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Numerical Solution for the Heat Equation Using Crank-Nicolson Difference Method

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Abstract: In this paper, the Crank-Nicolson difference method is applied to a simple problem involving one dimensional heat equation. Numerical solution to the heat equation using Crank-Nicolson difference equation is obtained. The method of solving the problem is implemented by using Python Programming. We have discussed the local truncation error and stability analysis of Crank-Nicolson difference method difference method of heat equation.
Keywords: Heat Equation, Crank-Nicolson Difference Equation, Matrix Equation, Boundary Conditions, Truncation Error, Stability Analysis.

I. INTRODUCTION

The heat diffusion equation is a parabolic partial differential equation which describes the heat distribution in a given region and provides the basic tool for heat conduction analysis. The heat equation is used to solve problems arising in the field of Engineering, Fluid Mechanics, Atmospheric Science, Climate Physics, Weather Forecasting and Solar Physics. Analytical and Numerical methods have gained the interest of researchers for finding approximate solution to partial differential equations. Numerical Methods have applied to calculate the approximate solutions using Crank-Nicolson difference method. The Crank-Nicolson method for solving heat equation was developed by John Crank and Phyllis Nicolson in 1947. This method is for numerically evaluating the partial differential equations which gives the accuracy of a second order approach in both space and time with the stability of an implicit method.

Liu J. and Hao Y. have applied the Crank-Nicolson difference method to obtain an analytical solution of uncertain heat equation. Besides, the two algorithms are investigated to compute two characteristics of uncertain heat equation solution the expected value and the extreme value with various examples [1]. Umar Ali et al. have described the Crank-Nicolson difference method for two – dimensional sub diffusion equation. They have found that the scheme is convergent and it is unconditionally stable. The result shows that the scheme is feasible and accurate [2]. Halil ANAC et al. have successfully applied Crank-Nicolson difference method to solve a random component heat equation. They have obtained the expected value and variance of this solution. The numerical solution shows that this method is very effective by implementing MAPLE software [3]. Cem Celik and Melda Duman have applied Crank-Nicolson difference method to a linear fractional diffusion equation and proved that the method is unconditionally stable and convergent. The numerical solution shows that the accuracy by using simulation method [4]. Chen Jing had developed for a one-dimensional and two-dimensional diffusion equation for Crank-Nicolson difference method. The alternating block technique is further extended to a three-dimensional space diffusion equation and he found that the new method is suitable to MIMD computers [5]. Fadugba S. E. et al. have demonstrated that the Crank-Nicolson difference method performs well and provides better accuracy for exact solution. This method is robust, unconditionally stable and converges faster than the two finite difference methods FTCS and BCTS [6]. Omar Abdullah Ajeel and A.M.Gaftan have solved the heat diffusion problem by using Crank-Nicolson difference method and ADI numerical method. Both the results were compared and it reveals that Crank-Nicolson difference method is more accurate than the ADI method [7].

F. Jamaluddin et al. have solved the homogeneous one-dimensional heat equation by using Implicit Crank-Nicolson difference method. The results are compared with analytical solution and exact solution. The results are very precise or remain the same as exact solution [8]. H. Zureigat et al. have obtained the results using Crank-Nicolson difference method satisfy the complex fuzzy number properties by taking the triangular fuzzy number shape for both the real part and imaginary part and have an accuracy. Further, it was found that the complex fuzzy approach is general and computationally efficient to transfer the information that happens periodically [9].

Irfan Raju et al. have proposed to solve the two-dimensional heat equation with initial conditions by using Crank-Nicolson difference method. They have suggested that the Digital Image Processing can be a valuable tool for numerical modeling of the heat equation by using MATLAB Codes [10].

S. K. Maritim had proposed that the modified Crank-Nicolson difference method produces more accurate results and unconditionally stable as compared to the ordinary Crank-Nicolson difference method [11]. This paper proposes the numerical solution for the heat equation using Crank-Nicolson difference method by Python programming. The paper is organized as follows: Section II presents the heat equation, Section III discusses the discrete grid, Section IV discusses the Discrete Initial and Boundary conditions, Section V presents the Crank- Nicolson difference method, Section VI focuses on implementation and results, Section VII discusses local truncation error, Section VIII focuses on stability analysis and finally the conclusion is presented in Section IX.

II. THE HEAT EQUATION

The heat equation is the first order in time (t) and second order in space (x) partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \dots\dots\dots\dots (1)$$

where $u = u(x, t)$ is the dependent variable. The equation describes heat transfer on a domain

$$\Omega = \{t \geq 0; 0 \leq x \leq 1\} \dots\dots\dots\dots (2)$$

With an initial condition at time $t = 0$ for all x and boundary condition on the left $x = 0$ and right side $x = 1$.

