bmatrix}+\ldots+\frac{1}{(m-1)!}\begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & t^{(m-1)} \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ & & & \ldots & & \\ 0 & \ldots & & & & 0 \end{bmatrix}$

$=\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \ldots & \frac{t^{(m-1)}}{(m-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & & \frac{t^{(m-2)}}{(m-2)!} \\ & & \ldots & & & \\ 0 & \ldots & & & & 1 \end{bmatrix}$

Thus, $e^{Jt}=e^{Dt}e^{Nt}$

$=e^{\lambda t}I_{m\times m}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \ldots & \frac{t^{(m-1)}}{(m-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{(m-2)}}{(m-2)!} \\ & & \ldots & & & \\ 0 & \ldots & & & & 1 \end{bmatrix}e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \ldots & \frac{t^{(m-1)}}{(m-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \ldots & \frac{t^{(m-2)}}{(m-2)!} \\ & & \ldots & & & \\ 0 & \ldots & & & & 1 \end{bmatrix}$ where J be a mxm Jordan block corresponding to eigen value λ.

- **When J is Jordan block corresponding to complex eigen value**

  $J=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is the Jordan block corresponding to complex eigen value $\lambda=a+ib$.

  It follows by induction that, $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^k=\begin{bmatrix} Re(\lambda^k) & -Im(\lambda^k) \\ Im(\lambda^k) & Re(\lambda^k) \end{bmatrix}$ where Re and Im denote the real and imaginary part of the complex eigen value $\lambda=a+ib$.

  Thus, $e^{J}=\sum_{k=0}^{\infty}\begin{bmatrix} Re(\frac{\lambda^k}{k!}) & -Im(\frac{\lambda^k}{k!}) \\ Im(\frac{\lambda^k}{k!}) & Re(\frac{\lambda^k}{k!}) \end{bmatrix}=\begin{bmatrix} Re(\lambda) & -Im(\lambda) \\ Im(\lambda) & Re(\lambda) \end{bmatrix}=e^{a}\begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$

  Therefore, $e^{Jt}=e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$ where J is the Jordan block corresponding to complex eigen value $\lambda=a+ib$

9.1.2 Illustrations through examples

We will find solution of $\dot{\mathbf{x}}=A\mathbf{x}$ with the initial condition $\mathbf{x}(0)=\mathbf{x}_0$. In each example we will vary A and find the solution of the linear system.

Example 1:
Solve the initial value problem $\frac{dx}{dt}=x+2y$, $\frac{dy}{dt}=y$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem $\dot{\mathbf{x}} = A\mathbf{x}$ for $A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ (where $\mathbf{x} = \frac{d\mathbf{x}}{dt}$) is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At} = Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are 1,1. $J=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$. $P=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 1 of 8.1.1)

Now, $e^{Jt}=e^{\lambda t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ when J is a 2x2 Jordan block corresponding to eigen value λ i.e. $J=\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$

$\therefore e^{Jt}=e^{t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ for $J=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$= e^{t}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\mathbf{x}_0=\begin{bmatrix}e^{t} & 2te^{t}\\ 0 & e^{t}\end{bmatrix}\mathbf{x}_0$

$\therefore \mathbf{x}(t)=\begin{bmatrix}e^{t} & 2te^{t}\\ 0 & e^{t}\end{bmatrix}\mathbf{x}_0$

Example 2:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 3\end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At} = Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are 1,1,3. $J=\begin{bmatrix}J_1 & \mathbf{0}\\ \mathbf{0} & J_2\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}$

$P=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 1 of 8.1.2)

Now, $e^{Jt}=e^{\lambda t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$ when J is a 2x2 Jordan block corresponding to eigen value λ.

