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Tackling Ternary Cubic Diophantine Equation $4(\alpha^2+\beta^2)-7(\alpha\beta)=64\gamma^3$

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Abstract: A few interesting characteristics among the solutions and the patterns of non-zero integral solutions to the non-homogeneous cubic equation with three unknowns represented by the Diophantine equation $4(\alpha^2+\beta^2)-7(\alpha\beta)=64\gamma^3$ are examined.

Keywords: Ternary cubic equation, Diophantine equation, Integer solution.

I. INTRODUCTION

Mathematics, conveys knowledge of number and is the global language of the world. Number theory is a field in pure mathematics that studies integral valued functions. In number theory, primes and prime factorization are highly important concepts. Fermat is recognized as the Father of modern number theory despite being lawyer by practice and an elementary "amateur" mathematician. Modern number theory is a broad subject that is classified into subheadings such as elementary number theory, algebraic number theory, analytic number theory, geometric number theory and probabilistic number theory. A Diophantine equation is a polynomial equation with two or more unknowns seeking only integer solutions. The word Diophantine refers to "Diophantus of Alexandria", a Hellenistic mathematician who lived in the century and is credited with being one of the first who introduced symbolism of algebra. A homogeneous degree 3 polynomial in three variables is called a Ternary Cubic forms. All cubic equation have either three real roots or one real root with two imaginary roots. Three degree polynomials are referred to as cubic equation. In [1-5] Elementary Number Theory cconcepts are studied, [6-9] examined quadratic Diophantine equation, [10-15] referred the cubic Diophantine equation and exponential Diophantine equation. A non-homogeneous ternary cubic equation with three unknowns $4(\alpha^2+\beta^2)-7(\alpha\beta)=64\gamma^3$ is solved in this paper and using some interesting number properties.

II. NOTATIONS

1) $P_n=2T_n=n(n+1)=$ Pronic number of rank n.
2) $TO_n=16n^3-33n^2+24n-6=$ Truncated octrahedral number of rank n.
3) $T_{8,n}=n(3n-2)=$ Octagonal number of rank n.
4) $S_n=6n(n-1)+1=$ Star number of rank n.
5) $T_{12,n}=n(5n-4)=$ Dodecagonal number of rank n.
6) $T_{26,n}=n(12n-11)=$ Icosihexagonal number of rank n.
7) $W_n=2^n n-1=$ Woodall number of rank n.
8) $Gno_n=(2n-1)=$ Gnomonic number of rank n.
9) $T_{30,n}=n(14n-13)=$ Triacontagonal number of rank n.
10) $T_{22,n}=n(10n-9)=$ Icosidigonal number of rank n.

III. METHOD OF ANALYSIS

Considering the Diophantine equation in ternary form

$$4(\alpha^2+\beta^2)-7(\alpha\beta)=64\gamma^3 \quad (1)$$

The linear transformation used is,

$$\alpha = \lambda + \eta \text{ and } \beta = \lambda - \eta \tag{2}$$

By (2), (1) reflects as,

$$\lambda^2 + 15\eta^2 = 64\gamma^3 \tag{3}$$

Below is the theorem derived from solving equation (3) using different substitutions.

Theorem: Multitude of solutions of $4(\alpha^2 + \beta^2) - 7(\alpha\beta) = 64\gamma^3$

Substitution: 1.1

Taking, $\gamma = \upsilon^2 + 15\omega^2$ (4)

We can write 64 as, $64 = (7 + i\sqrt{15})(7 - i\sqrt{15})$ (5)

Substitute the equation (4) and (5) in (3),

$$(\lambda + i\sqrt{15}\eta)(\lambda - i\sqrt{15}\eta) = (7 + i\sqrt{15})(7 - i\sqrt{15})(\upsilon + i\sqrt{15}\omega)^3(\upsilon - i\sqrt{15}\omega)^3$$

Equating the real and imaginary terms,

$$\lambda = 7\upsilon^3 - 315\upsilon\omega^2 - 45\upsilon^2\omega + 225\omega^3$$
$$\eta = \upsilon^3 - 45\upsilon\omega^2 + 21\upsilon^2\omega - 105\omega^3$$

Reducing equation (2) using (λ) and (η) gives,

$$\alpha = 8\upsilon^3 - 360\upsilon\omega^2 - 24\upsilon^2\omega + 120\omega^3$$
$$\beta = 6\upsilon^3 - 270\upsilon\omega^2 - 66\upsilon^2\omega + 330\omega^3$$
$$\gamma = \upsilon^2 + 15\omega^2$$