We will implement the Crank-Nicolson difference method for the heat equation with the initial conditions by considering the equation

$$u(x,0) = \begin{cases} 2x; & 0 \leq x \leq \frac{1}{2} \\ 2(1-x); & \frac{1}{2} \leq x \leq 1 \end{cases} \dots\dots\dots\dots\dots\dots (3)$$

and boundary conditions $u(0,t) = 0 \; and \; u(1,t) = 0$.

III. THE DISCRETE GRID

The region Ω discretized into a uniform mesh Ω_h. The discrete values of x are uniformly spaced in the interval $0 \leq x \leq 1$. In the space x direction into N steps giving a step size of $h = \frac{1-0}{N}$, resulting in $x[i] = 0 + ih$, $i = 0,1,2, \dots\dots\dots, N$ and into N_t steps in the time t direction giving a step size of $k = \frac{1-0}{N_t}$ resulting in $t[j] = 0 + jk$, $j = 0,1,2, \dots\dots\dots, 15$. The Figure 1 below shows the discrete grid points for $N = 10 \; and \; N_t = 100$, the known boundary conditions (Green), initial conditions (Blue) and the unknown values (Red) of the heat equation.

$$N = 10; \quad h = \frac{1}{N} = \frac{1}{10} = 0.1; \quad N_t = 100; \quad k = \frac{1}{N_t} = \frac{1}{100} = 0.01$$

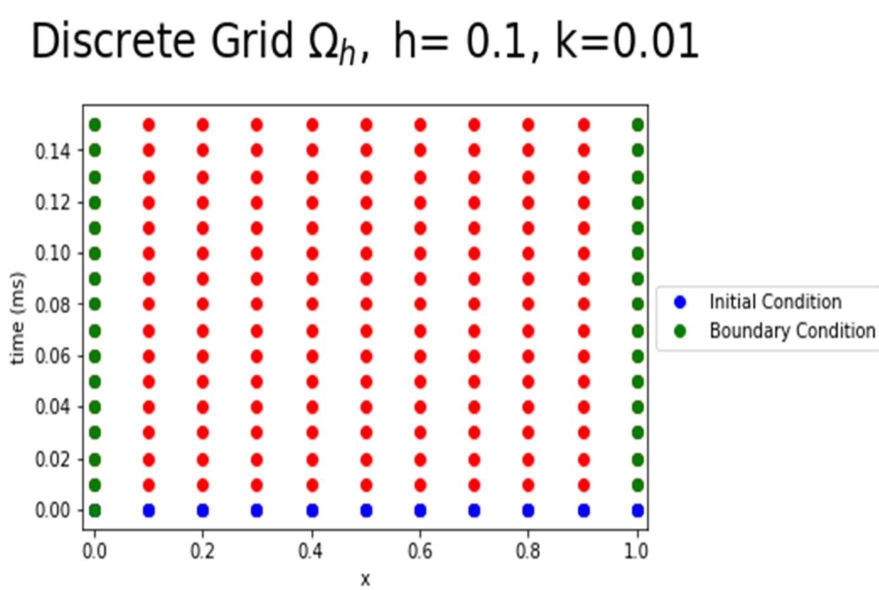

Figure 1.Discrete Grid points for initial and boundary conditions.

IV. DISCRETE INITIAL AND BOUNDARY CONDITIONS

The discrete initial conditions are

$$w[i,0] = \begin{cases} 2x[i]; & 0 \le x[i] \le \frac{1}{2} \\ 2(1-x[i]); & \frac{1}{2} \le x[i] \le 1 \end{cases} \ldots\ldots\ldots\ldots.(4)$$

and the discrete boundary conditions $w[0,j] = 0$ and $w[10,j] = 0$, where $w[i,j]$ is the numerical approximation of $U(x[i],t[j])$. The Figure 2 below plots the values of $w[i,0]$ for the initial (blue) and boundary (green) conditions for $t[0] = 0$.

