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Non-Trivial Integral Solutions of Ternary Quadratic Diophantine Equation

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Abstract: The ternary quadratic equation $\alpha^2 + \beta^2 - 14 \propto +10\beta = 74(\gamma^2 - 1)$ representing cone is analyzed by its nonzero distinct integer points on it. Employing the integer solutions, a few relations between the solutions and some patterns are presented.

I. INTRODUCTION

Diophantine equations have been studied for centuries and have practical applications in various areas of mathematics, including number theory, cryptography, and geometry. Solving Diophantine equations can be challenging, and depending on the complexity of the equation; finding solutions might not always be straightforward. There are numerous Diophantine equations that have finitely many, infinitely many, trivial, or none of these solutions. The intriguing ternary quadratic equation $\alpha^2 + \beta^2 - 14 \propto +10\beta = 74(\gamma^2 - 1)$ in this article that represents an endless cone is used to calculate numerous non-zero lattice points.

II. METHOD OF ANALYSIS

The ternary quadratic Diophantine equation studied for its non-zero distinct integer solutions is given by

$$\alpha^2 + \beta^2 - 14 \propto +10\beta = 74(\gamma^2 - 1)$$

$$\alpha^2 - 14 \propto +49 + \beta^2 + 10\beta + 25 = 74\,\gamma^2$$

$$(\propto -7)^2 + (\beta + 5)^2 = 74\,\gamma^2 \qquad (1)$$

Take $\varphi = \propto -7,\ \rho = \beta + 5$.

$$\varphi^2 + \rho^2 = 74\,\gamma^2 \qquad (2)$$

However, we have supplementary patterns of solutions which are illustrated as follows:

A. *Pattern 1.*

Let $\varphi = 7a + 5b$ and $\rho = 5a - 7b$. Then $74(a^2 + b^2) = 74\,\gamma^2$ and the equation (2) becomes $a^2 + b^2 = \gamma^2$.

We get $a = l\,(p^2 - q^2)$, $b = 2lpq$, $\gamma = l(p^2 + q^2)$, for some integers l, p and q with $p > q$, hence the solution is

$$\propto = l(7p^2 - 7q^2 + 10pq) + 7$$

$$\beta = l(5p^2 - 5q^2 - 14pq) - 5$$

$$\gamma = l(p^2 + q^2)$$

B. *Pattern 2 .*

Let $\varphi = 7a - 5b$ and $\rho = 5a + 7b$. Then $74(a^2+b^2) = 74\,\gamma^2$ and the equation (2) becomes $a^2+b^2 = \gamma^2$.

We obtain $a = l(p^2 - q^2), b = 2lpq, \gamma = l(p^2 + q^2)$, for some integers l, p and q with $p > q$, hence the solution is

$$\propto = l(7p^2 - 7q^2 - 10pq) + 7$$

$$\beta = l(5p^2 - 5q^2 + 14pq) - 5$$

$$\gamma = l(p^2 + q^2).$$

C. *Pattern 3.*

Assume $\gamma = \gamma(\vartheta, \mu) = \vartheta^2 + \mu^2$, where $\vartheta,\ \mu > 0$ (3)

Write $74 = 7^2 + 5^2$ as $7^2 + 5^2 = (7 + 5i)(7 - 5i)$ (4)

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Substituting (3) and (4) in (2) and employing the development of factorization.

Write $\varphi + i\rho = (7+5i)(\vartheta + i\mu)^2$

Equating the real and imaginary parts, we obtain

$\varphi = 7\vartheta^2 - 7\mu^2 - 10\vartheta\mu$, $\rho = 5\vartheta^2 - 5\mu^2 + 14\vartheta\mu$ here $\vartheta > \mu$ and $\vartheta, \mu \in Z - \{0\}$.

