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On Tensor Product of Standard Graphs-II

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Abstract: The characteristic properties of the graphs $K_m \wedge C_n$, $K_m \wedge P_n$, $C_m \wedge P_n$ are studied and mainly their Wiener Indices are obtained, wherever possible.

Index Terms: Tensor (Kronecker) product, Wiener index (number), connected graph, Hamiltonian graph.

I. INTRODUCTION

The Wiener index is initiated from the work of Wiener [5]. This Wiener number is an important topological index associated with the molecular graph of atoms which is a connected one. Further it is widely used to describe the molecular structures. Till now, no recursive method is known for the calculation of the Wiener number of a general connected graph.

In this paper, the Wiener numbers of $K_m \wedge C_n$, $K_m \wedge P_n$, $C_m \wedge P_n$, wherever possible are obtained. Some interesting observations are made. This paper is a continuation of our previous paper [3].

II. PRELIMINARIES

We present some known definitions and results (in the refined form, wherever necessary) for a ready reference to go through the work presented in the subsequent sections. For standard notation and further results, we refer Bondy & Murthy [1].

A. Definition 2.1 [4]

G, H are disjoint graphs. The Tensor product of G and H , denoted by $G \wedge H$ (that is isomorphic to $H \wedge G$) is the graph whose vertex set is $V(G) \times V(H)$ and the edge set being the set of all elements of the form $(u, v) (u^1, v^1)$ where $u, u^1 \in V(G)$, $v, v^1 \in V(H)$, $uu^1 \in E(G)$ and $vv^1 \in E(H)$.

B. Observations 2.2

- 1) If one of G, H is an empty graph (i.e. has no edges) then $G \wedge H$ is also an empty graph.
- 2) If G, H are finite, simple graphs with m, n vertices respectively, then $G \wedge H$ is a finite, simple graph with mn vertices. Further, if $u \in V(G)$ and $v \in V(H)$ then

$$\deg_{G \wedge H} (u, v) = \{ \deg_G u \} \cdot \{ \deg_H v \}.$$

C. Definition 2.3[5].

The Wiener index $W(G)$ of a finite, connected graph is defined to be

$$W(G) = \frac{1}{2} \sum_{u, v \in V(G)} d(u, v),$$

where $d(u, v)$ denotes the distance (the length of any shortest $u - v$ path) between u & v in G .

- 1) **Result 2.4[4]:** G_1, G_2 are connected graphs. Then $G_1 \wedge G_2$ is connected if and only if (iff) either G_1 or G_2 contains an odd cycle.
- 2) **Result 2.5 [4]:** If G_1, G_2 are connected graphs with no odd cycles, then $G_1 \wedge G_2$ has exactly two components.
- 3) **Result 2.6[1]:** A nonempty connected graph is Eulerian iff every vertex is of even degree.
- 4) **Result 2.7[1]:** If G is a simple graph with the number of vertices $v \geq 3$ and the minimum degree $\delta \geq v/2$ then G is Hamiltonian.
- 5) **Result 2.8[1]:** A simple graph is bipartite iff it contains no odd cycles.

In what follows m and n are positive integers.

§3. Results on $K_m \wedge C_n$ (m, n being positive integers & $n \geq 3$).

Initially, we have

III. OBSERVATIONS.

$K_1 \wedge C_n$ is an empty graph (with n vertices).

So, we consider $m \geq 2$ (and $n \geq 3$).

Denote $V(K_m) = \{u_1, u_2, \dots, u_m\}$ and $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Then $K_m \wedge C_n$ is the graph with $V(K_m \wedge C_n) = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 1, 2, \dots, n\}$ and the edge set being the set of elements of the form $(u_i, v_j) (u_{i'}, v_{j'})$ where $i, i' \in \{1, 2, \dots, m\}$ with $i \neq i'$; $j, j' \in \{1, 2, \dots, n\}$ with $j' = j - 1$ or $j + 1$ under the convention $v_0 = v_n, v_{n+1} = v_1$.

A. *Theorem.* $K_m \wedge C_n$ (isomorphic to $C_n \wedge K_m$) is a simple, finite and $2(m - 1)$ -regular graph (an even integer) with mn vertices and $(m-1)mn$ edges (observe that the degree does not depend on n).

B. *proof.* Since K_m, C_n are simple, finite graphs and so is $K_m \wedge C_n$. As K_m is $(m-1)$ -regular and C_n is 2-regular, it follows that $K_m \wedge C_n$ is $2(m-1)$ -regular. Since $K_m \wedge C_n$ has mn vertices, it follows that there are $(m-1)mn$ edges.

This proves the Theorem.

C. *Observations.* K_2, C_n are connected graphs and K_2 does not contain an odd cycle (in fact, any cycle).

a) By Result (2.4), it follows that $K_2 \wedge C_n$ is connected iff n is odd (since C_n contains the cycle C_n only).

b) By Result (2.5), it follows that $K_2 \wedge C_n$ has exactly two components iff n is even.

D. *Theorem.* $K_2 \wedge C_{2n+1}$ ($n \geq 1$) is isomorphic to $C_{2(2n+1)}$ and

$$W(K_2 \wedge C_{2n+1}) = (2n+1)^3.$$

E. *Proof.* By Th. (3.2) and Obs.(3.3) (a), $K_2 \wedge C_{2n+1}$ is a connected 2-regular graph with $2(2n+1)$ vertices and $(1)(2)(2n+1) = 2(2n+1)$ edges. So $K_2 \wedge C_{2n+1}$ is isomorphic to $C_{2(2n+1)}$. Hence, by a known result [see 2], it follows that

$$W(K_2 \wedge C_{2n+1}) = W(C_{2(2n+1)}) = (2n+1)^3.$$

In fact, in the usual notation, $K_2 \wedge C_{2n+1}$ is the cycle $\{(u_1, v_1), (u_2, v_2), (u_1, v_3), \dots, (u_2, v_{2n}), (u_1, v_{2n+1}), (u_2, v_1), (u_1, v_2), \dots, (u_1, v_{2n}), (u_2, v_{2n+1}), (u_1, v_1)\}$.