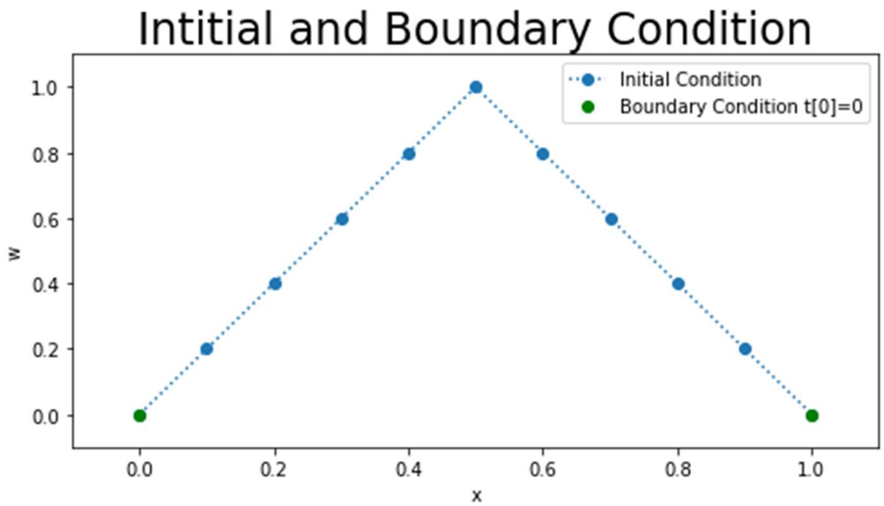

Figure 2.Plot of initial and boundary conditions for $t[0] = 0$.

V. THE CRANK-NICOLSON DIFFERENCE METHOD

The implicit Crank-Nicolson difference equation of the heat equation is derived by discretizing around $(x_i, t_{j+\frac{1}{2}})$

$$\frac{\partial u_{ij+\frac{1}{2}}}{\partial t} = \frac{\partial^2 u_{ij+\frac{1}{2}}}{\partial x^2} \ldots\ldots\ldots\ldots.(5)$$

The difference equation is given by

$$\frac{w_{ij+1} - w_{ij}}{k} = \frac{1}{2}\left\{\frac{w_{i+1j+1} + w_{i-1j+1} - 2w_{ij+1}}{h^2} + \frac{w_{i+1j} + w_{i-1j} - 2w_{ij}}{h^2}\right\} \ldots\ldots\ldots(6)$$

Rearranging the equation (6) we obtain,

$$2(w_{ij+1} - w_{ij}) = \frac{k}{h^2}\{w_{i+1j+1} + w_{i-1j+1} - 2w_{ij+1} + w_{i+1j} + w_{i-1j} - 2w_{ij}\}$$

$$\Rightarrow 2w_{ij+1} - 2w_{ij} = r\{w_{i+1j+1} + w_{i-1j+1} - 2w_{ij+1} + w_{i+1j} + w_{i-1j} - 2w_{ij}$$

$$\Rightarrow -rw_{i-1j+1} + (2+2r)w_{ij+1} - rw_{i+1j+1} = rw_{i-1j} + (2-2r)w_{ij} + rw_{i+1j} \ldots\ldots\ldots(7)$$

$$for\ i = 1,2,\ldots\ldots,9\ where\ r = \frac{k}{h^2}$$

This gives the formula for unknown term w_{ij+1} at the $(ij+1)$ mesh points in terms of $x[i]$ along the jth time row. Hence we can calculate the unknown pivotal values of w along the first row of $j = 1$ in terms of the known boundary conditions [12].

This can be written in the equation of matrix form $\boldsymbol{A}\boldsymbol{w}_{i+j} = \boldsymbol{B}\boldsymbol{w}_j + \boldsymbol{b}_j + \boldsymbol{b}_{j+1} \ldots\ldots\ldots\ldots(8)$

for which A is a 9×9 matrix.

$$\begin{bmatrix} 2+2r & -r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -r & 2+2r & -r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r & 2+2r & -r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 2+2r & -r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r & 2+2r & -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r & 2+2r & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r & 2+2r & -r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -r & 2+2r & -r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r & 2+2r \end{bmatrix} \begin{bmatrix} w_{1j+1} \\ w_{2j+1} \\ w_{3j+1} \\ w_{4j+1} \\ w_{5j+1} \\ w_{6j+1} \\ w_{7j+1} \\ w_{8j+1} \\ w_{9j+1} \end{bmatrix} =$$

$$\begin{bmatrix} 2-2r & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r & 2-2r & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 2-2r & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 2-2r & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 2-2r & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 2-2r & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & 2-2r & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 2-2r & r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 2-2r \end{bmatrix} \begin{bmatrix} w_{1j} \\ w_{2j} \\ w_{3j} \\ w_{4j} \\ w_{5j} \\ w_{6j} \\ w_{7j} \\ w_{8j} \\ w_{9j} \end{bmatrix} + \begin{bmatrix} rw_{0j} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ rw_{10j} \end{bmatrix} + \begin{bmatrix} rw_{0j+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ rw_{10j+1} \end{bmatrix}$$

It is assumed that the boundary values w_{0j} and w_{10j} are known for $j = 1,2, \dots \dots$ and $w_{i0} for$ $i = 0,1,2, \dots \dots \dots, 10$ is the initial condition. The Figure 3 below shows the values of the 9×9 matrix in colour plot form for $r = \frac{k}{h^2}$.