$\therefore e^{Jt}=\begin{bmatrix}e^{t} & te^{t} & 0\\ 0 & e^{t} & 0\\ 0 & 0 & e^{3t}\end{bmatrix}$ for $J=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}e^{t} & te^{t} & 0\\ 0 & e^{t} & 0\\ 0 & 0 & e^{3t}\end{bmatrix}\begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\mathbf{x}_0=\begin{bmatrix}e^{t} & te^{t} & -\frac{e^{t}}{2}-te^{t}+\frac{1}{2}e^{3t}\\ 0 & e^{t} & -e^{t}+e^{3t}\\ 0 & 0 & e^{3t}\end{bmatrix}\mathbf{x}_0$

$\therefore \mathbf{x}(t)=\begin{bmatrix}e^{t} & te^{t} & -\frac{e^{t}}{2}-te^{t}+\frac{1}{2}e^{3t}\\ 0 & e^{t} & -e^{t}+e^{3t}\\ 0 & 0 & e^{3t}\end{bmatrix}\mathbf{x}_0$

Example 3:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & -1 & 2\end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At} = Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are 2,2,2. $J=\begin{bmatrix}J_1 & \mathbf{0}\\ \mathbf{0} & J_2\end{bmatrix} = \begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$

$P=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 2 of 8.1.2)

Now, $e^{Jt}=e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ when J is a 2x2 Jordan block corresponding to eigen value λ

$\therefore e^{Jt}=\begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$ for $J=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$$=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\mathbf{x}_0=\begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & -te^{2t} & e^{2t} \end{bmatrix}\mathbf{x}_0$$

$$\therefore \mathbf{x}(t)=\begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & -te^{2t} & e^{2t} \end{bmatrix}\mathbf{x}_0$$

Example 4:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At}= Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are 2,2,2. $J=[J_1]=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$P=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 0 \end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 3 of 8.1.2)

Now, $e^{Jt}=e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$ when J is a 3x3 Jordan block corresponding to eigen value λ i.e.

$J=\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$

$\therefore e^{Jt}=\begin{bmatrix} e^{2t} & te^{2t} & e^{2t}\frac{t^2}{2!} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$ for $J=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$$=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & -3 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} & e^{2t}\frac{t^2}{2!} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\mathbf{x}_0=\begin{bmatrix} e^{2t} & 0 & e^{2t} \\ 0 & e^{2t} & 0 \\ \frac{5}{2}t^2e^{2t}-3te^{2t} & 5te^{2t} & e^{2t}+\frac{5}{2}t^2e^{2t}-3te^{2t} \end{bmatrix}\mathbf{x}_0$$

$$\therefore \mathbf{x}(t)=\begin{bmatrix} e^{2t} & 0 & e^{2t} \\ 0 & e^{2t} & 0 \\ \frac{5}{2}t^2e^{2t}-3te^{2t} & 5te^{2t} & e^{2t}+\frac{5}{2}t^2e^{2t}-3te^{2t} \end{bmatrix}\mathbf{x}_0$$

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Example 5:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At}= Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are 5,5,5,5. $J=[\,J_1\,] = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

$P=\begin{bmatrix} 6 & -4 & 4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 3 of 8.1.3)

Now, $e^{Jt}=e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$ when J is a 4x4 Jordan block corresponding to eigen value λ

$e^{Jt}=\begin{bmatrix} e^{5t} & te^{5t} & e^{5t}\frac{t^2}{2!} & e^{5t}\frac{t^3}{3!} \\ 0 & e^{5t} & te^{5t} & e^{5t}\frac{t^2}{2!} \\ 0 & 0 & e^{5t} & te^{5t} \\ 0 & 0 & 0 & e^{5t} \end{bmatrix}$ for $J=\begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$$=\begin{bmatrix} 6 & -4 & 4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} e^{5t} & te^{5t} & e^{5t}\frac{t^2}{2!} & e^{5t}\frac{t^3}{3!} \\ 0 & e^{5t} & te^{5t} & e^{5t}\frac{t^2}{2!} \\ 0 & 0 & e^{5t} & te^{5t} \\ 0 & 0 & 0 & e^{5t} \end{bmatrix}\begin{bmatrix} \frac{1}{6} & \frac{1}{9} & -\frac{8}{27} & 0 \\ 0 & \frac{1}{6} & -\frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x}_0$$

Example 6:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At}= Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are $\pm 2i$. $J=\begin{bmatrix} & \\ 2 & 0 \end{bmatrix}$, $P=\begin{bmatrix} & \\ 0 & 1 \end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 2 of 8.2.1)

$e^{Jt}=e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$ where J is the Jordan block corresponding to complex eigen value $\lambda=a+ib$ i.e. $J=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$e^{Jt}=\begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}$ for $J=\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$$=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}_0=\begin{bmatrix} \cos 2t & -\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix}\mathbf{x}_0$$

$$\therefore \mathbf{x}(t)=\begin{bmatrix} \cos 2t & -\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix}\mathbf{x}_0$$

