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The Algebraic Structure of an Implicit Runge-Kutta Type Method

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Abstract: In this paper, the theory of linear transformation (Homomorphism) and monomorphism is applied to a first-order Runge-Kutta Type Method illustrated in a Butcher Table and the extended second order Runge- Runge-Kutta type Method to substantiate their uniform order and error constants obtained. A homomorphism is a mapping from one group to another group which preserves the group operations. It's sometimes called the operation preserving function. The methods which initially are Linear Multistep were reformulated into Runge-Kutta (R-K) Type to establish the advantages the R-K has over Linear Multistep. The first-order Linear multistep was reformulated into first-order R-K type which was further extended to second order. This extension can be made to higher order. For this study, the extension was limited to the second order.

Keywords: Linear transformation, Monomorphism, Implicit, Runge-Kutta type

I. INTRODUCTION

A function T between two vector spaces $T: V \rightarrow W$ that preserves the operations of addition if v_1 and $v_2 \epsilon V$ then

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad (1)$$

And scalar multiplication if $v \epsilon V$ and $r \epsilon R$, then

$$T(rv) = rT(v) \quad (2)$$

is a homomorphism or linear Transformation, Agam (2013).

A homomorphism that is one-to-one or a mono is called a monomorphism. The monomorphism transformation preserves its algebraic structure and the order of the domain in its Range.

Reference [2,4] extended the general Linear method to the case in which the second derivative, as well as the first derivative, can be calculated. They constructed methods of third and fourth order which are A-stable, possess the Runge-Kutta stability property, and have a diagonally implicit structure for efficient implementation. However, they concentrated on only linear problems for which it is possible to compute accurate starting methods, the general-purpose starting methods for non-linear problems were not developed. Reference [5-7,10] presented a direct integration of second-order Ordinary Differential Equation using only the Explicit Runge-Kutta Nystrom (RKN) method with a higher derivative. They derived and tested various numerical schemes on standard problems. Due to the limitations of Explicit Runge-Kutta (ERK) in handling stiff problems, the extension to higher order Explicit Runge-Kutta Nystrom (RKN) was considered and results obtained showed an improvement over conventional Explicit Runge-Kutta schemes. The Implicit Runge-Kutta scheme was however not considered. [8] derived an implicit 6-stage block Runge-Kutta Type Method for direct integration of second-order (special or general), third-order (special or general) as well as first-order initial value problems and boundary value problems. The theory of Nystrom was adopted in the reformulation of the methods [6].

The convergence and stability analysis of the method were conducted, and the region of absolute stability was plotted. The method was A-stable, possessed the Runge-Kutta stability property, had an implicit structure for efficient implementation, and produced simultaneous approximation of the solution of both linear and nonlinear initial value problems.

A. *Consistency of Runge Kutta Methods*

The first and second-order Ordinary Differential Equation (ODE) methods are said to be consistent if

$$\varphi(x, y(x), 0) \equiv f(x, y(x)) \quad (3)$$

$$\varphi(x, y(x), y'(x), 0) \equiv f(x, y(x), y'(x)) \quad (4)$$

holds respectively.

Note that consistency demands that $\sum_1^s b_s = 1$ (5)

and

$$\sum_1^s b_s = \frac{1}{2} \quad (6)$$

for first and second order respectively. Also $\sum_1^s b_s$ is as shown in the butcher array table.

α	$\bar{A}$	A
	$\bar{b}^T$	b

$A = a_{ij} = \beta^2$ $\quad\bar{A} = \bar{a}_{ij} = \beta$ $\quad\beta = \beta e$

B. *Convergence of Runge –Kutta Methods*

If $y' = f(x, y(x))$; $y'' = f(x, y(x), y'(x))$ represents first and second order respectively, then for such method consistency is necessary and sufficient for convergence. Hence the methods are said to be convergent if and only if they are consistent [1].