Inference: 1.1

1. $\gamma(c,c) - 2T_{18,c} - 7Gno_c \equiv 0 \pmod 7$
2. $\beta(1,1) - \alpha(1,1)$ is a perfect square
3. $3\alpha(c,1) - 4\beta(c,1) - 16T_{26,c} - 88Gno_c + 872 = 0$
4. $\alpha(1,1) + 639$ is a woodall number
5. $5\gamma(c,1) - T_{12,c} - 2Gno_c \equiv 0 \pmod{77}$

Substitution: 1.2

Rewrite 64 as, $64 = (2 + i2\sqrt{15})(2 - i2\sqrt{15})$ (6)

From (4) and (6) in (3),

$$(\lambda + i\sqrt{15}\eta)(\lambda - i\sqrt{15}\eta) = (2 + i2\sqrt{15})(2 - i2\sqrt{15})(\upsilon + i\sqrt{15}\omega)^3(\upsilon - i\sqrt{15}\omega)^3$$

Equating the terms both real and imaginary,

$$\lambda = 2\upsilon^3 - 90\upsilon\omega^2 - 90\upsilon^2\omega + 450\omega^3$$
$$\eta = 2\upsilon^3 - 90\upsilon\omega^2 + 6\upsilon^2\omega - 30\omega^3$$

Reducing equation (2) using (λ) and (η) gives,

$$\alpha = 4\upsilon^3 - 180\upsilon\omega^2 - 84\upsilon^2\omega + 420\omega^3$$
$$\beta = -96\upsilon^2\omega + 480\omega^3$$
$$\gamma = \upsilon^2 + 15\omega^2$$

Inference: 1.2

1. $\beta(1,1) - 161$ is a carol number
2. $4\alpha(c,1) - TO_c + 101T_{8,c} + 473Gno_c \equiv 0 \pmod{1213}$
3. $\frac{\gamma(c,c)}{c^2} - 3$ is a star number

4. $6\gamma(c,1)+\beta(c,1)+18T_{12,c}+36Gno_c \equiv 0 \pmod{534}$
5. $\alpha(1,1)-50$ is a pronic number

Substitution: 1.3

Write equation (3) as,

$$\lambda^2+15\eta^2=64\gamma^3\times 1 \tag{7}$$

'1' can be written as,

$$1=\frac{(1+i\sqrt{15})(1-i\sqrt{15})}{16} \tag{8}$$

Equating (4), (5) and (8) in (7) we have,

$$(\lambda+i\sqrt{15}\eta)(\lambda-i\sqrt{15}\eta)=(7+i\sqrt{15})(7-i\sqrt{15})(\upsilon+i\sqrt{15}\omega)^3(\upsilon-i\sqrt{15}\omega)^3\frac{(1+i\sqrt{15})(1-i\sqrt{15})}{16}$$

Equating the terms both real and imaginary,

$$\lambda=\frac{1}{4}(-8\upsilon^3+1800\omega^3+360\upsilon\omega^2-360\upsilon^2\omega)$$

$$\eta=\frac{1}{4}(8\upsilon^3+120\omega^3-360\upsilon\omega^2-24\upsilon^2\omega)$$

Reducing equation (2) using (λ) and (η) gives,

$$\alpha=480\omega^3+84\upsilon^2\omega-180\upsilon\omega^2$$

$$\beta=-4\upsilon^3+420\omega^3+96\upsilon^2\omega$$

$$\gamma=\upsilon^2+15\omega^2$$

Inference: 1.3

1. $\alpha(1,1)-9\gamma(1,1)$ is a pronic number
2. $\alpha(c,1)-6T_{30,c}+51Gno_c \equiv 0 \pmod{429}$
3. $\frac{32\beta(c,c)}{c^3}-257$ is a carol number
4. $\alpha(c,1)-4\gamma(c,1)-8T_{22,c}+54Gno_c \equiv 0 \pmod{366}$
5. $3\gamma(1,1)-3$ is a hexagonal number

IV. CONCLUSION

In this paper, we've employed diverse substitutions to seek integral solutions for the non-homogeneous ternary cubic equation. These same substitutions can be applied to solve analogous equations.