Example 7:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At}= Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are $1 \pm i$. $J=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $P=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 1 of 8.2.1)

$e^{Jt}=e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$ where J is the Jordan block corresponding to complex eigen value $\lambda=a+ib$

$\therefore e^{Jt}=e^t\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ for $J=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$$=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^t\cos t & -e^t\sin t \\ e^t\sin t & e^t\cos t \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}_0=\begin{bmatrix} e^t\cos t & e^t\sin t \\ -e^t\sin t & e^t\cos t \end{bmatrix}\mathbf{x}_0$$

$$\therefore \mathbf{x}(t)=\begin{bmatrix} e^t\cos t & e^t\sin t \\ -e^t\sin t & e^t\cos t \end{bmatrix}\mathbf{x}_0$$

Example 8:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At}= Pe^{Jt}P^{-1}$(from Result 1)

For this A, eigen values are $1 \pm i$, $2 \pm i$. $J=\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, $P=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 1 of 8.2.2)

$e^{Jt}=e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$ where J is the Jordan block corresponding to complex eigen value $\lambda=a+ib$

$\therefore e^{Jt}=e^t\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ for $J_1=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $e^{Jt}=e^{2t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$ for $J_2=\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

$$e^{Jt}=\begin{bmatrix} e^t\cos t & -e^t\sin t & 0 & 0 \\ e^t\sin t & e^t\cos t & 0 & 0 \\ 0 & 0 & e^{2t}\cos t & -e^{2t}\sin t \\ 0 & 0 & e^{2t}\sin t & e^{2t}\cos t \end{bmatrix} \text{ for } J=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

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$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}e^t\cos t&-e^t\sin t&0&0\\e^t\sin t&e^t\cos t&0&0\\0&0&e^{2t}\cos t&-e^{2t}\sin t\\0&0&e^{2t}\sin t&e^{2t}\cos t\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}x_0$$

$$=\begin{bmatrix}e^t\cos t&-e^t\sin t&0&0\\e^t\sin t&e^t\cos t&0&0\\0&0&e^{2t}(\cos t+\sin t)&-2e^{2t}\sin t\\0&0&e^{2t}\sin t&e^{2t}(\cos t-\sin t)\end{bmatrix}\mathbf{x}_0$$

$$\mathbf{x}(t)=\begin{bmatrix}e^t\cos t&-e^t\sin t&0&0\\e^t\sin t&e^t\cos t&0&0\\0&0&e^{2t}(\cos t+\sin t)&-2e^{2t}\sin t\\0&0&e^{2t}\sin t&e^{2t}(\cos t-\sin t)\end{bmatrix}\mathbf{x}_0$$

Example 9:

Solve the initial value problem $\dot{\mathbf{x}}=A\mathbf{x}$ for $A=\begin{bmatrix}-3&0&0\\0&3&-2\\0&1&1\end{bmatrix}$ with initial condition $\mathbf{x}(0)=\mathbf{x}_0$.

Solution: The solution to the initial value problem is given by $\mathbf{x}(t)=e^{At}\mathbf{x}_0$.

$e^{At}= Pe^{Jt}P^{-1}$ (from Result 1)

For this A, eigen values are $-3, 2 \pm i$. $J=\begin{bmatrix}J_1&0\\0&J_2\end{bmatrix}=\begin{bmatrix}-3&0&0\\0&2&-1\\0&1&2\end{bmatrix}$. $P=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}$ such that $P^{-1}AP=J$. (Refer to Example 1 of 8.3.1)

$e^{Jt}=e^{at}\begin{bmatrix}\cos bt&-\sin bt\\\sin bt&\cos bt\end{bmatrix}$ where J is the Jordan block corresponding to complex eigen value $\lambda=a+ib$

$\therefore e^{Jt}=e^{2t}\begin{bmatrix}\cos t&-\sin t\\\sin t&\cos t\end{bmatrix}$ for $J_2=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$

$\therefore e^{Jt}=\begin{bmatrix}e^{-3t}&0&0\\0&e^{2t}\cos t&-e^{2t}\sin t\\0&e^{2t}\sin t&e^{2t}\cos t\end{bmatrix}$ for $J=\begin{bmatrix}-3&0&0\\0&2&-1\\0&1&2\end{bmatrix}$