Hence the solution is

$$\propto = 7(\vartheta^2 - \mu)^2 - 10\vartheta\mu + 7$$
$$\beta = 5(\vartheta^2 - \mu^2) + 14\vartheta\mu - 5$$
$$\gamma = \vartheta^2 + \mu^2.$$

D. Pattern 4.

We can also write 74 = $7^2 + 5^2$ as $7^2 + 5^2 = (5+7i)(5-7i)$ (5)

Substituting (3) and (5) in (1) and employing the technique of factorization,

Write $\varphi + i\rho = (5+7i)(\vartheta + i\mu)^2$.

Equating the real and imaginary parts, we get $\propto = 5(\vartheta^2 - \mu)^2 - 14\vartheta\mu + 7$

$$\beta = 7(\vartheta^2 - \mu^2) + 10\vartheta\mu - 5$$
$$\gamma = \vartheta^2 + \mu^2.$$

Which represents the dissimilar integer points on the cone (1), $(\propto -7)^2 + (\beta + 5)^2 = 74\gamma^2$.

E. Pattern 5.

Equation (4) can also be written in the subsequent technique:

$$74 = 7^2 + 5^2 = (-7+5i)(-7-5i)$$

Proceeding as above, we get

$$\propto = -7(\vartheta^2 - \mu^2) - 10\vartheta\mu + 7$$
$$\beta = 5(\vartheta^2 - \mu^2) - 14\vartheta\mu - 5$$
$$\gamma = \vartheta^2 + \mu^2, \quad \vartheta^2 > \mu^2 \text{ And } \vartheta, \mu \in Z - \{0\}.$$

F. Pattern 6.

Equation (4) can also be written in the following technique:

$$74 = 7^2 + 5^2 = (-5+7i)(-5-7i)$$

Proceeding as above, we get

$$\propto = -5(\vartheta^2 - \mu^2) - 14\vartheta\mu + 7$$
$$\beta = 7(\vartheta^2 - \mu^2) - 10\vartheta\mu - 5$$
$$\gamma = \vartheta^2 + \mu^2, \quad \vartheta^2 > \mu^2 \text{ And } \vartheta, \mu \in Z - \{0\}.$$

G. Pattern 7.

Equation (1) can be written as $\frac{\varphi+7\gamma}{5\gamma+\rho} = \frac{5\gamma-\rho}{\varphi-7\gamma} = \frac{c}{d}(say), d \neq 0$ (6)

This equation is equal to the following two equations:

$$d\varphi - c\rho + (7d - 5c)\gamma = 0.$$
$$c\varphi + d\rho - (7c + 5d)\gamma = 0.$$

By using the system of cross-multiplication, we get the integral solutions of (1) to be

$$\propto = 7(c^2 - d^2) + 10cd + 7$$
$$\beta = 5(c^2 - d^2) - 14cd - 5$$
$$\gamma = c^2 + d^2.$$

H. Pattern 8.

Equation (1) can be written as $\frac{5\gamma+\rho}{\varphi+7\gamma} = \frac{\varphi-7\gamma}{5\gamma-\rho} = \frac{c}{d}(say), d \neq 0$

This equation is consequent to the following two equations:

$$c\varphi - d\rho + (7c - 5d)\gamma = 0$$
$$d\varphi + c\rho - (7d + 5c)\gamma = 0$$

By the way of cross-multiplication, we get the integral points of (1) to be

$$\propto = d(7d + 5c) - c(7c - 5d) + 7$$

$$\beta = c(7d + 5c) + d(7c - 5d) - 5$$
$$\gamma = c^2 + d^2.$$

Note 1. Equation (2) can be written as

$$\frac{\varphi - 7\gamma}{5\gamma + \rho} = \frac{5\gamma - \rho}{\varphi + 7\gamma} = \frac{c}{d}(say), d \neq 0$$

Proceeding as above, $\propto = -c\,(7c - 5d) + d\,(7d + 5c) + 7$

$$\beta = -c\,(7d + 5c) - d(7c - 5d) - 5$$
$$\gamma = c^2 + d^2.$$

Note 2. Applying the above technique to the case,

$$\frac{\varphi - 7\gamma}{5\gamma - \rho} = \frac{5\gamma + \rho}{\varphi + 7\gamma} = \frac{c}{d}(say), d \neq 0$$