F. *Theorem.* $K_2 \wedge C_{2n}$ ($n \geq 2$) is isomorphic to the (disjoint) union of C_{2n} & C_{2n} and the Wiener number of each component is n^3 .

By Th.(3.2), $K_2 \wedge C_{2n}$ is a 2-regular graph with $4n$ vertices and $4n$ edges. By observation (3.3)(b), this has exactly two components.

Now follows that each component is a cycle. Clearly the components are the cycles $\{(u_1, v_1), (u_2, v_2), (u_1, v_3), \dots, (u_1, v_{2n-1}), (u_2, v_{2n}), (u_1, v_1)\}$ and $\{(u_2, v_1), (u_1, v_2), (u_2, v_3), \dots, (u_2, v_{2n-1}), (u_1, v_{2n}), (u_2, v_1)\}$. Each is C_{2n} . Hence by a known result [see 2] follows the theorem.

G. *Observations.*

Since $K_2 \wedge C_{2n+1}$ ($n \geq 1$) is an even cycle, follows that this graph is bipartite, Eulerian and Hamiltonian.

Since $K_2 \wedge C_{2n}$ ($n \geq 2$) is union of C_{2n} and C_{2n} , follows that the graph is bipartite and each component (C_{2n}) is Eulerian and Hamiltonian.

H. *Theorem.* For $m, n \geq 3$, $K_m \wedge C_n$ is a) connected b) Eulerian and c) bipartite iff n is even.

Proof. Since K_m, C_n are connected and K_m ($m \geq 3$) contains the odd cycles K_3 , by Result (2.4), it follows that $K_m \wedge C_n$ is connected.

This proves (a).

Since the degree of each vertex of $K_m \wedge C_n$ is even (see Th.(3.2)), by the characterization result (2.6), it follows that $K_m \wedge C_n$ is Eulerian.

This proves (b).

Suppose n is even ($\Rightarrow n \geq 4$).

In the usual notation,

$$X = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 1, 3, \dots, (n - 1)\},$$

and

$$Y = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 2, 4, \dots, n\}$$

are such that $\{X, Y\}$ is a bipartition of the vertex set $K_m \wedge C_n$. So the graph is bipartite.

When n is odd,

$\{(u_1, v_1), (u_2, v_2), (u_3, v_3), \dots, (u_n, v_n), (u_1, v_1)\}$ is a cycle of length n (odd) in $K_m \wedge C_n$. So it is not bipartite.

This completes the proof of the Theorem.

I. *Observations*

$K_2 \wedge C_n$ ($n \geq 3$) is discussed in this article.

$K_m \wedge C_3 = K_m \wedge K_3$ and this is discussed in [3].

a) $K_3 \wedge C_n = C_3 \wedge C_n$ ($n \geq 3$) and this is discussed in [3].

Thus, we are left with the graphs. $K_m \wedge C_n$ ($m, n \geq 4$) and we discuss about these graphs.

J. Result. $W(K_m \wedge C_4) = 4m(3m + 2)$ ($m \geq 4$).

K. Justification. Since the graph is regular, it follows that the graph is symmetric w.r.t. all $4m$ vertices (u_i, v_j) ($i = 1, 2, \dots, m; j = 1, 2, 3, 4$).

On Calculation

$$d\{(u_1, v_1), (u_1, v_1)\} = 0, d\{(u_1, v_1), (u_1, v_3)\} = 2,$$

$$d\{(u_1, v_1), (u_i, v_j)\} = 2 \text{ for } i = 2, 3, \dots, m; j = 1, 3.$$

$$d\{(u_1, v_1), (u_1, v_2)\} = 3 = d\{(u_1, v_1), (u_1, v_4)\},$$

$$d\{(u_1, v_1), (u_i, v_j)\} = 1 \text{ for } i = 2, 3, \dots, m; j = 2, 4.$$

So

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^3 d\{(u_1, v_1), (u_i, v_j)\} &= 1(0) + \{1 + 2(m-1)\}(2) + 2(3) + 2(m-1)(1) \\ &= (4m-2) + 6 + (2m-2) \\ &= 6m + 2. \end{aligned}$$

We get the same sum for all the $4m$ vertices. Hence

$$\begin{aligned} W(K_m \wedge C_4) &= (1/2)(4m)(6m + 2) \\ &= 4m(3m + 1). \end{aligned}$$

