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A Note on the Wedderburn Decomposition and Unit Group of the Semisimple Group Algebra $\mathbb{F}_p S_n$

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***Abstract*: In this short note, we give an algorithm to compute the unit group of a semisimple group algebra $\mathbb{F}_p S_n$ for any $n \geq 5$. To show the practicality of the algorithm, we explicitly find the unit groups of the semisimple group algebras $\mathbb{F}_p S_5$, $\mathbb{F}_p S_6$, $\mathbb{F}_p S_7$ and $\mathbb{F}_p S_8$.**

Keywords: Group algebra, Wedderburn decomposition, Unit group, Symmetric group, Young-Tableaux

I. UNIT GROUP OF $\mathbb{F}_p S_n$

Let $\mathbb{F}_p$ be a field, where p is a prime. The study of unit groups of the semisimple group algebras is a classical research problem (cf. [1, 7, 15, 8, 9, 10, 11, 13] and the references therein). The unit groups of semisimple group algebras of symmetric groups S_3 and S_4 have been studied in [5, 14]. In this paper, we continue in this direction and deduce the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_p S_n$ for any $n \geq 5$ and as a by-product obtain the unit group. It is well known that for a finite group G, the group algebra $\mathbb{F}_p G$ is semisimple if and only if p does not divide $|G|$ [6]. Consequently, $\mathbb{F}_p G$ can be written as a direct sum of matrix rings over finite fields of characteristic p. To be more precise, we have $\mathbb{F}_p G \cong \bigoplus_{t=1}^{j} M_{n_t}(\mathbb{F}_{p_t})$ for any semisimple group algebra $\mathbb{F}_p G$. Let ℓ be the exponent of G and $\Im$ be the primitive ℓ^{th} root of unity. In accordance to [2], we define $I_{\mathbb{F}} = \{r \mid \Im \mapsto \Im^r$ is an automorphism of $\mathbb{F}(\Im)$ over $\mathbb{F}\}$. An element $h \in G$ is a p-regular element if its order is not divisible by p. For a p-regular element $h \in G$, let the sum of all the conjugates of h be denoted by γ_h, and the cyclotomic-$\mathbb{F}$ class of γ_h be denoted by $S(\gamma_h) = \{\gamma_{h^r} \mid r \in I_{\mathbb{F}}\}$. On utilizing [2, Proposition 1.2] and the result that symmetric group S_n splits over every field, one can see that $\mathbb{F}_p S_n \cong \bigoplus_{t=1}^{j} M_{n_t}(\mathbb{F}_p)$ (i.e. $p_t = p$ for all $t = 1$ to j), where j is the number of conjugacy classes of S_n that are further equal to partitions $p(n)$ of n. Consequently, the n_t's appearing in the Wedderburn decomposition of $\mathbb{F}_p S_n$ are nothing but the dimensions of irreducible representations of $\mathbb{F}_p S_n$. To this end, we now show the following.

Theorem 1. *The n_t's appearing in the Wedderburn decomposition of $\mathbb{F}_p S_n$ are equal to the number of standard Young-Tableaux of all the partitions of n.*

Proof. It is known that the p-restricted partitions of n indexes the irreducible representations of $\mathbb{F}_p S_n$ (cf. [4]). The p-restricted partitions of n are all those integer partitions in which the difference between successive parts can be at most $p-1$. It turns out that in the semisimple case (since $p \geq n+1$), the irreducible representations are given by the Specht modules (cf. [4] for their definition) - in the modular case, the irreducible representations are quotients of Specht modules (obtained by modular reduction of a lattice in **Q**-irreducible). Moreover, the quotients of Specht modules are indexed by partitions of n and the respective dimension is equal to the number of standard Young-Tableaux of the respective partition type. Further, the number of standard Young-Tableaux of a partition can be obtained by Hook length formula [12] associated with a Young diagram [3] obtained from the partition. All in all, n_t's are equal to the number of standard Young-Tableaux of all the partitions of n. □

To this end, due to Theorem 1, we are now ready to give an algorithm to compute the unit group of semisimple group algebra $\mathbb{F}_p S_n$ for any n. Using this algorithm, we explicitly calculate the unit groups of the semisimple group algebras $\mathbb{F}_p S_5$, $\mathbb{F}_p S_6$, $\mathbb{F}_p S_7$ and $\mathbb{F}_p S_8$. As usual, we denote the group of general linear matrices of order r over field $\mathbb{F}_p$ by $GL(r, \mathbb{F}_p)$.