We get hold of the solution of (1) as

$$\propto = c\,(7c - 5d) - d\,(7d + 5c) + 7$$
$$\beta = -c\,(7d + 5c) - d(7c - 5d) - 5$$
$$\gamma = -d^2 - c^2.$$

I. Pattern 9.

Equation (2) can be written as $(7^2 + 5^2)\gamma^2 - \rho^2 = \varphi^2 * 1$ (7)

Assume $\varphi = (7^2 + 5^2)(\vartheta^2 - \mu^2), where\ \vartheta, \mu > 0$ (8)

Write 1 as $1 = \frac{(\sqrt{7^2+5^2}+7)(\sqrt{7^2+5^2}-7)}{5^2}$ (9)

Using (8) and (9) in (7) and applying the manner of factorization,

$$(\sqrt{7^2 + 5^2}\,\gamma + \rho = \frac{\left(\sqrt{(7^2 + 5^2)}\,\vartheta + \mu\right)^2 (\sqrt{7^2 + 5^2} + 7)}{5}$$

Equating the rational and irrational factors, we get

$$\alpha = (7^2 + 5^2)(\vartheta^2 - \mu^2) + 7$$
$$\beta = \frac{1}{5}[7(7^2 + 5^2)(\vartheta^2 + \mu^2) + 2(7^2 + 5^2)\vartheta\mu] - 5$$
$$\gamma = \frac{1}{5}[(7^2 + 5^2)(\vartheta^2 + \mu^2) + 14\vartheta\mu].$$

Because our concentration centers on integral solutions, substitute ϑ by 5χ and μ by 5ω in the equations above. Consequently the equivalent solutions to (1) are given by

$$\alpha = 5^2(7^2 + 5^2)\chi^2 - \omega^2 + 7$$
$$\beta = 5[7(74)(\chi^2 + \omega^2) + 2(74)\chi\omega] - 5$$
$$\gamma = 5[(74)\chi^2 + \omega^2 + 14\chi\omega].$$

Note 3. Equation (9) can be written as

$$1 = \frac{(-\sqrt{7^2 + 5^2} + 7)(-\sqrt{7^2 + 5^2} - 7)}{5^2}$$

Proceeding as above, we get

$$\alpha = 5^2(74)\chi^2 - \omega^2] + 7$$
$$\beta = 5[7(74)\chi^2 + \omega^2) - 2(7^2 + 5^2)\chi\omega] - 5$$
$$\gamma = 5[6\chi\omega - (74)\chi^2 - \omega^2]$$

J. Pattern 10.

Instead of (8) we can also write 1 as

$$1 = \frac{(\sqrt{(7^2 + 5^2)} + 5)(\sqrt{(7^2 + 5^2)} - 5)}{7^2}$$

Thus the similar solutions to (1) are agreed for the choice $\vartheta = 7\chi$ and $\mu = 7\omega$ by

$$\alpha = 7^2[(74)\chi^2 - \omega^2] + 7$$
$$\beta = 7[5((74)\chi^2 + \omega^2) + 2(74)\chi\omega] - 5$$
$$\gamma = 7[(74)\chi^2 + \omega^2 + 10\chi\omega]$$

Note 4 . Equation (8) can be written as

$$1 = \frac{\left(-\sqrt{(7^2 + 5^2)} + 5\right)\left(-\sqrt{(7^2 + 5^2)} - 5\right)}{7^2}$$

Proceeding as above, we get,

$$\alpha = 7^2[(74)\chi^2 - \omega^2] + 7$$
$$\beta = 7[5(74)\chi^2 + \omega^2) - 2(74)\chi\omega] - 5$$
$$\gamma = 7[10\chi\omega - (74)\chi^2 - \omega^2].$$

III. CONCLUSION

In this paper, we have made an attempt to obtain infinitely many non-zero distinct integer solutions to the equation given by $\alpha^2 + \beta^2 - 14 \propto +10\beta = 74(\gamma^2 - 1)$. As ternary quadratic equations are rich in variety, one may search for the other choice of ternary quadratic Diophantine equations and determine their integer solutions along with suitable properties.

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