L. Result. $W(K_m \wedge C_5) = 5m(4m + 1)$ ($m \geq 4$).

M. Justification. As the graph is regular, follows the graph is symmetric w.r.t. all the $5m$ vertices.

On Calculation,

$$d\{(u_1, v_1), (u_1, v_1)\} = 0,$$

$$d\{(u_1, v_1), (u_1, v_j)\} = 3 \text{ for } j = 2, 5,$$

$$d\{(u_1, v_1), (u_1, v_j)\} = 2 \text{ for } j = 3, 4.$$

$$d\{(u_1, v_1), (u_i, v_j)\} = 1 \text{ for } i = 2, 3, \dots, m; j = 2, 5.$$

So

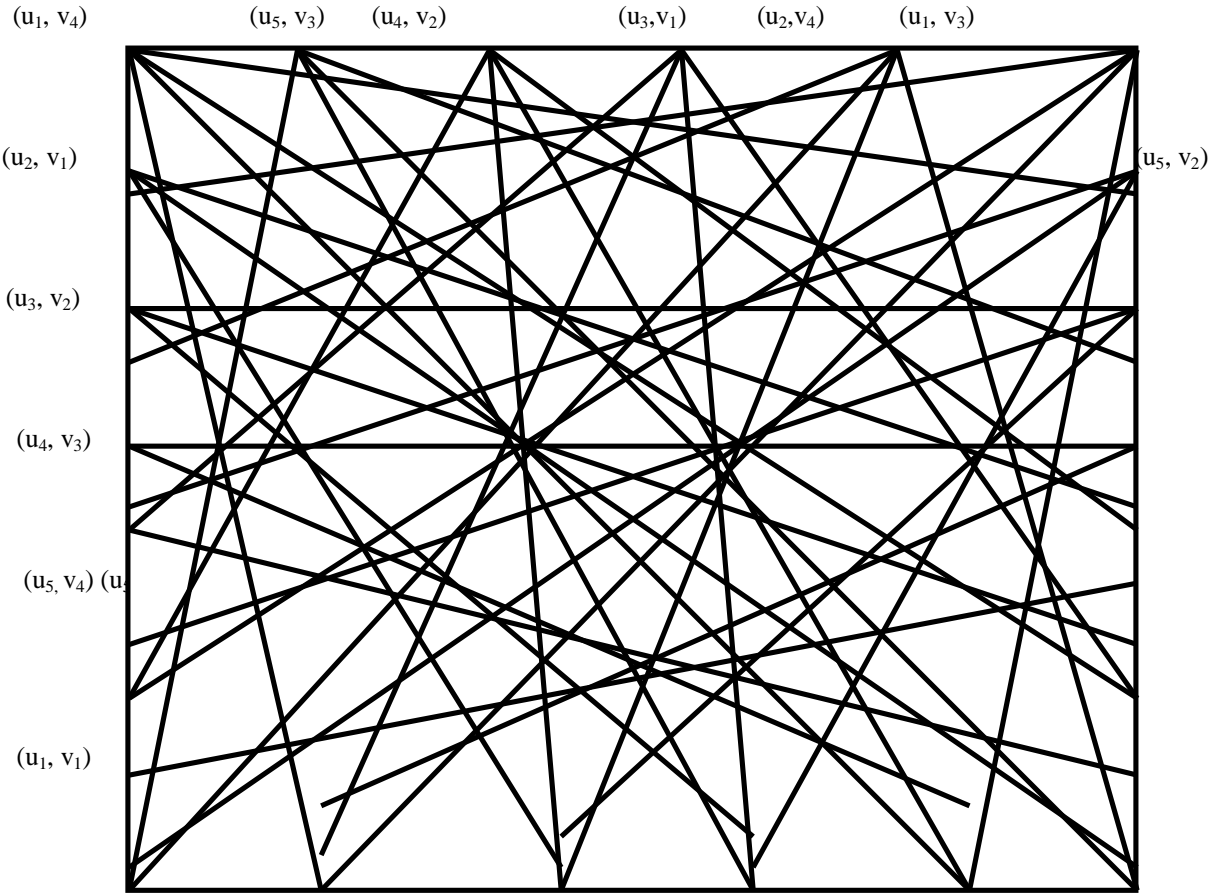
$$\begin{aligned} d\{(u_1, v_1), (u_i, v_j)\} &= \sum_{i=1}^m \sum_{j=1}^3 d\{(u_1, v_1), (u_i, v_j)\} \\ &= 1(0) + \{2 + 3(m-1)\}(2) + 2(3) + 2(m-1)(1) \\ &= (6m-2) + 6 + (2m-2) = 8m + 2. \end{aligned}$$

We get the same sum for all the $5m$ vertices. Hence

$$W(K_m \wedge C_5) = (1/2)(5m)(8m + 2) = 5m(4m + 1).$$

Finally, we exhibit the following:

N. A diagrammatic representation of $k_4 \wedge c_5$.



O. Open problem. To find a general formula for the Wiener number of $K_m \wedge C_n$ ($m, n \geq 4$).

IV. RESULTS

ON $K_m \wedge P_n$ (m, n being positive integers).

Primarily, we have

A. Observations.

1) If atleast one of m, n is 1, then $K_m \wedge P_n$ is an empty graph.

So, we consider $m, n \geq 2$.

2) $K_m \wedge P_2 = K_m \wedge K_2$ ($m \geq 2$) and this is discussed in [3].

So, we take $n \geq 3$.

3) $K_2 \wedge P_n = P_2 \wedge P_n$ ($n \geq 2$) and this is discussed in [3].

So, we take $m \geq 3$.

Thus, we discuss about the graphs where $m, n \geq 3$

4) Denote $V(K_m) = \{u_1, u_2, \dots, u_m\}$ and $V(P_n) = \{v_1, v_2, \dots, v_n\}$, then $K_m \wedge P_n$ is the graph with $V(K_m \wedge P_n) = \{(u_i, v_j) : i=1, 2, \dots, m; j=1, 2, \dots, n\}$ and the edge set being the set of elements of the form $(u_i, v_j) (u_{i'}, v_{j'})$ where $i, i' \in \{1, 2, \dots, m\}$ with $i' \neq i, j, j' \in \{1, 2, \dots, n\}$, $j' = 2$ when $j=1$, $j' = n-1$ when $j=n$ and $j' = j-1$ or $j+1$ when $2 \leq j \leq n-1$.

Since $\deg_{K_m}(u_i) = m-1$ and $\deg_{P_n}(v_j) = 1$ or 2 according as $j=1, n$ or $j=2, \dots, (n-1)$, it follows that

$$\deg_{K_m \wedge P_n}(u_i, v_j) = \begin{cases} 1(m-1) & \text{for } i=1, 2, \dots, m; j=1 \text{ or } n, \end{cases}$$

$$2(m - 1) \text{ for } i=1, 2, \dots, m; j=2, 3, \dots, (n - 1).$$

(Observe that the degree does not depend on 'n').

B. *Theorem.* $K_m \wedge P_n$ ($m, n \geq 3$) (isomorphic to $P_n \wedge K_m$) is a simple, finite graph with mn vertices and $m(m - 1)(n - 1)$ edges.

1) *Proof.* Since K_m, P_n are simple, finite graphs and so is $K_m \wedge P_n$. It has $2m$ vertices of degree $(m - 1)$ and has $(n - 2)m$ vertices of degree $2(m - 1)$; it follows that the number of edges in $K_m \wedge P_n$ is $\frac{1}{2} [2m(m - 1) + (n - 2)m + 2(m - 1)] = m(m - 1)(n - 1)$.