Thus, $\mathbf{x}(t)= Pe^{Jt}P^{-1}\mathbf{x}_0$

$$=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}e^{-3t}&0&0\\0&e^{2t}\cos t&-e^{2t}\sin t\\0&e^{2t}\sin t&e^{2t}\cos t\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-1\\0&0&1\end{bmatrix}\mathbf{x}_0=\begin{bmatrix}e^{-3t}&0&0\\0&e^{2t}(\cos t+t\sin t)&-2e^{2t}\sin t\\0&e^{2t}\sin t&e^{2t}(\cos t-\sin t)\end{bmatrix}\mathbf{x}_0$$

$$\therefore \mathbf{x}(t)=\begin{bmatrix}e^{-3t}&0&0\\0&e^{2t}(\cos t+t\sin t)&-2e^{2t}\sin t\\0&e^{2t}\sin t&e^{2t}(\cos t-\sin t)\end{bmatrix}\mathbf{x}_0$$

B. Phase Portraits of Linear Systems

Consider a system of linear differential equation $\dot{\mathbf{x}}$=Ax. Its phase portrait is a representative set ofits solutions, plotted as parametric curves(with t as the parameter)on the Cartesian plane tracing the path of each particular solution $(x, y)=(x_1(t), x_2(t))$, $-\infty < t < \infty$.Similar to a direction field, a phase portrait is a graphical tool to visualize how the solutions of a given system of differential equations would behave in the long run.

In this context, the Cartesian plane where the phase portrait resides is called the phase plane. The parametric curves traced by the solutions are sometimes called their trajectories.

It is quite labour-intensive, but it is possible to sketch the phase portrait by hand without first having to solve the system of equations that it represents. Just like a direction field, a phase portrait can be a tool to predict the behaviours of a system's solutions.

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1) Equilibrium Solution (Critical point or Stationary point)

An equilibrium solution of the system $\dot{x}$=Ax is a point (x1, x2) where $\dot{x}$=0, that is, where $\dot{x}_1$ =0=$\dot{x}_2$ .An equilibrium solution is a constant solution of the system and is usually called a critical point. For a linear system $\dot{x}$=Ax, an equilibrium solution occurs at each solution of the system (of homogeneous algebraic equations) Ax=0. As we have seen, such a system has exactly one solution, located at the origin, if det(A)≠0. If det(A)=0, then there are infinitely many solutions.

For our purpose, and unless otherwise noted, we will consider systems of linear differential equations whose coefficient matrix A has nonzero determinant. That is, we will consider systems where origin is the only critical point.

2) Classification of Critical Points

We will presently classify the critical points of various systems of first order linear differential equations by their stability. In addition, due to the truly two-dimensional nature of the parametric curves, we will also classify the type of those critical points by their shapes (or rather, by the shape formed by trajectories about each critical point).

3) Stability Classification of Critical Points

a) *Asymptotically Stable:* All trajectories of its solutions converge to the critical point as t→ ∞.A critical point is asymptotically stable if all of A's eigenvalues have negative real part for complex eigenvalues.
b) *Unstable:* All trajectories (or all but a few, in the case of saddle point) start out at the critical point at t→ −∞,then move away to infinitely distant out as t→ ∞.A critical point is unstable if one of A's eigenvalues is positive and other negative or has positive real part for complex eigenvalues.
c) *Stable (or neutrally stable):* Each trajectory moves about the critical point within a finite range of distance. It never moves out to infinitely distant, nor (unlike in the case of asymptotically stable) does it ever go to the critical point. A critical point is stable if A's eigenvalues are purely imaginary.

In short, as t increases, if all (or almost all) trajectories

- Converge to the critical point - asymptotically stable
- Move away from the critical point to infinitely far away – unstable
- Stay in a fixed orbit within a finite (i.e., bounded) range of distance away from the critical point stable (or neutrally stable)

Here, we will discuss the various phase portraits that are possible for the linear system

$$\dot{\mathbf{x}}=A\mathbf{x} \quad \ldots (1)$$

when $\mathbf{x} \in \mathbf{R}^2$ and A is a 2x2 matrix.