Algorithm: **Input**: positive integer n and a prime p such that $\mathbb{F}_p S_n$ is semisimple.

- write all the partitions of n (total partitions are $p(n)$)
- draw "Young's diagram" for each partition
- calculate "number of standard Young-Tableaux" for each partition using Hook length formula. Let these be $(r_1, \cdots, r_{p(n)})$.

Output: The Wedderburn decomposition of $\mathbb{F}_p S_n$ is isomorphic to $\bigoplus_{i=1}^{p(n)} M_{r_i}(\mathbb{F}_p)$ and the unit group of $\mathbb{F}_p S_n$ is isomorphic to $\bigoplus_{i=1}^{p(n)} GL(r_i, \mathbb{F}_p)$.

In order to show the worthiness of above algorithm, first, we deduce the unit group of $\mathbb{F}_p S_6$ for any $p > 5$. Clearly, the partitions $p(6)$ of 6 are 11 and listed as follows: 6, (5,1), (4,2), (4,1,1), (3,3), (3,2,1), (3,1,1,1), (2,2,2), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1). The respective Young's diagram corresponding to these partitions are:

6	5	4	3	2	1

6	4	3	2	1
1				

5	4	2	1
2	1		

6	3	2	1
2			
1			

4	3	2
3	2	1

5	3	1
3	1	
1		

6	2	1
3		
2		
1		

4	3
3	2
2	1

5	2
4	1
2	
1	

6	1
4	
3	
2	
1	

6
5
4
3
2
1

Using Hook length formula, the respective number of standard Young-Tableaux of a partition corresponding to above diagrams are $1,5,9,10,5,16,10,5,9,5$ and 1. Therefore, the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_p S_6$ is isomorphic to $\mathbb{F}_p^2 \oplus M_5(\mathbb{F}_p)^4 \oplus M_9(\mathbb{F}_p)^2 \oplus M_{10}(\mathbb{F}_p)^2 \oplus M_{16}(\mathbb{F}_p)$. Consequently, the unit group

$$U(\mathbb{F}_p S_6) \cong (\mathbb{F}_p^*)^2 \oplus GL(5, \mathbb{F}_p)^4 \oplus GL(9, \mathbb{F}_p)^2 \oplus GL(10, \mathbb{F}_p)^2 \oplus GL(16, \mathbb{F}_p).$$

Next, we deduce the unit group of $\mathbb{F}_p S_5$ for any $p > 5$. Clearly, the partitions $p(5)$ of 5 are 7 and listed as follows: $5, (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1)$, and $(1,1,1,1,1)$. Further, one can verify that the respective number of standard Young-Tableaux of a partition corresponding to Young diagrams in this case are $1,4,5,6,5,4$ and 1. Therefore, the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_p S_5$ is isomorphic to $\mathbb{F}_p^2 \oplus M_4(\mathbb{F}_p)^2 \oplus M_5(\mathbb{F}_p)^2 \oplus M_6(\mathbb{F}_p)$. Consequently, the unit group

$$U(\mathbb{F}_p S_5) \cong (\mathbb{F}_p^*)^2 \oplus GL(4, \mathbb{F}_p)^2 \oplus GL(5, \mathbb{F}_p)^2 \oplus GL(6, \mathbb{F}_p).$$

Finally, using our algorithm, one can verify that for the groups S_7 and S_8, the unit groups of semisimple group algebras $\mathbb{F}_p S_7$ and $\mathbb{F}_p S_8$ are as follows:

- for $p > 7$, $U(\mathbb{F}_p S_7)$ is isomorphic to

$(\mathbb{F}_p^*)^2 \oplus GL(6,\mathbb{F}_p)^2 \oplus GL(14,\mathbb{F}_p)^4 \oplus GL(15,\mathbb{F}_p)^2 \oplus GL(20,\mathbb{F}_p) \oplus GL(21,\mathbb{F}_p)^2 \oplus GL(35,\mathbb{F}_p)^2$.

• for $p > 7$, $U(\mathbb{F}_p S_8)$ is isomorphic to
$(\mathbb{F}_p^*)^2 \oplus GL(7,\mathbb{F}_p)^2 \oplus GL(14,\mathbb{F}_p)^2 \oplus GL(20,\mathbb{F}_p)^2 \oplus GL(21,\mathbb{F}_p)^2 \oplus GL(28,\mathbb{F}_p)^2 \oplus GL(35,\mathbb{F}_p)^2 \oplus GL(42,\mathbb{F}_p) \oplus GL(56,\mathbb{F}_p)^2 \oplus GL(64,\mathbb{F}_p)^2 \oplus GL(70,\mathbb{F}_p)^2 \oplus GL(90,\mathbb{F}_p)$.