II. METHODOLOGY

Proposition: Let T be a linear transformation that is continuously differentiable on a set of ordered three-tuple vector in $\mathbb{R}^3$ as follows.

$$V_i = (x + c_i h, y + \sum_{j=1}^s a_{ij} T(v_j), y' + \sum_{j=1}^s a_{ij} T'(v_j)) \in \mathbb{R}^3 \quad (7)$$

$$T(V_i) = h(y' + \sum_{j=1}^s a_{ij} T'(v_j)) \quad (8)$$

and

$$T'(v_i) = hf\left(x + c_i h, y + \sum_{j=1}^s a_{ij} T(v_j), y' + \sum_{j=1}^s a_{ij} T'(v_j)\right) = hm_i \quad (9)$$

$$i{:}e\ m_i = f\left(x + c_i h, y + \sum_{j=1}^s a_{ij} T(v_j), y' + \sum_{j=1}^s a_{ij} T'(v_j)\right) \quad (10)$$

Then the Transformation $T: \mathbb{R}^3 \to \mathbb{R}$ is a well-defined monomorphism:

Proof:

Let $u, v \in \mathbb{R}$ be defined by

$U = (x + c_i h, y_1 + \sum_{j=1}^{s} a_{ij}\, T(u_j), y_1' + \sum_{j=1}^{s} a_{ij} T'(u_j))$ (11)

$V = (x + c_i h, y_2 + \sum_{j=1}^{s} a_{ij}\, T(v_j), y_2' + \sum_{j=1}^{s} a_{ij} T'(v_j))$ (12)

$T(U + V) = h(y_1' + y_2' + \sum_{j=1}^{s} a_{ij}\, (T'(u_j) + T'(v_j))$ (13)

By the definition of T on $\mathbb{R}^3$

$T(U + V) = h(y_1' + \sum_{j=1}^{s} a_{ij} T'(u_j)) + h(y_2' + \sum_{j=1}^{s} a_{ij} T'(v_j))$ (14)

$T(U + V) = T(U) + T(V)$ (15)

$T(k.U) = k.T(U)$ (16)

Hence T is a homomorphism

Now we show that T is $1 - 1$

Let $u, v \in \mathbb{R}^3$ with

$T(u) = T(v)$ (17)

By definition of T

$\Rightarrow h(y_1' + \sum_{j=1}^{s} a_{ij}\, T'(u_j)) = h(y_2' + \sum_{j=1}^{s} a_{ij}\, T'(v_j))$ (18)

Since

$T(u) = T(v)$ then $T(u_j) = T(v_j)$ and $T'(u_j) = T'(v_j)$ (19)

$y_1 = y_2$ and $x_1 + c_i h = x_2 + c_i h$ $i{:}e$ $x_1 = x_2$ (20)

Hence $U = V$ (21)

Thus, T is $1 - 1 \Leftrightarrow$ a monomorphism from $\mathbb{R}^3 \to \mathbb{R}$

Remark: The necessity for the above proposition is to ensure that the algebraic structure and the order do not change during the transformation. We consider the case $k = 1$. The Runge-Kutta type method is obtained from the Reformulation of a Linear Multistep Method of the same step number and is given as:

$$\left.\begin{aligned} y_{n+\frac{1}{2}} &= y_n + h(\tfrac{3}{4}k_2 - \tfrac{1}{4}k_3) \\ y_{n+1} &= y_n + hk_2 \end{aligned}\right\} \quad (22)$$

Where

$k_1 = f(x_n, y_n)$

$k_2 = f(x_n + \frac{1}{2}h, y_n + h\left\{0k_1 + \frac{3}{4}k_2 - \frac{1}{4}k_3\right\})$

$k_3 = f(x_n + h, y_n + h\{0k_1 + k_2 + 0k_3)$

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Putting the coefficients of (22) in the Butcher Table we obtained Table 1

Table I: The Butcher Table for $k = 1$

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{4} & \frac{-1}{4} \\ 1 & 0 & 1 & 0 \\ \hline & 0 & 1 & 0 \end{array}$$

Extending the method (22) with the coefficients as characterized in the butcher array for second order as

$$\begin{array}{c|c|c} \alpha & \bar{A} & A \\ \hline & \bar{b}^T & b \end{array}$$

$A = a_{ij} = \beta^2$ $\quad \bar{A} = \bar{a}_{ij} = \beta$ $\quad \beta = \beta e$ gives