We begin by describing the phase portraits for the linear system

$$\dot{\mathbf{x}}=J\mathbf{x} \quad \ldots \quad (2)$$

where the matrix $P^{-1}AP=J$ where J has the following forms: $J=\begin{bmatrix} & \\ 0 & \mu \end{bmatrix}$, $J=\begin{bmatrix} & \\ 0 & \lambda \end{bmatrix}$ or $J=\begin{bmatrix} & \\ b & a \end{bmatrix}$

The phase portrait for the linear system (1) above then is obtained from the phase portrait (2) We have seen the solution of linear system (1) with the initial value $\mathbf{x}(0)=\mathbf{x}_0$ is given by

$$\mathbf{x}(t)= \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0 , \quad \mathbf{x}(t)= e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0 \quad \text{or } \mathbf{x}(t)= e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$$

We now list the various phase portraits that result from these solutions, grouped according to their topological type with a finer classification of sources, sinks into various types of stable, unstable nodes and foci.

Given $\dot{\mathbf{x}}=A\mathbf{x}$, where there is only one critical point at (0,0).

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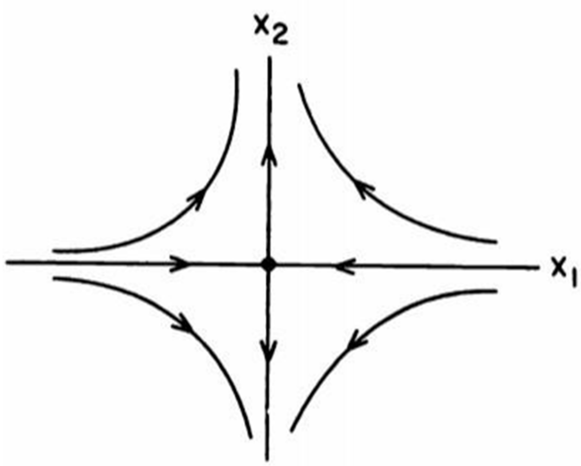
Case 1: Distinct real eigen values λ, μ with $\lambda < 0 < \mu$

$$J=\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

In this case, the trajectories given by the eigen vectors of the negative eigen value initially start at the infinite-distant away, move toward and eventually converge at the critical point. The trajectories that represent the eigen vectors of the positive eigenvalue move in exactly the opposite way: start at the critical point, then diverge to infinite-distant out. Every other trajectory starts at infinite-distant away, moves toward but never converges to the critical point, before changing direction and moves back to infinite distant away. All the while it would roughly follow the 2 sets of eigenvectors.
This type of critical point is called a saddle point. It is always unstable.

The phase portrait in this case is given in the figure.



A saddle at the origin

If $\mu < 0 < \lambda$, the arrows in the figure are reversed.

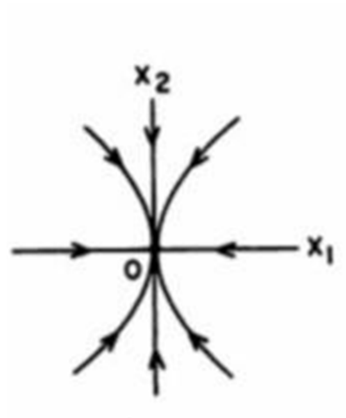
Case 2: Distinct real eigen values λ, μ, both positive or both negative

$$J=\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

In this case, the phase portrait shows trajectories either moving away from the critical point to infinite-distant away (when both positive) or moving directly toward and converge to the critical point (when both negative). The trajectories that are the eigenvectors move in straight lines. The rest of the trajectories move, initially when near the critical point, roughly in the same direction as the eigenvector of the eigenvalue with the smaller absolute value. Then, further away, they would bend toward the direction of the eigenvector of the eigenvalue with the larger absolute value. The trajectories either move away from the critical point to infinite-distant away (when both are positive) or move toward infinite-distant out and eventually converge to the critical point (when both are negative).

This type of critical point is called improper node. It is asymptotically stable if both eigenvalues are negative and unstable if both eigenvalues are positive.

The phase portrait in this case is given in the figure below.