II. DISCUSSION

We have given a simple algorithm with which one can easily characterize the unit group of the semisimple group algebra $\mathbb{F}_p S_n$ for any n. Therefore, this paper settles one of the stated problems of [8] related to the unit group of the group algebras of symmetric groups. The results proved in this paper are for finite fields of the form $\mathbb{F}_p$, however, one can see that the Wedderburn decomposition of the semisimple group algebra $\mathbb{F}_p S_n$ is independent of the order of finite field. To be more precise, the field under consideration can have p elements or p^k elements for some k.

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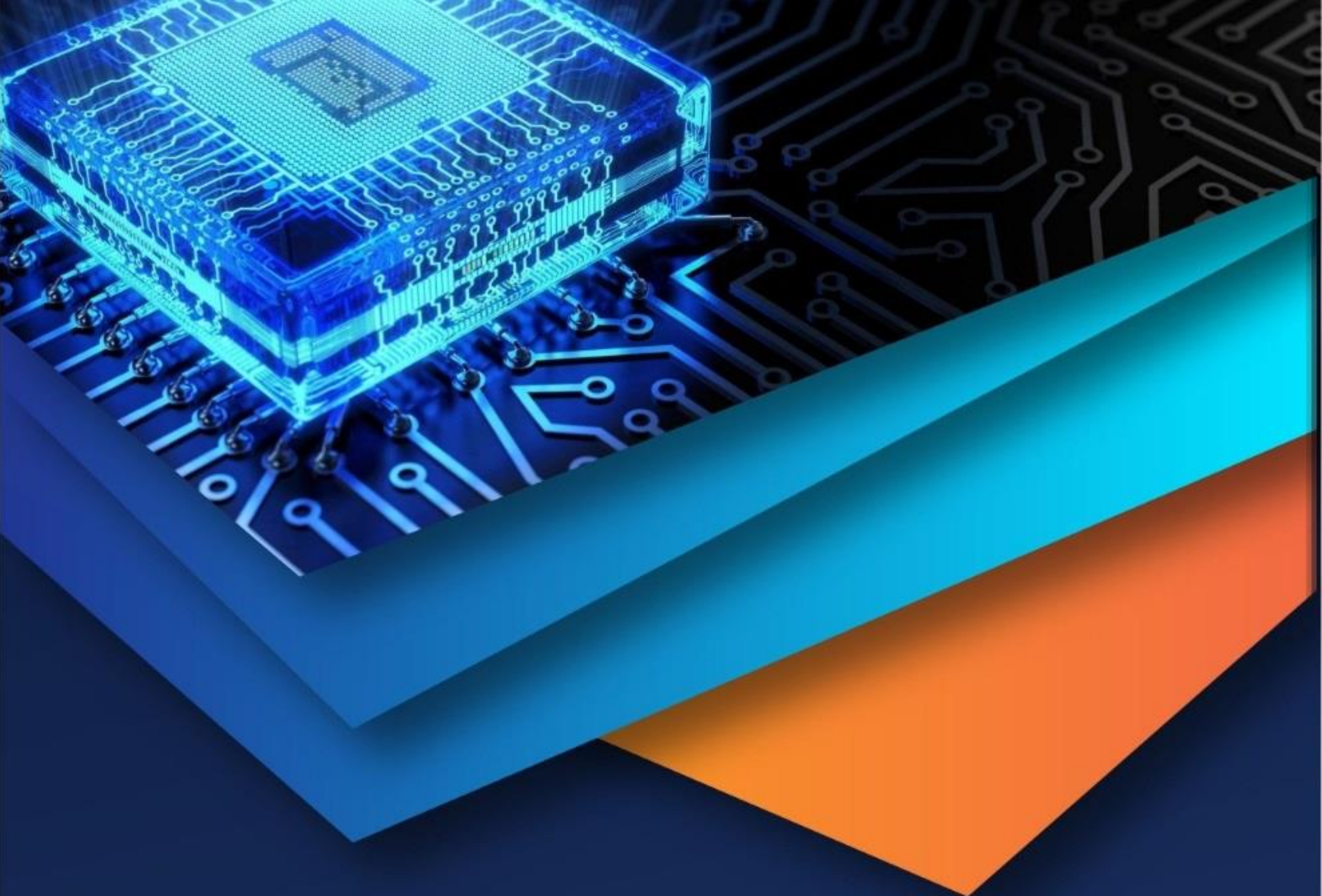

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