REFERENCES

[1] David M. Burton "Elementary Number Theory", Sixth edition, Tata mc graw hill edition.
[2] Neville Robbins "Beginning Number Theory", Second edition, Naroa Publishing House.
[3] Gareth A. Jones and J. Marry Jones "Elementary Number Theory", Springer international edition.
[4] Dr. Sudhir K. Pundir and Dr. Rimple Pundir "Theory of Numbers", Second edition, Published by: K.K. Mittal prakashan, meerut-250001, Laser typesetting.
[5] Thomas Koshy "Elementary Number Theory with Applications", Published by Elsevier, a division of Reed Elsevier India private limited, 17-A/1, Main Ring Road, Lajpat Nagar-Iv,New Delhi-110 024,India.
[6] M.A.Gopalan, P.Shanmuganadham and A.Vijayashankar, (2008), "On Binary Quadratic Equation $x^2-5xy+y^2+8x-20y+15=0$",Acta Ciencia Indica, Vol .XXXIVM.NO.4,pp.1803-1805.
[7] P.Saranya, G. Janaki, "On the Exponential Diophantine Equation $36^x+3^y=z^2$", International Research Journal of Engineering and Technology, Vol 4, Issue 110, pp: 1042-1044, 2017.
[8] C.Saranya, G, Janaki, " Observations on the Ternary Quadratic Diophantine Equation $5x^2+7y^2=972z^2$", Jamal Academic Research Journal- An Interdisciplinary,Special Issue,February 2016, pg no.305-308,Impact Factor, ISSN No: 0973-0303.

[9] G. Janaki, P. Saranya, "On the Ternary Quadratic Diophantine Equation $5(x^2+y^2)-6xy=4z^2$", Imperial Journal of Interdisciplinary Research, Vol 2, Issue 3, pp: 396-397, 2016.

[10] M.A.Gopalan, S.Vidhyalakshmi, A.Kavitha, " Observation on the Ternary Cubic Equation $x^2+y^2+xy=12z^3$", Antartica J.Math 10(5),2013,453-460.

[11] S.Vidhyalakshmi, T.R. Usha Rani and M.A. Gopalan "Integral Solution of Non-Homogenous Ternary Cubic Equation $ax^2+by^2=(a+b)z^3$", Diophantus J.Math., 2(1), 31-38, 2013.

[12] G. Janaki, P. Saranya, "On the Ternary Cubic Diophantine Equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$", Imperial Journal of Interdisciplinary research.Vol 5, Issue 3, pp: 227-229, 2016.

[13] C. Saranya, G. Janaki, "Integral Solutions of the Ternary Cubic Equation $3(x^2+y^2)-4xy+2(x+y+1)=972z^3$", International Journal of Engineering and Techonology, Vol 4, Issue 3, March 2017.pg no.665-669.

[14] P.Saranya, G. Janaki, On the Exponential Diophantine Equation $36^x+3^y=z^2$, International Research Journal of Engineering and Technology, Vol 4, Issue 11 , Nov 2017, Pg No: 1042-1044.

[15] P.Saranya, G. Janaki, Ascertainment on The Exponential Equation $p^{3c}-a(p^{2c}-p^b)=p^{b+c}$,Compliance Engineering Journal, Vol 10 , Issue 8 , August 2019, Pg No:225-229.

[16] G.Janaki, P.Saranya, On the Ternary Cubic Diophantine Equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$, International Journal of Science and Research-Online, Vol 5, Issue 3 , March 2016, Pg No: 227-229.

[17] P.Saranya, G. Janaki, Explication of The Transcendental Equation $p+\sqrt{p^3+q^3-pq}+\sqrt[3]{r^2+s^2}=(m^2+1)t^3$, Adalya Journal, Vol 8, Issue 9, September 2019, Pg No: 70-73.

[18] Saranya, P., G. Janaki, and M. Shri Padmapriya. "Elucidation of the Transcendence Equation $j+\sqrt{(j^3+k^3-jk)}+\sqrt[3]{(l^2+m^2)}=h^3(2^{2n}+1)$." (2023): Pg No: 28-36.

[19] Saranya, P., G. Janaki, and K. Poorani. "On the Ternion Cubical Diophantine Equation $5(m^2+n^2)-6(mn)+8(m+n)+16=370p^3$." (2023): Pg No: 13-21.

[20] Saranya, P., and M. Priya. "Transcendental Equation Involving Palindrome Number." Arya Bhatta Journal of Mathematics and Informatics 15, no. 1 (2023): 61-66.