An asymptotically stable node at the origin ($\lambda < \mu < 0$)

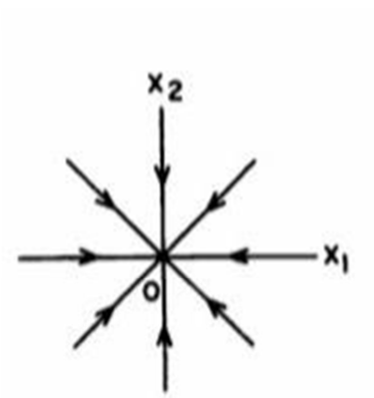
Case 3: Repeated eigen values, there are two linearly independent eigenvectors corresponding to λ

$$J=\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

The phase portrait in this case has a distinct star-burst shape. The trajectories either move directly away from the critical point to infinite-distant away (when $\lambda>0$) or move directly outward and converge to the critical point (when $\lambda<0$)

This type of critical point is called a proper node. It is asymptotically stable if $\lambda<0$, unstable if $\lambda>0$.

The phase portrait in this case is given in the figure below



An asymptotically stable node at the origin. ($\lambda < 0$)

If $\lambda>0$, the arrows are reversed and the origin is referred to as an unstable node.

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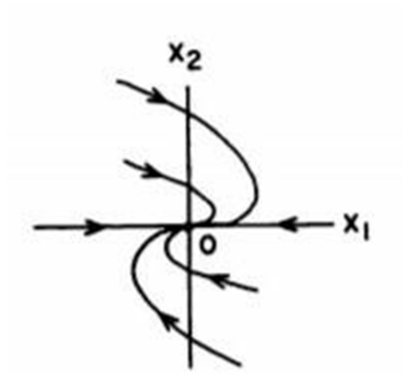
Case 4: Repeated eigen values, there is only one linearly independent eigenvectors corresponding to λ

$$J=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

The phase portrait in this case shares characteristics with that of a node. With only one eigenvector, it is a degenerated-looking node that is a cross between a node and spiral point. The trajectories either all diverge away from the critical point to infinite-distant away (when $\lambda>0$), or all converge to the critical point (when $\lambda<0$)

This type of critical point is called an improper node. It is asymptotically stable if $\lambda<0$ and unstable if $\lambda>0$.



An asymptotically stable node at the origin ($\lambda< 0$)

If $\lambda>0$, the arrows are reversed and the origin is referred to as an unstable node.

Case 5: Complex conjugate eigenvalues

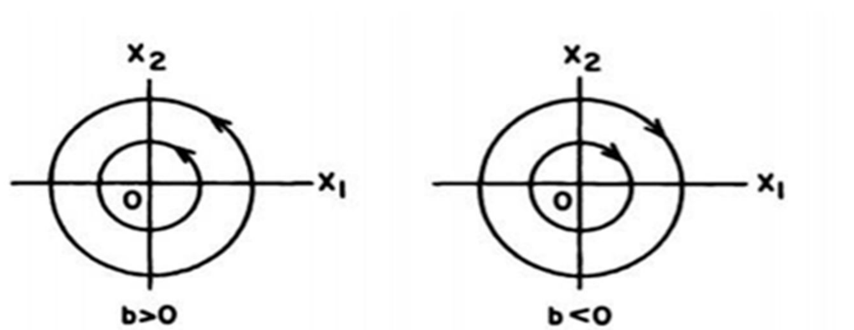
i) When the real part is zero

$$J=\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$$

In this case, trajectories neither converge to the critical point nor move to infinite-distant away. Rather, they stay in constant, elliptical (or rarely circular) orbits.

This type of critical point is called a centre. It is always stable (or neutrally stable)

The phase portrait in this case is given in the figure below



A center at the origin

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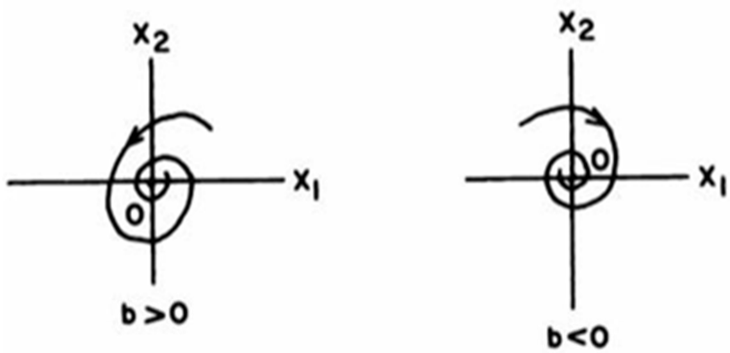
ii) When the real part is nonzero $J=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

