



# IJRASET

International Journal For Research in  
Applied Science and Engineering Technology



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# INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

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**Volume:** 10    **Issue:** VIII    **Month of publication:** August 2022

**DOI:** <https://doi.org/10.22214/ijraset.2022.46253>

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# A Note on the Wedderburn Decomposition and Unit Group of the Semisimple Group Algebra $\mathbb{F}_p\mathcal{S}_n$

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**Abstract:** In this short note, we give an algorithm to compute the unit group of a semisimple group algebra  $\mathbb{F}_p\mathcal{S}_n$  for any  $n \geq 5$ . To show the practicality of the algorithm, we explicitly find the unit groups of the semisimple group algebras  $\mathbb{F}_p\mathcal{S}_5$ ,  $\mathbb{F}_p\mathcal{S}_6$ ,  $\mathbb{F}_p\mathcal{S}_7$  and  $\mathbb{F}_p\mathcal{S}_8$ .

**Keywords:** Group algebra, Wedderburn decomposition, Unit group, Symmetric group, Young-Tableaux

## I. UNIT GROUP OF $\mathbb{F}_p\mathcal{S}_n$

Let  $\mathbb{F}_p$  be a field, where  $p$  is a prime. The study of unit groups of the semisimple group algebras is a classical research problem (cf. [1, 7, 15, 8, 9, 10, 11, 13] and the references therein). The unit groups of semisimple group algebras of symmetric groups  $\mathcal{S}_3$  and  $\mathcal{S}_4$  have been studied in [5, 14]. In this paper, we continue in this direction and deduce the Wedderburn decomposition of the semisimple group algebra  $\mathbb{F}_p\mathcal{S}_n$  for any  $n \geq 5$  and as a by-product obtain the unit group. It is well known that for a finite group  $G$ , the group algebra  $\mathbb{F}_pG$  is semisimple if and only if  $p$  does not divide  $|G|$  [6]. Consequently,  $\mathbb{F}_pG$  can be written as a direct sum of matrix rings over finite fields of characteristic  $p$ . To be more precise, we have  $\mathbb{F}_pG \cong \bigoplus_{t=1}^j M_{n_t}(\mathbb{F}_{p^t})$  for any semisimple group algebra  $\mathbb{F}_pG$ . Let  $\ell$  be the exponent of  $G$  and  $\zeta$  be the primitive  $\ell^{\text{th}}$  root of unity. In accordance to [2], we define  $I_{\mathbb{F}} = \{r \mid \mathfrak{S} \mapsto \zeta^r \mathfrak{S}$  is an automorphism of  $\mathbb{F}(\mathfrak{S})$  over  $\mathbb{F}\}$ . An element  $h \in G$  is a  $p$ -regular element if its order is not divisible by  $p$ . For a  $p$ -regular element  $h \in G$ , let the sum of all the conjugates of  $h$  be denoted by  $\gamma_h$ , and the cyclotomic- $\mathbb{F}$  class of  $\gamma_h$  be denoted by  $\mathcal{S}(\gamma_h) = \{\gamma_{h^r} \mid r \in I_{\mathbb{F}}\}$ . On utilizing [2, Proposition 1.2] and the result that symmetric group  $\mathcal{S}_n$  splits over every field, one can see that  $\mathbb{F}_p\mathcal{S}_n \cong \bigoplus_{t=1}^j M_{n_t}(\mathbb{F}_p)$  (i.e.  $p_t = p$  for all  $t = 1$  to  $j$ ), where  $j$  is the number of conjugacy classes of  $\mathcal{S}_n$  that are further equal to partitions  $p(n)$  of  $n$ . Consequently, the  $n_t$ 's appearing in the Wedderburn decomposition of  $\mathbb{F}_p\mathcal{S}_n$  are nothing but the dimensions of irreducible representations of  $\mathbb{F}_p\mathcal{S}_n$ . To this end, we now show the following.

**Theorem 1.** The  $n_t$ 's appearing in the Wedderburn decomposition of  $\mathbb{F}_p\mathcal{S}_n$  are equal to the number of standard Young-Tableaux of all the partitions of  $n$ .