C. *Theorem.* $K_m \wedge P_n$ ($m, n \geq 3$) is

connected b) bipartite and c) Eulerian iff m is odd.

1) *Proof.* Since K_m, P_n are connected graphs and K_m ($m \geq 3$) contains the odd cycle K_3 , by Result (2.4), it follows that $K_m \wedge P_n$ is connected. This proves (a).

In the usual notation, let

$$V_1 = \{(u_i, v_j) : i = 1, 2, \dots, m, j = 1, 3, \dots, \overline{n-1} \text{ or } n \text{ as according } n \text{ is even or odd}\},$$

$$V_2 = \{(u_i, v_j) : i = 1, 2, \dots, m, j = 2, 4, \dots, \overline{n-1} \text{ or } n \text{ as according } n \text{ is odd or even}\}.$$

Clearly no two vertices of either V_1 or V_2 are adjacent in $K_m \wedge P_n$. This implies that $\{V_1, V_2\}$ is a bipartition of the vertex set of $K_m \wedge P_n$. Thus $K_m \wedge P_n$ is bipartite. This proves (b).

By the characterization Result (2.6), $K_m \wedge P_n$ is Eulerian iff each of its vertex is of even degree and $\Leftrightarrow m$ is odd. This proves (c).

Thus the proof of the theorem is complete.

D. *REMARK.* $|V_1| = mn/2 = |V_2|$ when n is even and $|V_1| = m(n + 1)/2$ & $|V_2| = m(n - 1)/2$ when n is odd.

E. *Theorem.* $K_m \wedge P_3$ ($m \geq 3$) is a $((m - 1), 2(m - 1))$ -biregular graph and $W(K_m \wedge P_3) = m(7m + 1)$.

1) *PROOF.* By Th.(4.3), it follows that the graph is bipartite with a bipartition $\{V_1, V_2\}$, where

$$V_1 = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 1, 3\}$$

and

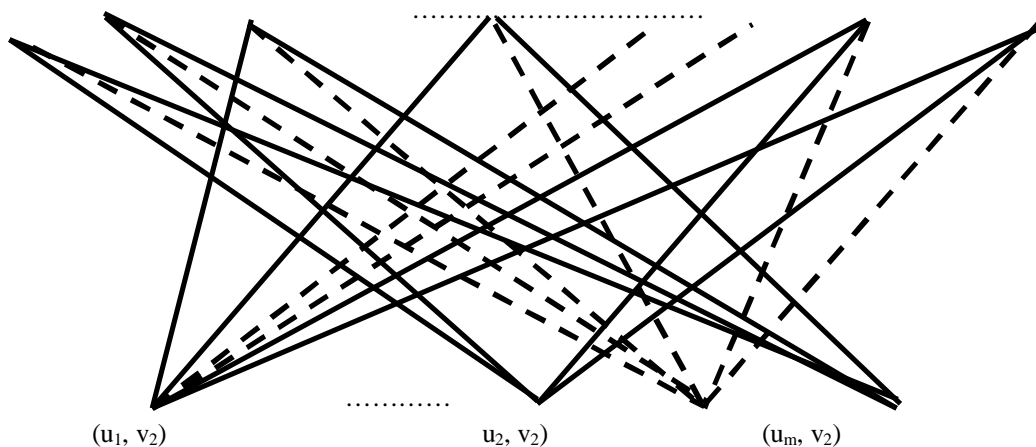
$$V_2 = \{(u_i, v_2) : i = 1, 2, \dots, m\}.$$

Clearly every vertex of V_1 is of degree $(m - 1)$ and that of V_2 is $2(m - 1)$. Thus the graph is a $((m - 1), 2(m - 1))$ -biregular graph.

Clearly $|V_1| = 2m$ and $|V_2| = m$.

Its diagrammatic representation is

$$(u_1, v_1) \quad (u_1, v_3) \quad (u_2, v_1) \quad \dots \quad (u_m, v_1) \quad (u_m, v_3)$$



Now,

$$d\{(u_1, v_1), (u_1, v_1)\} = 0$$

$$d\{(u_1, v_1), (u_i, v_1)\} = 2 \quad (i=2, \dots, m),$$

$$d\{(u_1, v_1), (u_i, v_3)\} = 2 \quad (i=2, \dots, m),$$

$$d\{(u_1, v_1), (u_1, v_2)\} = 3$$

and

$$d\{(u_1, v_1), (u_i, v_2)\} = 1 \quad (i=2,3, \dots, m);$$

$$\therefore \sum_{i=1}^m \sum_{j=1}^3 d\{(u_1, v_1), (u_i, v_j)\} = 0 + 2(m-1) + 2(m) + 3 + (m-1) = 5m.$$

Since, interchanging any two vertices in V_1 , does not affect the graph follows that we get the same sum for all the $2m$ points in V_1 .
Also

$$d\{(u_1, v_2), (u_1, v_1)\} = 3 = d\{(u_1, v_2), (u_1, v_3)\},$$

$$d\{(u_1, v_2), (u_i, v_j)\} = 1 \quad \text{for } i=2, \dots, m \text{ and } j=1, 3$$

$$\text{and } d\{(u_1, v_2), (u_1, v_2)\} = 0, d\{(u_1, v_2), (u_i, v_2)\} = 2 \text{ for } i=2, \dots, m-1.$$

$$\begin{aligned} \text{Thus } \sum_{i=1}^m \sum_{j=1}^n d\{(u_1, v_2), (u_i, v_j)\} &= (3+3) + (2m-2)1 + 0 + (m-1)(2) \\ &= 6 + 2m - 2 + 2m - 2 \\ &= 4m + 2. \end{aligned}$$

As in V_1 , we get the same sum for all points of V_2 .

$$\begin{aligned} \text{Thus } W(K_m \wedge P_3) &= \frac{1}{2} [(2m)(5m) + m(4m+2)] \\ &= 5m^2 + m(2m+1) \\ &= m(7m + 1). \end{aligned}$$

