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On Tensor Product of Standard Graphs-II

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Abstract: The characteristic properties of the graphs $K_m \wedge C_n$, $K_m \wedge P_n$, $C_m \wedge P_n$ are studied and mainly their Wiener Indices are obtained, wherever possible.

Index Terms: Tensor (Kronecker) product, Wiener index (number), connected graph, Hamiltonian graph.

I. INTRODUCTION

The Wiener index is initiated from the work of Wiener [5]. This Wiener number is an important topological index associated with the molecular graph of atoms which is a connected one. Further it is widely used to describe the molecular structures. Till now, no recursive method is known for the calculation of the Wiener number of a general connected graph.

In this paper, the Wiener numbers of $K_m \wedge C_n$, $K_m \wedge P_n$, $C_m \wedge P_n$, wherever possible are obtained. Some interesting observations are made. This paper is a continuation of our previous paper [3].

II. PRELIMINARIES

We present some known definitions and results (in the refined form, wherever necessary) for a ready reference to go through the work presented in the subsequent sections. For standard notation and further results, we refer Bondy & Murthy [1].

A. *Definition 2.1 [4]*

G, H are disjoint graphs. The Tensor product of G and H, denoted by $G \wedge H$ (that is isomorphic to $H \wedge G$) is the graph whose vertex set is V(G) x V(H) and the edge set being the set of all elements of the form $(u, v)(u^1, v^1)$ where $u, u^1 \in V(G)$, $v,v^1 \in V(H)$, $uu^1 \in E(G)$ and $vv^1 \in E(H)$.

B. *Observations 2.2*

1) If one f G , H is an empty graph (i.e. has no edges) then $G \wedge H$ is also an empty graph.
2) If G, H are finite, simple graphs with m, n vertices respectively, then $G \wedge H$ is a finite, simple graph with mn vertices. Further, if $u \in V(G)$ and $v \in V(G)$ then

$$\deg_{G \wedge H}(u, v) = \{\deg_G u\} . \{\deg_H v\}.$$

C. *Definition 2.3[5].*

The Wiener index W(G) of a finite, connected graph is defined to be

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d(u, v),$$

where d(u, v) denotes the distance (the length of any shortest u – v path) between u & v in G.

1) *Result 2.4[4]:* G_1, G_2 are connected graphs. Then $G_1 \wedge G_2$ is connected if and only if (iff) either G_1 or G_2 contains an odd cycle.
2) *Result 2.5 [4]:* If G_1, G_2 re connected graphs with no odd cycles, then $G_1 \wedge G_2$ has exactly two components.
3) *Result 2.6[1]:* A nonempty connected graph is Eulerian iff every vertex is of even degree.
4) *Result 2.7[1]:* If G is a simple graph with the number of vertices $\nu \geq 3$ and the minimum degree $\delta \geq \nu/2$ then G is Hamiltonian.
5) *Result 2.8[1]:* A simple graph is bipartite iff it contains no odd cycles.

In what follows m and n are positive integers.

§3. Results on $K_m \wedge C_n$ (m, n being positive integers & $n \geq 3$).

Initially, we have

III. OBSERVATIONS.

$K_1 \wedge C_n$ is an empty graph (with n vertices).

So, we consider $m \geq 2$ (and $n \geq 3$).

Denote $V(K_m) = \{ u_1,u_2,\ldots,u_m \}$ and $V(C_n) = \{ v_1,v_2,\ldots,v_n \}$. Then $K_m \wedge C_n$ is the graph with $V(K_m \wedge C_n) = \{ (u_i,v_j) : i = 1,2,\ldots,m ;\ j = 1,2,\ldots,n \}$ and the edge set being the set of elements of the form $(u_i, v_j)\ (u_{i'}, v_{j'})$ where $i, i^1 \in \{1,2,\ldots,m \}$ with $i^1 \neq i$; $j, j^1 \in \{ 1,2,\ldots,n\}$ with $j^1 = j-1$ or $j+1$ under the convention $v_0 = v_n$, $v_{n+1} = v_1$.