Table II: The Butcher Table for Second Order for K=1

$$\begin{array}{c|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{3}{4} & \frac{-1}{4} & 0 & \frac{5}{16} & \frac{-3}{16} \\ 1 & 0 & 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} \\ \hline & 0 & 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} \end{array}$$

The Table I satisfies the Runge-Kutta conditions for the solution of first-order ODE since

(i) $\quad \sum_{j=1}^{s} a_{ij} = c_i$ (23)

(ii) $\quad \sum_{j=1}^{s} b_j = 1$ (24)

We consider the general second-order differential equation in the form

$y'' = f(x, y, y'), y(x_0) = y_0 \quad y'(x_0) = \quad y'_0$ (25)

$y'' = f(v), \qquad v = (x, y, y')$ (26)

$T(V_i) = T\left(x + c_i h, y + \sum_{j=1}^{3} a_{ij} T(V_j), y' + \sum_{j=1}^{3} a_{ij} T'(V_j)\right)$ (27)

$= h\ (y' + \sum_{j=1}^{3} a_{ij}\ T'\ (V_j))$ (28)

$=h\ (y' + \sum_{j=1}^{3} a_{ij}\ hm_j)$ (29)

$T(V_1) = h(y' + 0hm_1 + 0hm_2 + 0hm_3) \Rightarrow T(V_1) = h(y')$ (30)

$$T(V_2) = h\left(\left(y' + 0hm_1 + \frac{3}{4}hm_2 - \frac{1}{4}hm_3\right)\right)$$

$\Rightarrow T(V_2) = h\left(\left(y' + \frac{3}{4}hm_2 - \frac{1}{4}hm_3\right)\right)$ (31)

$T(V_3) = h(y' + 0hm_1 + hm_2 + 0hm_3)$

$\Rightarrow T(V_3) = h(y' + hm_2)$ (32)

Also

$m_i = T'(V_j) = f\left(x + c_i h, y + \sum_{j=1}^{s} a_{ij} T(V_j), y' + \sum_{j=1}^{s} a_{ij} T'(V_j)\right)$ (33)

$m_1 = f(x + 0h, y + 0 + 0 + 0) = \ m_1 = f(x, y)$ (34)

$m_2 = f(x + \frac{1}{2}h, y + \frac{1}{2}hy' + \frac{5}{16}h^2 m_2 - \frac{3}{16}h^2 m_3, y' + \frac{3}{4}hm_2 - \frac{1}{4}hm_3)$ (35)

$m_3 = f(x + h, y + hy' + \frac{3}{4}h^2 m_2 - \frac{1}{4}h^2 m_3, y' + hm_2\ )$ (36)

The direct method for solving $y'' = f(x, y, y')$ is now

$y_{n+1} = y_n + b_1 T(V_1) + b_2 T(V_2) + b_3 T(V_3)$ (37)

$y_{n+1} = y_n + 0T(V_1) + T(V_2) + 0T(V_3)$ (38)

$y_{n+1} = y_n + T(V_2)$ (39)

$y_{n+1} = y_n + h\left(y_n' + \frac{3}{4}hm_2 - \frac{1}{4}hm_3\right)$ (40)

$y_{n+1} = y_n + hy_n' + \frac{h^2}{4}(3m_2 - m_3)$ (41)

$y_{n+1}' = y_n' + b_1 T'(V_1) + b_2 T'(V_2) + b_3 T'(V_3)$ (42)

$y_{n+1}' = y_n' + 0hm_1 + hm_2 + 0hm_3$ (43)

$y_{n+1}' = y_n' + hm_2$ (44)

III. RESULTS AND DISCUSSION

We made use of the coefficients of the Butcher table of the first-order RKTM to prove the second-order RKTM. Equations (40) and (44) satisfy the Runge-Kutta consistency conditions of second and first order respectively. This further shows that it is a monomorphism.

IV. CONCLUSION

This research work established the reason behind the uniform order and error constant of the first-order Runge-kutta type method and the extended second-order Runge-kutta type method. Also why the algebraic structure and the order of the two methods are preserved and not changed during the transformation.

Data Availability: The article includes the information necessary to understand the results of this investigation.