F. Result. $W(K_m \wedge P_n) = \frac{m}{6} [mn(n^2 + 5) + 6(n-2)]. \quad (m \geq 3 \ \& \ n \geq 3 \ \text{and } n \text{ is even}).$

In the usual notation, $K_m \wedge P_n$ is a bipartite graph with a bipartition, (X, Y) where

$$X = \{(u_i, v_j): i=1, 2, \dots, m; j=1, 3, \dots, (n-1)\},$$

and

$$Y = \{(u_i, v_j): i=1, 2, \dots, m; j=2, 4, \dots, n\}.$$

Clearly $|X| = |Y| = mn/2$. As the graph is symmetric w.r.t X and Y , we observe that

$$\sum_{i'=1}^m \sum_{j' \text{ odd}}^n d\{(u_{i'}, v_{j'}), (u_i, v_j)\} = \sum_{i'=1}^m \sum_{j' \text{ even}}^n d\{(u_{i'}, v_{j'}), (u_i, v_j)\}$$

(That means sum taken over the vertices in X is same as the sum taken over the vertices in Y).

On Calculation,

$$\begin{aligned} d\{(u_1, v_1), (u_i, v_1)\} &= \begin{cases} 0 & \text{if } i = 1, \\ 2 & \text{if } i \neq 1. \end{cases} \\ d\{(u_1, v_1), (u_i, v_2)\} &= \begin{cases} 3 & \text{if } i = 1, \\ 1 & \text{if } i \neq 1. \end{cases} \\ d\{(u_1, v_1), (u_i, v_j)\} &= (j-1) \text{ for all } i \text{ and } j = 3, \dots, n. \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^n d\{(u_1, v_1), (u_i, v_j)\} &= [1(0) + (m-1)2 + 1(3) + (m-1)1 + m \sum_{j=3}^n (j-1)] \\ &= [(2m-2) + 3 + (m-1) + m \sum_{j=2}^{n-1} j] \end{aligned}$$

$$= \frac{m}{2}(n^2 - n + 4) \quad \text{---(i)-->}$$

Since u_1 is adjacent with all $u_{j'}$ ($j' \neq 1$), it follows that we get the same sum when u_1 is replaced by $u_{j'}$.

Further

$d\{(u_1, v_3), (u_i, v_j)\} = (j - 3)$ for all i and $j = 5, \dots, n$ (when $n \geq 5$).

$$\Rightarrow \sum_{i=1}^m \sum_{j=1}^n d\{(u_1, v_3), (u_i, v_j)\} = \frac{m(n^2 - 5n + 16) + 4}{2} \quad \text{---(ii)-->}$$

For $j = 5, 7, \dots, (m - 1)$ (when $m \geq 8$)

$d\{(u_1, v_{j'}), (u_i, v_1)\} = (j' - 1)$ for all i ,

$d\{(u_1, v_{j'}), (u_i, v_2)\} = (j' - 2)$ for all i ,

$d\{(u_1, v_{j'}), (u_i, v_{j'-2})\} = 2$ for all i .

$d\{(u_1, v_{j'}), (u_i, v_{j'-1})\} = d\{(u_1, v_{j'}), (u_i, v_{j'+1})\} =$

if $i = 1$,

$d\{(u_1, v_{j'}), (u_i, v_{j'-1})\} = d\{(u_1, v_{j'}), (u_i, v_{j'+1})\} =$

$\left. \begin{array}{l} 3 \text{ if } i = 1, \\ 1 \text{ if } i \neq 1. \end{array} \right\}$
 $\left. \begin{array}{l} 2 \text{ if } i \neq 1. \end{array} \right\}$

$d\{(u_1, v_{j'}), (u_i, v_{j'+2})\} = 2$ for all i ,

$d\{(u_1, v_{j'}), (u_i, v_n)\} = (n - j')$ for all i .

$$\begin{aligned} \therefore \sum_{i=1}^m \sum_{j=1}^n d\{(u_1, v_{j'}), (u_i, v_j)\} &= m[(j' - 1) + (j' - 2) + \dots + 2] + \\ &2\{1(3) + (m - 1)(1)\} + \{1(0) + (m - 1)(2) + m[2 + 3 + \dots + (n - j')]\} \\ &= m[2 + \dots + (j' - 1)] + (4m + 2) + m[2 + \dots + (n - j')] \\ &= m\left[\frac{(j' - 1)j'}{2} - 1\right] + (4m + 2) + m\left[\frac{(n - j')(n - j' + 1)}{2} - 1\right] \\ &= m\left(\frac{n^2 + n + 4}{2}\right) + 2 + m[j'^2 - (n + 1)j']. \end{aligned}$$

$$\begin{aligned} \therefore \sum_{j'=5,7,\dots,(m-1)} d\{(u_1, v_{j'}), (u_i, v_j)\} \\ = m\left\{\frac{(n^2 + n + 4)}{4} + 4\right\}(n - 4) - \frac{m(n + 1)(n^2 - 16)}{4} - 10m + \frac{mn(n^2 - 1)}{6} \quad \text{---(iii)-->} \end{aligned}$$

Now follows from (i), (ii) & (iii),

$$\begin{aligned}
 W(K_m \wedge P_n) &= \frac{1}{2} (2) m \left[\frac{m}{2} (n^2 - n + 4) + \frac{m(n^2 - 5n + 16) + 4}{2} \right. \\
 &\quad \left. + \left\{ \frac{m(n^2 + n + 4) + 4}{4} \right\} (n - 4) - \frac{m(n + 1)(n^2 - 16)}{4} \right. \\
 &\quad \left. - 10m + \frac{mn(n^2 - 16)}{6} \right] \\
 &= \frac{1}{6} [m^2(n^3 + 5n) + 6m(n - 2)] \text{ (On simplification)} \\
 &= \frac{m}{6} [mn(n^2 + 5) + 6(n - 2)].
 \end{aligned}$$

This completes the proof of the result.