A. *Theorem.* $K_m \wedge C_n$ (isomorphic to $C_n \wedge K_m$) is a simple, finite and $2(m-1)$ – regular graph (an even integer) with mn vertices and (m-1) mn edges (observe that the degree does not depend on n).

B. *proof.* Since K_m, C_n are simple, finite graphs and so is $K_m \wedge C_n$. As K_m is (m-1) - regular and C_n is 2-regular, it follows that $K_m \wedge C_n$ is 2(m-1)- regular. Since $K_m \wedge C_n$ has mn vertices, it follows that there are (m-1)mn edges.

This proves the Theorem.

C. *Observations.* K_2, C_n are connected graphs and K_2 does not contain an odd cycle (in fact, any cycle).

a) By Result (2.4), it follows that $K_2 \wedge C_n$ is connected iff n is odd (since C_n contains the cycle C_n only).

b) By Result (2.5), it follows that $K_2 \wedge C_n$ has exactly two components iff n is even.

D. *Theorem.* $K_2 \wedge C_{2n+1}$ ($n \geq 1$) is isomorphic to $C_{2(2n+1)}$ and $W(K_2 \wedge C_{2n+1}) =(2n+1)^3$.

E. *Proof.* By Th. (3.2) and Obs.(3.3) (a), $K_2 \wedge C_{2n+1}$ is a connected 2-regular graph with 2(2n+1) vertices and (1)(2)(2n+1) = 2(2n+1) edges. So $K_2 \wedge C_{2n+1}$ is isomorphic to $C_{2(2n+1)}$. Hence, by a known result [see 2], it follows that $W(K_2 \wedge C_{2n+1}) = W (C_{2(2n+1)}) =(2n+1)^3$.

In fact, in the usual notation, $K_2 \wedge C_{2n+1}$ is the cycle $\{(u_1,v_1),(u_2,v_2),(u_1,v_3),\ldots,(u_2,v_{2n}),(u_1,v_{2n+1}),(u_2,v_1),(u_1,v_2),\ldots,(u_1,v_{2n}),(u_2,v_{2n+1}),(u_1,v$

F. *Theorem.* $K_2 \wedge C_{2n}$ ($n \geq 2$) is isomorphic to the (disjoint) union of C_{2n} & C_{2n} and the Wiener number of each component is n^3.

By Th.(3.2), $K_2 \wedge C_{2n}$ is a 2-regular graph with 4n vertices and 4n edges. By observation (3.3)(b), this has exactly two components. Now follows that each component is a cycle. Clearly the components are the cycles $\{(u_1,v_1),(u_2,v_2),(u_1,v_3),\ldots,(u_1,v_{2n-1}),(u_2,v_{2n}),(u_1,v_1)\}$ and $\{(u_2,v_1),(u_1,v_2),(u_2,v_3)\ldots,(u_2,v_{2n-1}),(u_1,v_{2n}),(u_2,v_1)\}$. Each is C_{2n}. Hence by a known result [see 2] follows the theorem.

G. *Observations.*

Since $K_2 \wedge C_{2n+1}$ ($n \geq 1$) is an even cycle, follows that this graph is bipartite, Eulerian and Hamiltonian.

Since $K_2 \wedge C_{2n}$ ($n \geq 2$) is union of C_{2n} and C_{2n}, follows that the graph is bipartite and each component (C_{2n}) is Eulerian and Hamiltonian.

H. *Theorem.* For $m, n \geq 3$, $K_m \wedge C_n$ is a) connected b) Eulerian and c) bipartite iff n is even.

Proof. Since K_m, C_n are connected and K_m ($m \geq 3$) contains the odd cycles K_3, by Result (2.4), it follows that $K_m \wedge C_n$ is connected. This proves (a).

Since the degree of each vertex of $K_m \wedge C_n$ is even (see Th.(3.2)), by the characterization result (2.6), it follows that $K_m \wedge C_n$ is Eulerian.

This proves (b).

Suppose n is even ($\Rightarrow n \geq 4$).