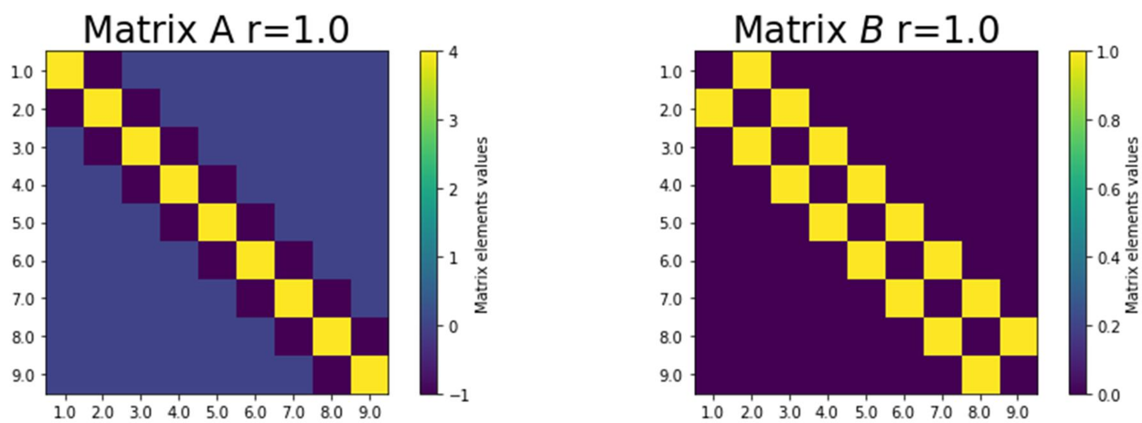

Figure 3.The values of the matrix of order 9×9

VI. IMPLEMENTATION AND RESULTS

To approximate the solution numerically at $t[1]$, the matrix equation becomes
$w_1 = A^{-1}(Bw_0 + b_1 + b_0)$ where all the right hand side is known.
To approximate the solution numerically at $t[2]$, the matrix equation becomes
$w_2 = A^{-1}(Bw_1 + b_2 + b_1)$ where all the right hand side is known.
To approximate the solution numerically at $t[3]$, the matrix equation becomes
$w_3 = A^{-1}(Bw_2 + b_3 + b_2)$ where all the right hand side is known.
To approximate the solution numerically at $t[4]$, the matrix equation becomes
$w_4 = A^{-1}(Bw_3 + b_4 + b_3)$ where all the right hand side is known and so on.

Each set of numerical solution $w[i,j]$ for all i at the previous time step is used to approximate the solution $w[i,j+1]$. The Figure 4 below shows the numerical approximation $w[i,j]$ of the heat equation using the Crank-Nicolson method at $x[i]$ for $i = 0,1,2, \dots \dots \dots, 10$ and $time$ $steps$ $t[j]$ for $j = 1,2,3, \dots \dots \dots, 15$.

The left plot shows the numerical approximation $w[i,j]$ as a function of $x[i]$ with each colour representing the different time steps $t[j]$.

The right plot shows the numerical approximation $w[i,j]$ as colour plot as a function of $x[i]$ on the X - axis and time $t[j]$ on the Y-axis.