G. *Open problem.* To find a general formula for the Wiener Number of $K_m \wedge P_n$ when $m \geq 3$ and n is odd.

H. *Result.* $W(K_m \wedge P_5) = m(25m + 3)$ ($m \geq 3$).

1) *ROOF.* Clearly $K_m \wedge P_5$ is a bipartite graph with bipartition X, Y where

$$X = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 1, 3, 5\},$$

and

$$Y = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 2, 4\}.$$

On calculation

$$d\{(u_1, v_1), (u_1, v_1)\} = 0, \quad d\{(u_1, v_1), (u_i, v_1)\} = 2 \text{ for } i \neq 1.$$

$$d\{(u_1, v_1), (u_i, v_3)\} = 2 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ for all } i.$$

$$d\{(u_1, v_1), (u_i, v_5)\} = 4$$

$$\text{Also } d\{(u_1, v_1), (u_i, v_j)\} = 3 \text{ for } j = 2, 4;$$

$$d\{(u_1, v_1), (u_i, v_4)\} = 3 \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ for } i \neq 1.$$

$$d\{(u_1, v_1), (u_i, v_2)\} = 1$$

So

$$\begin{aligned}
 \sum_{i=1}^m \sum_{j=1}^5 d\{(u_1, v_1), (u_i, v_j)\} &= 1(0) + \{(m - 1) + m\}(2) + m(4) + \{2 + (m - 1)3\} + (m - 1)1 \\
 &= (4m - 2) + 4m + (3m + 3) + (m - 1) \\
 &= 12m.
 \end{aligned}$$

We observe that we get the same sum with all the $2m$ vertices (u_i, v_j) ($i = 1, 2, \dots, m; j = 1, 4$).

Now

$$d\{(u_1, v_3), (u_i, v_j)\} = 2 \text{ for all } i \text{ and } j = 1, 5$$

$$d\{(u_1, v_3), (u_1, v_3)\} = 0 \text{ and } d\{(u_1, v_3), (u_i, v_3)\} = 2 \text{ for } i \neq 1.$$

$$d\{(u_1, v_3), (u_1, v_j)\} = 1 \text{ for } j = 2, 4,$$

$$d\{(u_1, v_3), (u_i, v_j)\} = 1 \text{ for } i \neq 1 \text{ and } j = 2, 4.$$

So

$$\sum_{i=1}^m \sum_{j=1}^5 d\{(u_i, v_3), (u_i, v_j)\} = 1(0) + \{2m + (m - 1)\}2 + 2(3) + 2(m - 1) \quad (1)$$

$$= (6m - 2) + 6 + (2m - 2)$$

$$= 8m + 2.$$

We observe that we get the same sum with all the m vertices (u_i, v_3) ($i=1, 2, \dots, m$).

Further

$$d\{(u_1, v_2), (u_i, v_j)\} = 3 \quad \text{for } j = 1, 3.$$

$$d\{(u_1, v_2), (u_i, v_j)\} = 1 \quad \text{for } i \neq 1 \text{ and } j = 1, 3.$$

$$d\{(u_1, v_2), (u_i, v_5)\} = 3 \quad \text{for all } i.$$

$$d\{(u_1, v_2), (u_i, v_2)\} = 0; \quad d\{(u_1, v_2), (u_i, v_2)\} = 2 \quad \text{for } i \neq 1,$$

$$d\{(u_1, v_2), (u_i, v_4)\} = 2 \quad \text{for all } i.$$

So

$$\sum_{i=1}^m \sum_{j=1}^5 d\{(u_1, v_2), (u_i, v_j)\} = (2+m)(3) + 2(m - 1)(1) + 1(0) + \{(m - 1) + m\} \quad (2)$$

$$= (6 + 3m) + (2m - 2) + (4m - 2)$$

$$= 9m + 2.$$

We observe that we get the same sum with all the $2m$ vertices (u_i, v_j) ($i=1,2,\dots,m; j=2, 4$).

Hence

$$W(K_m \wedge P_5) = \frac{1}{2} [2m(12m) + m(8m + 2) + 2m(9m + 2)]$$

$$= \frac{1}{2} [50m^2 + 6m]$$

$$= m(25m + 3).$$

V. RESULTS ON $C_M \wedge P_N$ (M, N BEING POSITIVE INTEGERS WITH $M \geq 3$)

Initially we have

A. *Observations.*

- 1) $C_m \wedge P_1$ is an empty graph (with m vertices).
So we take $n \geq 2$.
- 2) $C_m \wedge P_2 = C_m \wedge K_2 = K_2 \wedge C_m$ and this is considered in § 2
- 3) $C_3 \wedge P_n = K_3 \wedge P_n$ and this is considered in § 4 when $n=3$ or 4.

So, we are left with the graphs for which $m \geq 4$ and $n \geq 3$

Denote $V(C_m) = \{u_1, u_2, \dots, u_m\}$ and $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Then $C_m \wedge P_n$ is the graph with $V(C_m \wedge P_n) = \{(u_i, v_j) : i = 1, 2, \dots, m; j = 1, 2, \dots, n\}$ and the edge set being the set of edges of the form $(u_i, u_j)(u_{i'}, v_{j'})$ where $i, i' \in \{1, 2, \dots, m\}$ with $i' = i-1$ or $i+1$ under the convention $u_0 = u_m$ and $u_{m+1} = u_1, j, j' \in \{1, 2, \dots, n\}$ with $j' = 2$ when $j = 1, j' = n-1$ when $j=n$ and $j = j+1$ or $j-1$ when $2 \leq j \leq n-1$.

e) Since $\deg_{C_m}(u_i) = 2$ and $\deg_{P_n}(v_j) = 1$ or 2 according as $j \in \{1, n\}$ or $2 \leq j \leq (n - 1)$ it follows that

$$\deg_{C_m \wedge P_n}(u_i, v_j) = \begin{cases} 2 & \text{for } i=1, 2, \dots, m; j=1 \text{ or } n, \\ 4 & \text{for } i=1, 2, \dots, m; 2 \leq j \leq n - 1. \end{cases}$$

(Thus the degree of each vertex is even & is independent of m and n).