In the usual notation,

$$X = \{(u_i, v_j): i =1, 2, \ldots, m; j = 1, 3, \ldots, (n-1)\},$$

and

$$Y = \{(u_i, v_j): i =1, 2, \ldots, m; j = 2, 4, \ldots, n\}$$

are such that $\{X, Y\}$ is a bipartition of the vertex set $K_m \wedge C_n$. So the graph is bipartite.

When n is odd,

$\{(u_1, v_1), (u_2, v_2), (u_3, v_3), \ldots, (u_n, v_n), (u_1, v_1)\}$ is a cycle of length n (odd) in $K_m \wedge C_n$. So it is not bipartite.

This completes the proof of the Theorem.

I. *Observations*

$K_2 \wedge C_n$ ($n \geq 3$) is discussed in this article.

$K_m \wedge C_3 = K_m \wedge K_3$ and this is discussed in [3].

a) $K_3 \wedge C_n = C_3 \wedge C_n$ ($n \geq 3$) and this is dicussed in [3].

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Thus, we are left with the graphs. $K_m \wedge C_n$ ($m, n \geq 4$) and we discuss about these graphs.

J. *Result.* $W(K_m \wedge C_4) = 4m\,(3m + 2)$ $(m \geq 4)$.

K. *Justification.* Since the graph is regular, it follows that the graph is symmetric w.r.t. all 4m vertices (u_i, v_j) ($i = 1, 2, \ldots, m$; $j = 1, 2, 3, 4$).

On Calculation

$d\{(u_1, v_1), (u_1, v_1)\} = 0$, $d\{(u_1, v_1), (u_1, v_3)\} = 2$,

$d\{(u_1, v_1), (u_i, v_j)\} = 2$ for $i = 2, 3, \ldots, m$; $j = 1, 3$.

$d\{(u_1, v_1), (u_1, v_2)\} = 3 = d\{(u_1, v_1), (u_1, v_4)\}$,

$d\{(u_1, v_1), (u_i, v_j)\} = 1$ for $i = 2, 3, \ldots, m$; $j = 2, 4$.

So

$$\sum_{i=1}^{m}\sum_{j=1}^{3} d\{(u_1, v_1), (u_i, v_j)\} = 1(0) + \{ 1+ 2(m-1)\}(2) + 2(3) + 2(m-1)\,(1)$$

$$= (4m - 2) + 6 + (2m - 2)$$

$$= 6m + 2.$$

We get the same sum for all the 4m vertices. Hence

$$W(K_m \wedge C_4) = (1/2)(4m)\,(6m + 2)$$

$$= 4m\,(3m + 1).$$

L. ***Result.*** $W(K_m \wedge C_5) = 5m(4m + 1)$ $(m \geq 4)$.

M. *Justification.* As the graph is regular, follows the graph is symmetric w.r.t. all the 5m vertices.

On Calculation,

$d\{(u_1, v_1), (u_1, v_1)\} = 0$,

$d\{(u_1, v_1), (u_1, v_j)\} = 3$ for $j = 2, 5$,

$d\{(u_1, v_1), (u_1, v_j)\} = 2$ for $j = 3,4$.

$d\{(u_1, v_1), (u_i, v_j)\} = 1$ for $i = 2, 3, \ldots, m$; $j = 2, 5$.

So

$$d\{(u_1, v_1)\,(u_i, v_j)\} = \sum_{i=1}^{m}\sum_{j=1}^{3} d\{(u_1, v_1), (u_i, v_j)\}$$