The trajectories still retain the elliptical traces as in the previous case. However, with each revolution, their distances from the critical point grow/decay exponentially according to the term e^{at}.Therefore, the phase portrait shows trajectories that spiral away from the critical point to infinite-distant away (when a>0) or trajectories that spiral toward and converge to the critical point (when a<0).
This type of critical point is called a spiral point. It is asymptotically stable if a<0 and unstable if a>0.

The phase portrait in this case is given in the figure below



An asymptotically stable spiral at the origin(a<0)

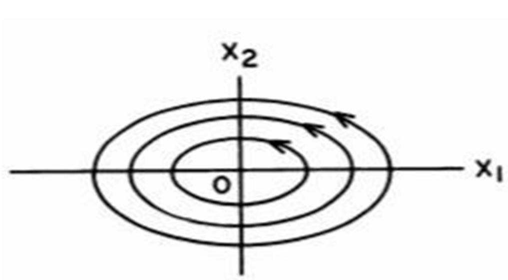
If a>0, the trajectories spiral away from the origin with increasing t and the origin is called an unstable spiral.

Let us see through an **example**.
The solution to the linear system $\dot{\mathbf{x}}=A\mathbf{x}$ where $A=\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$ (eigen values are $\pm 2i$)is given by

$\mathbf{x}(t)=\begin{bmatrix} \cos 2t & -\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix}\mathbf{c}$ where $\mathbf{c}=\mathbf{x}_0$ or equivalently, $x_1(t)=c_1\cos 2t-2c_2\sin 2t$

$$x_2(t)=\frac{1}{2c_1}\sin 2t+c_2\cos 2t$$

(Refer to Example 6 of 9.1.2)
Now, the solutions satisfy $x_1^2(t)+4\,x_2^2(t)=c_1^2+4\,c_2^2$ for all $t\in R$ i.e., the trajectories of this system lie on ellipses as shown in the figure below.



A center at the origin

X. CONCLUSION

The Jordan Canonical Form describes the structure of an arbitrary linear transformation on a finite-dimensional vector space over an algebraically closed field. Here we develop it only using the basicconcepts of linear algebra, with no reference to determinants or ideals of polynomials.

In this project, we have talked about how to explicitly compute Jordan forms.

A. Uses of JCF

1) Over an algebraically closed field, which matrices are diagonalisable? Diagonalisable matrices are those matrices which have all regular eigenvalues i.e. geometric multiplicity of each eigen value is equal to its algebraic multiplicity. But there exist non-diagonalisable matrices too. For such non-diagonalisable matrices, there exist atleast one eigenvalue for which geometric multiplicity is less than its algebraic multiplicity. Diagonalisable matrices are similar to a diagonal matrix and non-diagonalisable matrices are similar to JCF of the matrix. Diagonal matrix is special form of JCF where each Jordan block is of size 1. The question posed above can also be answered by looking at the structure of JCF. JCF of a diagonalisable matrix has all Jordan blocks of size 1. JCF of a non-diagonalisable matrix has atleast one Jordan block of size ≥ 2.

2) The JCF presents all the important data about a matrix-the list of eigenvalues, eigendimension , generalised eigendimension associated to each eigen value and the minimal and characteristic polynomials in a readable form.

3) When does minimal polynomial coincide with the characteristic polynomial? The characteristic polynomial of a matrix Anxn equals the minimal polynomial of Anxn if and only if the dimension of each eigenspace of A is 1 i.e. the matrix has n distinct eigenvalues. If a matrix has n distinct eigenvalues, then JCF of the matrix will have all Jordan blocks of size 1 corresponding to n distinct eigenvalues. Therefore, also by looking at the structure of JCF of a matrix, we can say whether the minimal polynomial coincide with the characteristic polynomial or not?

4) Over an algebraically closed field F, the JCF is a complete invariant for conjugacy. This means the following for A, B $\in$ M (n, F), we have that A is conjugate to B iff the JCF of A and B are the same upto the permutation of the blocks. The fact that JCF is a complete invariant for conjugacy is all the more interesting since the minimal and the characteristic polynomial together do not form a complete conjugacy invariant. Also, the JCF over C is a complete conjugacy invariant for square matrices over R even though R is not algebraically closed.