*Proof.* It is known that the  $p$ -restricted partitions of  $n$  indexes the irreducible representations of  $\mathbb{F}_p\mathcal{S}_n$  (cf. [4]). The  $p$ -restricted partitions of  $n$  are all those integer partitions in which the difference between successive parts can be at most  $p - 1$ . It turns out that in the semisimple case (since  $p \geq n + 1$ ), the irreducible representations are given by the Specht modules (cf. [4] for their definition) - in the modular case, the irreducible representations are quotients of Specht modules (obtained by modular reduction of a lattice in  $\mathbb{Q}$ -irreducible). Moreover, the quotients of Specht modules are indexed by partitions of  $n$  and the respective dimension is equal to the number of standard Young-Tableaux of the respective partition type. Further, the number of standard Young-Tableaux of a partition can be obtained by Hook length formula [12] associated with a Young diagram [3] obtained from the partition. All in all,  $n_t$ 's are equal to the number of standard Young-Tableaux of all the partitions of  $n$ .  $\square$

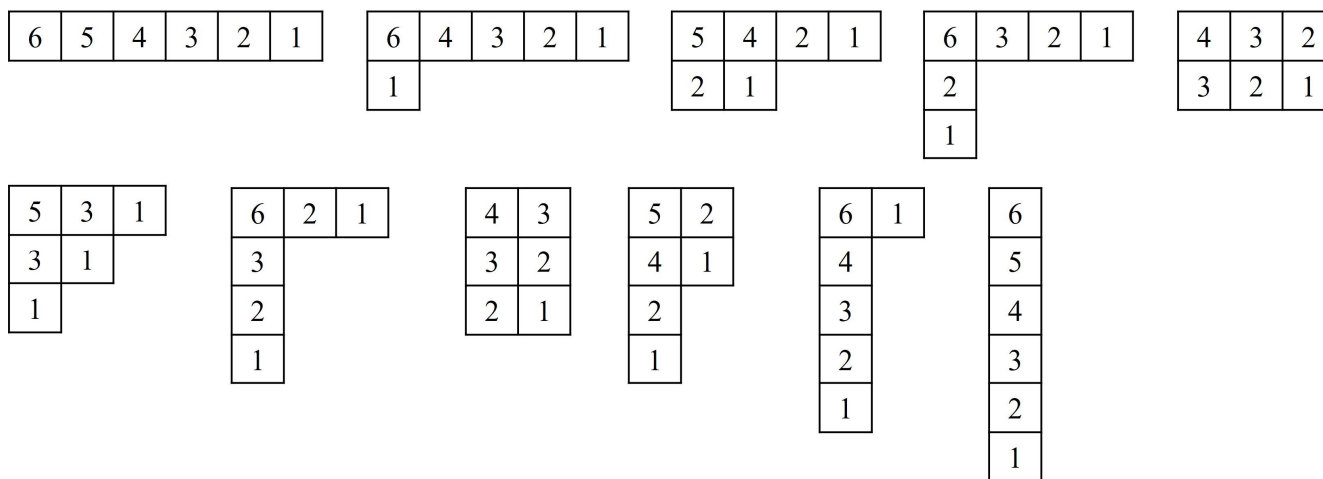
To this end, due to Theorem 1, we are now ready to give an algorithm to compute the unit group of semisimple group algebra  $\mathbb{F}_p\mathcal{S}_n$  for any  $n$ . Using this algorithm, we explicitly calculate the unit groups of the semisimple group algebras  $\mathbb{F}_p\mathcal{S}_5$ ,  $\mathbb{F}_p\mathcal{S}_6$ ,  $\mathbb{F}_p\mathcal{S}_7$  and  $\mathbb{F}_p\mathcal{S}_8$ . As usual, we denote the group of general linear matrices of order  $r$  over field  $\mathbb{F}_p$  by  $GL(r, \mathbb{F}_p)$ .

**Algorithm: Input:** positive integer  $n$  and a prime  $p$  such that  $\mathbb{F}_p S_n$  is semisimple.

- write all the partitions of  $n$  (total partitions are  $p(n)$ )
- draw "Young's diagram" for each partition
- calculate "number of standard Young-Tableaux" for each partition using Hook length formula. Let these be  $(r_1, \dots, r_{p(n)})$ .

**Output:** The Wedderburn decomposition of  $\mathbb{F}_p S_n$  is isomorphic to  $\bigoplus_{i=1}^{p(n)} M_{r_i}(\mathbb{F}_p)$  and the unit group of  $\mathbb{F}_p S_n$  is isomorphic to  $\bigoplus_{i=1}^{p(n)} GL(r_i, \mathbb{F}_p)$ .