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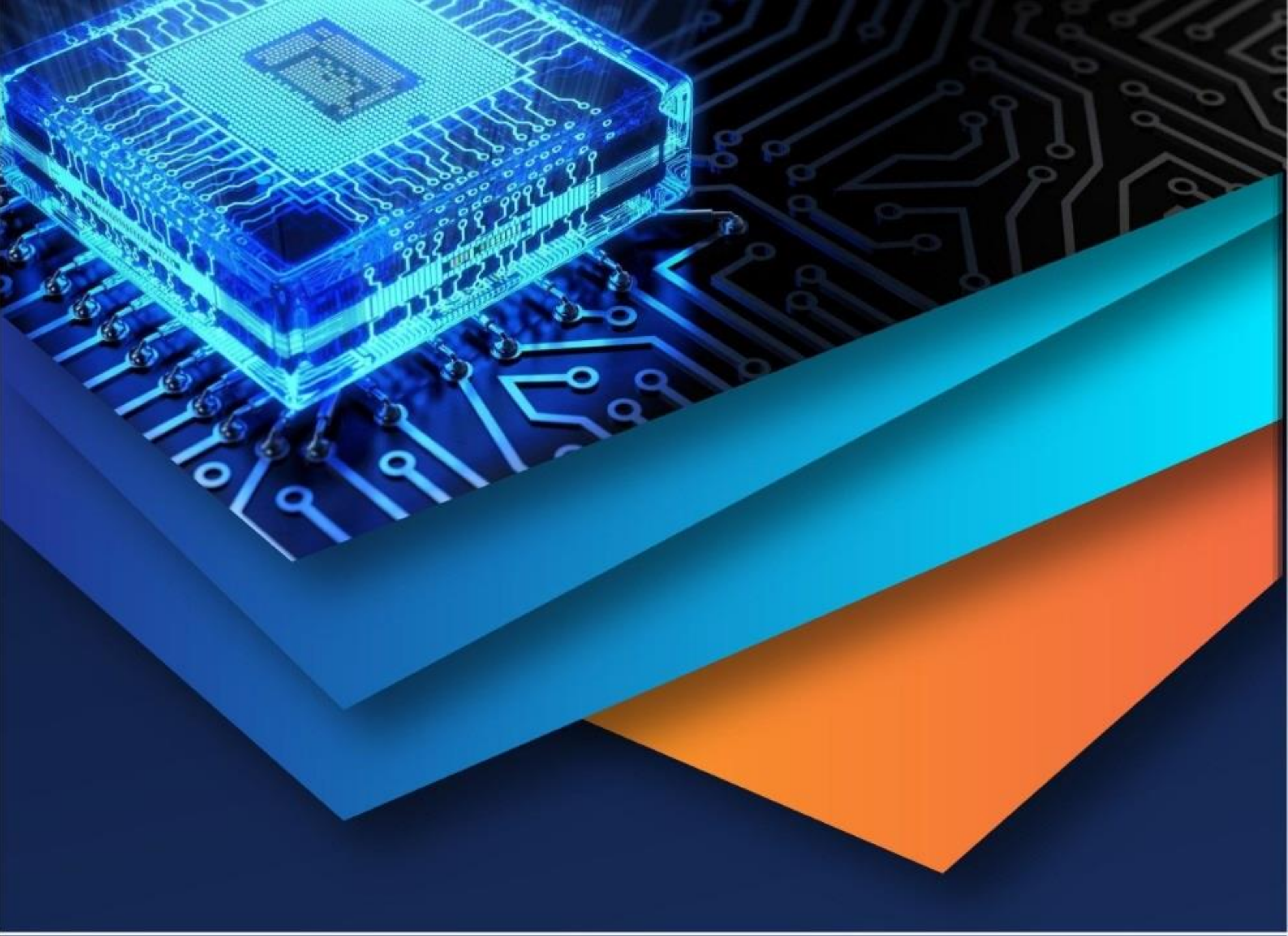
Statement and Declarations

Declarations

Conflict of Interest The authors hereby state that they have no conflict of interest in or out of the publishing of this work.

Ethical approval: This article contains no studies with human participants or animals performed by any authors.

Competing Interest: The authors hereby state that they have no conflict of interest in or out of the publishing of this work.



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