B. Theorem.

For $m, n \geq 3$, $C_m \wedge P_n$ (isomorphic to $P_n \wedge C_m$) is a simple, finite graph such that the degree of each vertex is either 2 or 4 with mn vertices and $2m(m-1)$ edges and is bipartite.

Since C_m, P_n are simple, finite and so is $C_m \wedge P_n$. Clearly it has mn vertices. From observation (5.1)(e), it follows that the degree of each vertex is either 2 or 4. Further, there are $2m$ vertices of degree 2 and $(n-2)m$ vertices of degree 4. Hence, the number of edges is $\frac{1}{2} [2m(2) + (n-2)m(4)] = \frac{1}{2} [4m + 4mn - 8m]$

$$= 2mn - 2m = 2m(n-1).$$

Let $V(C_m) = \{u_1, u_2, \dots, u_m\}$ and $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Denote

$$V_1 = \{(u_i, v_j) : i = 1, 2, \dots, m, j = 1, 3, \dots, \overline{n-1} \text{ or } n \text{ according as } n \text{ is even or odd}\}$$

$$\text{and } V_2 = \{(u_i, v_j) : i = 1, 2, \dots, m, j = 2, 4, \dots, \overline{n-1} \text{ or } n \text{ according as } n \text{ is odd or even}\}.$$

Clearly, no two vertices of either V_1 or V_2 are adjacent in $C_m \wedge P_n$. Now, follows that $\{V_1, V_2\}$ is a bipartition of this graph. Hence, the graph is bipartite.

This completes the proof of the theorem.

C. Observations

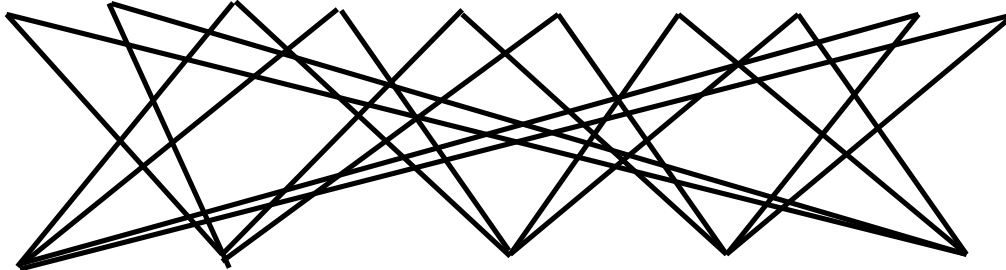
- 1) follows that $C_m \wedge P_n$ is connected when and only when m is odd.
- 2) Since C_m, P_n are connected, P_n does not contain any cycle and C_m does not contain an odd cycle when m is even, by Result (2.5), it follows that $C_m \wedge P_n$ contain exactly two components, when m is even.
- 3) Since, each vertex in $C_m \wedge P_n$ is of even degree, it follows that $C_m \wedge P_n$ is Eulerian when m is odd and is a union of two disjoint Eulerian graphs when m is even. (Since each component is Eulerian).
- 4) $C_m \wedge P_n$ ($m \geq 4, n \geq 3$) is not connected when m is even and is connected when m is odd ($\Rightarrow m \geq 5$).

D. Open problem. To find a general formula for the Wiener number of $C_m \wedge P_n$ for m odd & ≥ 5 and $n \geq 3$. We end up this by finding the following:

C. Result. $W(C_5 \wedge P_3) = 280$.

1) **Justification.** A diagrammatic representation of $C_5 \wedge P_3$ is

$$(u_1, v_1) \quad (u_1, v_3) \quad (u_2, v_1) \quad (u_2, v_3) \quad (u_3, v_1) \quad (u_3, v_3) \quad (u_4, v_1) \quad (u_4, v_3) \quad (u_5, v_1) \quad (u_5, v_3)$$



$$(u_1, v_2) \quad (u_2, v_2) \quad (u_3, v_2) \quad (u_4, v_2) \quad (u_5, v_2)$$

We observe that the graph is symmetric w.r.t. the vertices of degree two, namely (u_i, v_j) ($i = 1, 2, \dots, 5; j = 1, 3$) as well as w.r.t. the vertices of degree 4, namely (u_i, v_j) ($i = 1, 2, \dots, 5; j = 2$).

Now,

$$\begin{aligned} d\{(u_1, v_1), (u_1, v_1)\} &= 0, \quad d\{(u_1, v_1), (u_1, v_3)\} = 2, \\ d\{(u_1, v_1), (u_i, v_1)\} &= 2 = d\{(u_1, v_1), (u_i, v_3)\} \quad (i=3, 4), \\ d\{(u_1, v_1), (u_i, v_1)\} &= 4 = d\{(u_1, v_1), (u_i, v_3)\} \quad (i=2, 5); \end{aligned}$$

Also

$$\begin{aligned} d\{(u_1, v_1), (u_1, v_2)\} &= 5, \\ d\{(u_1, v_1), (u_i, v_2)\} &= 2 \quad (i=3, 4), \\ d\{(u_1, v_1), (u_i, v_2)\} &= 1 \quad (i=2, 5). \end{aligned}$$

$$\text{So } \sum_{i=1}^5 \sum_{j=1}^4 d\{(u_i, v_1), (u_i, v_j)\} = 1(0) + 2(1) + 7(2) + 4(4) + 1(5) = 37.$$

There are 10 points having the same sum.