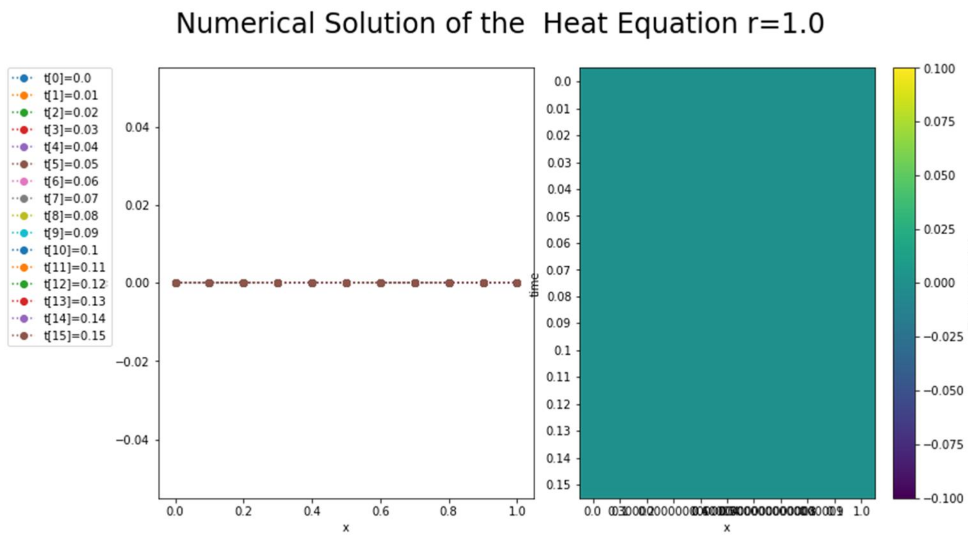

Figure 4.Numerical approximation $w[i,j]$ of the heat equation using the Crank-Nicolson method

VII. TRUNCATION ERROR

In Crank-Nicolson method, we approximate the local truncation error of the classical implicit difference approach to the equation $\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0\ with$

$$F_{ij+\frac{1}{2}}(w) = \frac{w_{ij+1} - w_{ij}}{k} - \frac{1}{2}\left\{\frac{w_{i+1j+1} + w_{i-1j+1} - 2w_{ij+1}}{h^2} + \frac{w_{i+1j} + w_{i-1j} - 2w_{ij}}{h^2}\right\} = 0$$

$Let\ T_{ij+\frac{1}{2}} = F_{ij+\frac{1}{2}}(U) =$

$$\frac{U_{ij+1} - U_{ij}}{k} - \frac{1}{2}\left\{\frac{U_{i+1j+1} + U_{i-1j+1} - 2U_{ij+1}}{h^2} + \frac{U_{i+1j} + U_{i-1j} - 2U_{ij}}{h^2}\right\} \dots\dots\dots\dots \quad (9)$$

$$U_{ij+\frac{1}{2}} = \frac{U_{ij+1} + U_{ij}}{2}$$

$$T_{ij+\frac{1}{2}} = F_{ij+\frac{1}{2}}(U) = \frac{U_{ij+1} - U_{ij}}{k} - \frac{U_{i+1j+\frac{1}{2}} + U_{i-1j+\frac{1}{2}} - 2U_{ij+\frac{1}{2}}}{h^2}$$

By Taylor's expansion, we have

$$U_{i+1j+\frac{1}{2}} = U\left((i+1)h, (j+\frac{1}{2})k\right)$$

$$U_{i+1j+\frac{1}{2}} = U\left(x_i + h, t_j + \frac{1}{2}k\right)$$

$$U_{i+1j+\frac{1}{2}} = U_{ij+\frac{1}{2}} + h\left(\frac{\partial U}{\partial x}\right)_{ij+\frac{1}{2}} + \frac{h^2}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_{ij+\frac{1}{2}} + \frac{h^3}{6}\left(\frac{\partial^3 U}{\partial x^3}\right)_{ij+\frac{1}{2}} + \cdots \quad (10)$$

$$U_{i-1j+\frac{1}{2}} = U\left((i-1)h, (j+\frac{1}{2})k\right)$$

$$U_{i-1j+\frac{1}{2}} = U\left(x_i - h, t_j + \frac{1}{2}k\right)$$

$$U_{i-1j+\frac{1}{2}} = U_{ij+\frac{1}{2}} - h\left(\frac{\partial U}{\partial x}\right)_{ij+\frac{1}{2}} + \frac{h^2}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_{ij+\frac{1}{2}} - \frac{h^3}{6}\left(\frac{\partial^3 U}{\partial x^3}\right)_{ij+\frac{1}{2}} + \cdots \quad (11)$$

$$U_{ij+1} = U(ih, (j+1)k)$$

$$U_{ij+1} = U\left(x_i, t_j + k\right)$$

$$U_{ij+1} = U_{ij+\frac{1}{2}} + \frac{k}{2}\left(\frac{\partial U}{\partial t}\right)_{ij+\frac{1}{2}} + \frac{\left(\frac{k}{2}\right)^2}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij+\frac{1}{2}} + \frac{\left(\frac{k}{2}\right)^3}{6}\left(\frac{\partial^3 U}{\partial t^3}\right)_{ij+\frac{1}{2}} + \cdots \quad (12)$$