In order to show the worthiness of above algorithm, first, we deduce the unit group of  $\mathbb{F}_p S_6$  for any  $p > 5$ . Clearly, the partitions  $p(6)$  of 6 are 11 and listed as follows: 6, (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1), (2,2,2), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1). The respective Young's diagram corresponding to these partitions are:



Using Hook length formula, the respective number of standard Young-Tableaux of a partition corresponding to above diagrams are 1,5,9,10,5,16,10,5,9,5 and 1. Therefore, the Wedderburn decomposition of the semisimple group algebra  $\mathbb{F}_p S_6$  is isomorphic to  $\mathbb{F}_p^2 \oplus M_5(\mathbb{F}_p)^4 \oplus M_9(\mathbb{F}_p)^2 \oplus M_{10}(\mathbb{F}_p)^2 \oplus M_{16}(\mathbb{F}_p)$ . Consequently, the unit group

$$U(\mathbb{F}_p S_6) \cong (\mathbb{F}_p^*)^2 \oplus GL(5, \mathbb{F}_p)^4 \oplus GL(9, \mathbb{F}_p)^2 \oplus GL(10, \mathbb{F}_p)^2 \oplus GL(16, \mathbb{F}_p).$$

Next, we deduce the unit group of  $\mathbb{F}_p S_5$  for any  $p > 5$ . Clearly, the partitions  $p(5)$  of 5 are 7 and listed as follows: 5, (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), and (1,1,1,1,1). Further, one can verify that the respective number of standard Young-Tableaux of a partition corresponding to Young diagrams in this case are 1,4,5,6,5,4 and 1. Therefore, the Wedderburn decomposition of the semisimple group algebra  $\mathbb{F}_p S_5$  is isomorphic to  $\mathbb{F}_p^2 \oplus M_4(\mathbb{F}_p)^2 \oplus M_5(\mathbb{F}_p)^2 \oplus M_6(\mathbb{F}_p)$ . Consequently, the unit group

$$U(\mathbb{F}_p S_5) \cong (\mathbb{F}_p^*)^2 \oplus GL(4, \mathbb{F}_p)^2 \oplus GL(5, \mathbb{F}_p)^2 \oplus GL(6, \mathbb{F}_p).$$

Finally, using our algorithm, one can verify that for the groups  $S_7$  and  $S_8$ , the unit groups of semisimple group algebras  $\mathbb{F}_p S_7$  and  $\mathbb{F}_p S_8$  are as follows:

- for  $p > 7$ ,  $U(\mathbb{F}_p S_7)$  is isomorphic to

$$(\mathbb{F}_p)^2 \oplus GL(6, \mathbb{F}_p)^2 \oplus GL(14, \mathbb{F}_p)^4 \oplus GL(15, \mathbb{F}_p)^2 \oplus GL(20, \mathbb{F}_p) \oplus GL(21, \mathbb{F}_p)^2 \oplus GL(35, \mathbb{F}_p)^2.$$

• for  $p > 7$ ,  $U(\mathbb{F}_p S_8)$  is isomorphic to

$$(\mathbb{F}_p^*)^2 \oplus GL(7, \mathbb{F}_p)^2 \oplus GL(14, \mathbb{F}_p)^2 \oplus GL(20, \mathbb{F}_p)^2 \oplus GL(21, \mathbb{F}_p)^2 \oplus GL(28, \mathbb{F}_p)^2 \oplus GL(35, \mathbb{F}_p)^2 \oplus GL(42, \mathbb{F}_p) \oplus GL(56, \mathbb{F}_p)^2 \oplus GL(64, \mathbb{F}_p)^2 \oplus GL(70, \mathbb{F}_p)^2 \oplus GL(90, \mathbb{F}_p).$$

## II. DISCUSSION

We have given a simple algorithm with which one can easily characterize the unit group of the semisimple group algebra  $\mathbb{F}_p S_n$  for any  $n$ . Therefore, this paper settles one of the stated problems of [8] related to the unit group of the group algebras of symmetric groups. The results proved in this paper are for finite fields of the form  $\mathbb{F}_p$ , however, one can see that the Wedderburn decomposition of the semisimple group algebra  $\mathbb{F}_p S_n$  is independent of the order of finite field. To be more precise, the field under consideration can have  $p$  elements or  $p^k$  elements for some  $k$ .

## III. ACKNOWLEDGMENT

I would like to thank Dr. Gaurav Mittal for his valuable suggestions and unsolicited support to organize this paper.

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