Further

$$\begin{aligned} d\{(u_1, v_2), (u_1, v_j)\} &= 5 & (j = 1, 3), \\ d\{(u_1, v_2), (u_i, v_j)\} &= 1 & (i = 2, 5 \text{ and } j = 1, 3), \\ d\{(u_1, v_2), (u_i, v_j)\} &= 3 & (i = 3, 4 \text{ and } j = 1, 3), \\ d\{(u_1, v_2), (u_1, v_2)\} &= 0, \\ d\{(u_1, v_2), (u_i, v_2)\} &= 2 & (i = 3, 4), \\ d\{(u_1, v_2), (u_i, v_2)\} &= 4 & (i = 2, 5). \end{aligned}$$

So

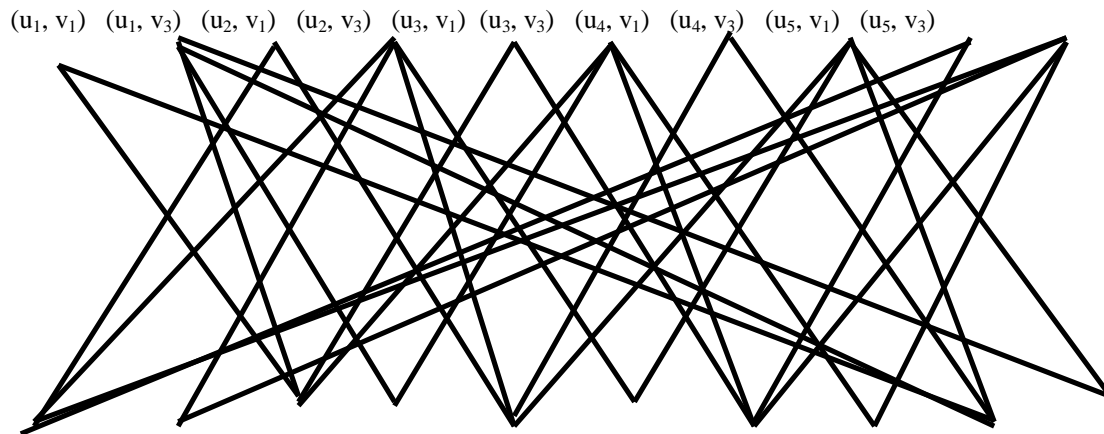
$$\sum_{i=1}^5 \sum_{j=1}^4 d\{(u_1, v_2), (u_i, v_2)\} = 2(5) + 4(1) + 4(5) + 1(10) + 2(2) + 2(4) = 38.$$

There are '5' points having the same sum.

$$\text{Hence, } W(C_5 \wedge P_3) = (1/2) [10(37) + 5(38)] = 280.$$

D. Result. $W(C_5 \wedge P_4) = 540$.

A diagrammatic representation of $C_5 \wedge P_4$ is



$$(u_1, v_2) \quad (u_1, v_4) \quad (u_2, v_2) \quad (u_2, v_4) \quad (u_3, v_2) \quad (u_3, v_4) \quad (u_4, v_2) \quad (u_4, v_4) \quad (u_5, v_2) \quad (u_5, v_4)$$

We observe that the graph is symmetric w.r.t. the vertices of degree two, namely (u_i, v_j) ($i = 1, 2, \dots, 5; j = 1, 3$) as well as w.r.t. the vertices of degree 4, namely (u_i, v_j) ($i = 1, 2, \dots, 5, j = 2, 3$).

Now

$$\begin{aligned} d\{(u_1, v_1), (u_1, v_1)\} &= 0, \quad d\{(u_1, v_1), (u_1, v_3)\} = 2, \\ d\{(u_1, v_1), (u_i, v_j)\} &= 4 & (i = 2, 5; j = 1, 3), \\ d\{(u_1, v_1), (u_i, v_j)\} &= 2 & (i = 3, 4; j = 1, 3); \\ d\{(u_1, v_1), (u_i, v_2)\} &= 1 & (i = 2, 5), \\ d\{(u_1, v_1), (u_i, v_j)\} &= 3 & (i = 3, 4; j = 2, 4), \\ d\{(u_1, v_1), (u_i, v_4)\} &= 3 & (i = 1, 2), \\ d\{(u_1, v_1), (u_1, v_j)\} &= 5 & (j = 2, 4). \end{aligned}$$

So

$$\sum_{i=1}^5 \sum_{j=1}^4 d\{(u_1, v_1), (u_i, v_j)\} = 1(0) + 2(1) + 5(2) + 6(3) + 4(4) + 2(5)$$

$$= 2 + 10 + 18 + 16 + 10$$

$$= 56.$$

There are 10 such points. We get the same sum for all these points.

Also

$$d\{(u_1, v_3), (u_1, v_1)\} = 2, d\{(u_1, v_3), (u_1, v_3)\} = 0,$$

$$d\{(u_1, v_3), (u_i, v_j)\} = 2 \quad (i = 3, 4; j = 1, 3),$$

$$d\{(u_1, v_3), (u_i, v_j)\} = 4 \quad (i = 2, 5; j = 1, 3),$$

$$d\{(u_1, v_3), (u_i, v_j)\} = 5 \quad (i = 2, 5; j = 2, 4),$$

$$d\{(u_1, v_3), (u_i, v_j)\} = 1 \quad (i = 2, 5; j = 2, 4),$$

$$d\{(u_1, v_3), (u_i, v_j)\} = 3 \quad (i = 3, 4; j = 2, 4),$$

$$\text{So } \sum_{i=1}^5 \sum_{j=1}^4 d\{(u_1, v_3), (u_i, v_j)\} = 1(0) + 4(1) + 5(2) + 4(3) + 4(4) + 2(5)$$

$$= 4 + 10 + 12 + 16 + 10$$

$$= 52.$$

There are 10 such points. We get the same sum for all these points.

$$\text{Hence, } W(C_5 \wedge P_4) = \frac{1}{2} (10) [56 + 52]$$

$$= 5(108) = 540.$$

VI. CONCLUSIONS.

As there is significant use of Tensor product graphs in computational Chemistry, an attempt is made to obtain Wiener index of $K_m \wedge K_n$, $P_m \wedge P_n$ and $C_m \wedge C_n$ in the preceding paper [see 3]. Now we attempted to determine the Wiener index of $K_m \wedge P_n$, $K_m \wedge P_n$ and $C_m \wedge P_n$ wherever possible.

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