5) JCF is useful in solving the system of linear differential equations $\dot{\mathbf{x}}$=Ax, where A need not be diagonalisable.

XI. ACKNOWLEDGEMENT

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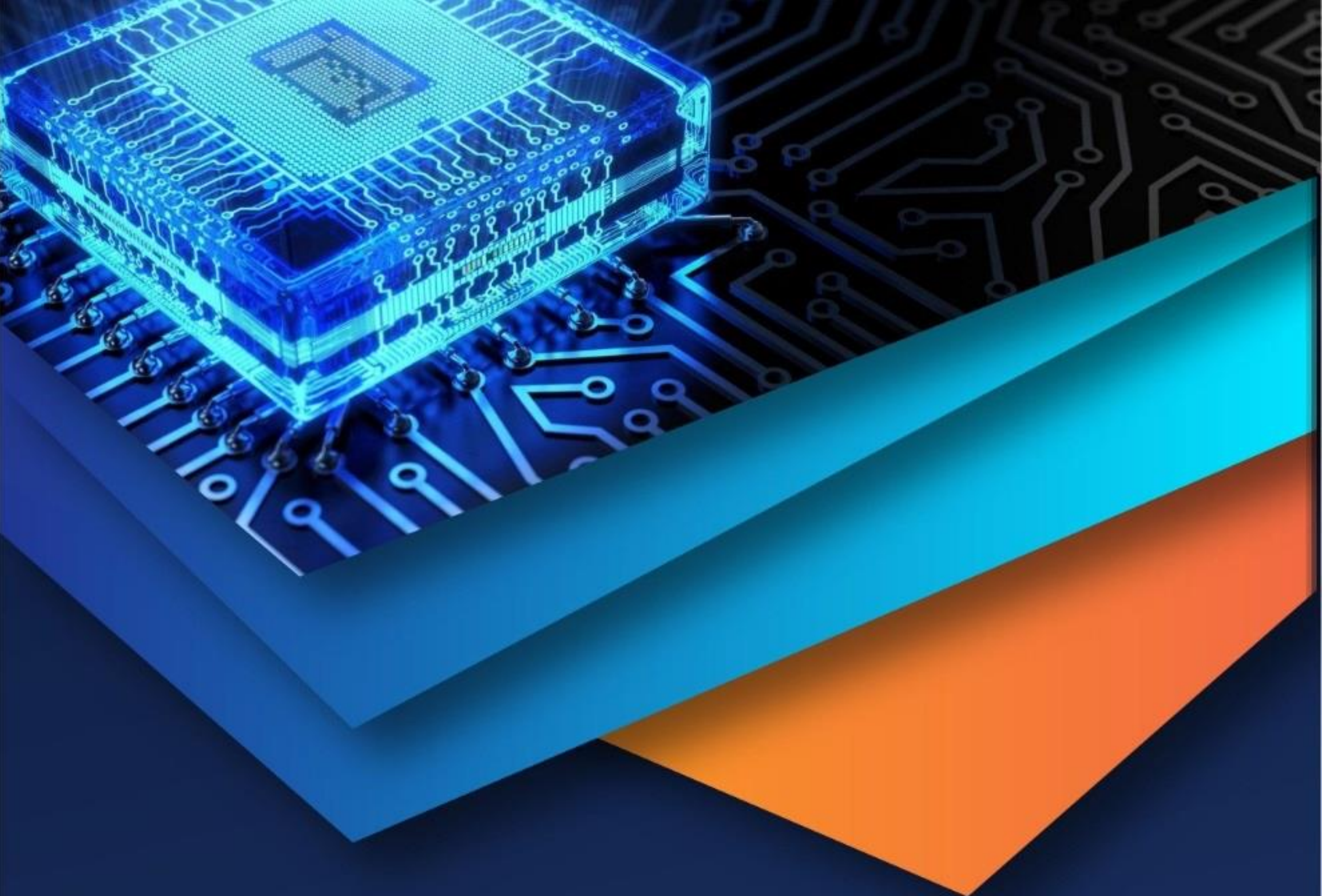

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