$$U_{ij-1} = U(ih, (j-1)k)$$

$$U_{ij-1} = U(x_i, t_j - k)$$

$$U_{ij-1} = U_{ij+\frac{1}{2}} - \frac{k}{2}\left(\frac{\partial U}{\partial t}\right)_{ij+\frac{1}{2}} + \frac{\left(\frac{k}{2}\right)^2}{2}\left(\frac{\partial^2 U}{\partial t^2}\right)_{ij+\frac{1}{2}} - \frac{\left(\frac{k}{2}\right)^3}{6}\left(\frac{\partial^3 U}{\partial t^3}\right)_{ij+\frac{1}{2}} + \cdots \quad (13)$$

Substituting (10), (11), (12) and (13) in (9), we obtain

$$T_{ij+\frac{1}{2}} = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{ij+\frac{1}{2}} - \frac{h^2}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_{ij+\frac{1}{2}} + \frac{k^2}{24}\left(\frac{\partial^3 U}{\partial t^3}\right)_{ij+\frac{1}{2}} - \frac{h^4}{360}\left(\frac{\partial^6 U}{\partial x^6}\right)_{ij+\frac{1}{2}} + \cdots$$

But U is the solution of the differential equation $\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{ij+\frac{1}{2}} = 0$

The principal part of local truncation error is

$$\frac{k^2}{24}\left(\frac{\partial^3 U}{\partial t^3}\right)_{ij+\frac{1}{2}} - \frac{h^2}{12}\left(\frac{\partial^4 U}{\partial x^4}\right)_{ij+\frac{1}{2}}$$

Hence the truncation error is $T_{ij} = O(k^2) + O(h^2) \ldots\ldots\ldots (14)$

VIII. STABILITY ANALYSIS

To discuss the stability of the fully implicit Crank-Nicolson difference method of the heat equation, we will use the von Neumann method. The difference equation is

$$\frac{w_{pq+1} - w_{pq}}{k} = \frac{1}{2}\left\{\frac{w_{p+1q+1} + w_{p-1q+1} - 2w_{pq+1}}{{h_x}^2} + \frac{w_{p+1q} + w_{p-1q} - 2w_{pq}}{{h_x}^2}\right\} \ldots\ldots\ldots\ldots (15)$$

Approximating $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ at $(ph, (q + \frac{1}{2})k)$

Substituting $w_{pq} = \alpha^q e^{i\beta x}$ into the difference equation, we obtain

$$\alpha^{q+1} e^{i\beta ph} - \alpha^q e^{i\beta ph} =$$

$$\frac{1}{2}\left[r\left\{\alpha^{q+1} e^{i\beta(p-1)h} - 2\alpha^{q+1} e^{i\beta ph} + \alpha^{q+1} e^{i\beta(p+1)h} + \alpha^q e^{i\beta(p-1)h} - 2\alpha^q e^{i\beta ph} + \alpha^q e^{i\beta(p+1)h}\right\}\right] \; where\; r = \frac{k}{h^2}$$

Dividing by $\alpha^q e^{i\beta ph}$, we obtain $\alpha - 1 = r\alpha\left[e^{i\beta h} + e^{-i\beta h} - 2\right] + r\left[e^{i\beta h} + e^{-i\beta h} - 2\right]$

$$\Rightarrow \alpha - \alpha r[2\cos\beta h - 2] = 1 + r[2\cos\beta h - 2]$$

$$\Rightarrow \alpha\left[1 + 4r\sin^2\beta\left(\frac{h}{2}\right)\right] = 1 - 4r\sin^2\beta\left(\frac{h}{2}\right)$$

$$\Rightarrow \alpha = \frac{1-4r\left[\sin^2\beta\left(\frac{h}{2}\right)\right]}{1+4r\left[\sin^2\beta\left(\frac{h}{2}\right)\right]} \leq 1 \ldots\ldots\ldots (16)$$

The equation is unconditionally stable as $0 \leq \alpha \leq 1$ $for\ all\ r\ and\ all\ \beta$.

IX. CONCLUSION

We first introduced the one-dimensional heat equation to obtain the numerical approximation using Crank-Nicolson difference method. We have proposed the initial value problem with boundary conditions to find the numerical solution. The region is discretized into a uniform mesh. The solution is obtained by implementing Python programming by using initial and boundary conditions. The numerical solution for the heat equation is shown in Figure 4. We have investigated the local truncation error and stability analysis of the Crank-Nicolson difference method of the heat equation.

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