$$= 1(0) + \{ 2+ 3(m-1)\}(2) + 2(3) + 2(m-1)\,(1)$$

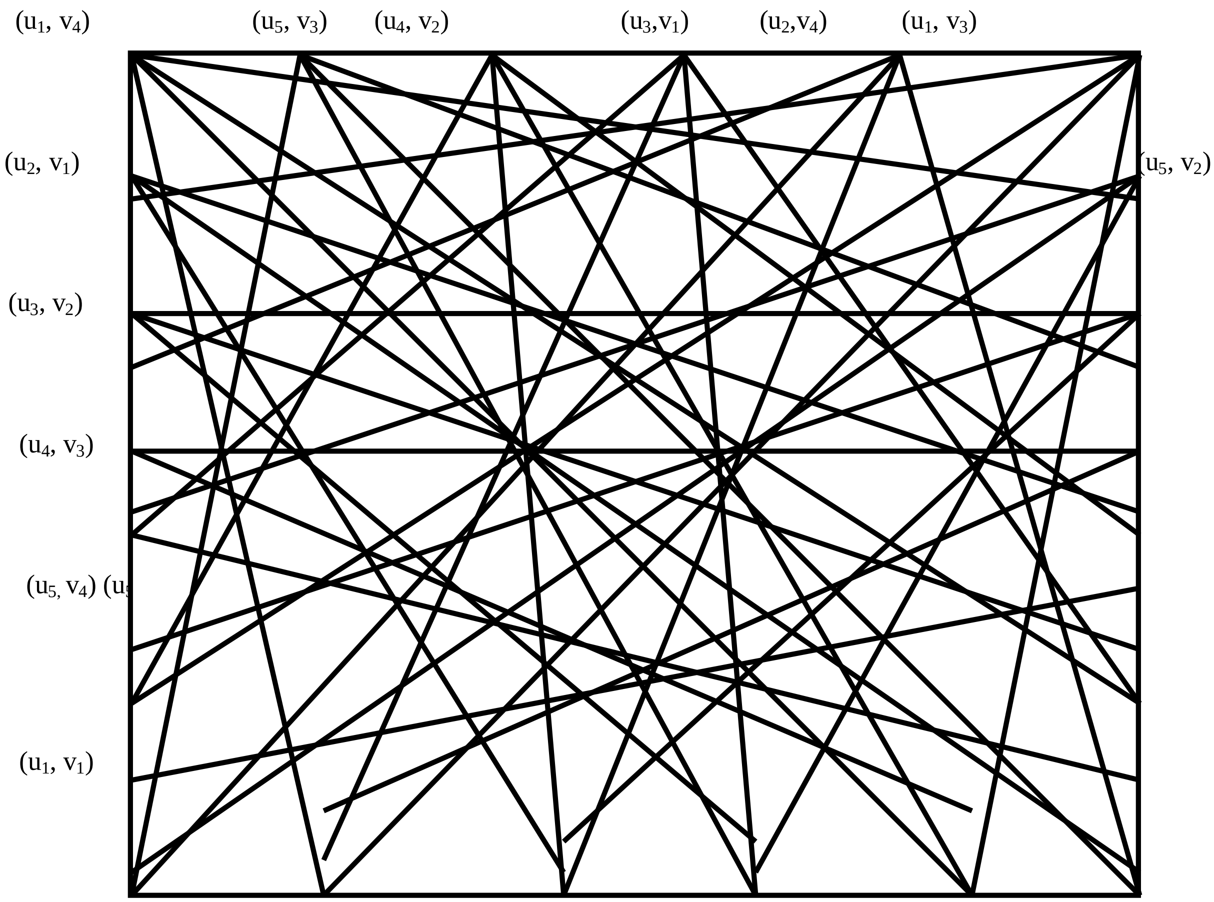
$$= (6m - 2) + 6 +(2m - 2) = 8m + 2.$$

We get the same sum for all the 5m vertices. Hence

$W(K_m \wedge C_5) = (1/2)\,(5m)\,(8m + 2) = 5m(4m + 1)$.

Finally, we exhibit the following:

N. *A diagrammatic representation of $k_4 \wedge c_5$.*



O. Open problem. To find a general formula for the Wiener number of $K_m \wedge C_n$ (m, n ≥ 4).

IV. RESULTS

ON $K_m \wedge P_n$ (m, n being positive integers).
Primarily, we have

A. Observations.

1) If atleast one of m, n is 1, then Km ^ Pn is an empty graph.
So, we consider m, n ≥ 2.

2) $K_m \wedge P_2 = K_m \wedge K_2$ (m ≥ 2) and this is discussed in [3].
So, we take n ≥ 3.

3) $K_2 \wedge Pn = P_2 \wedge P_n$ (n ≥ 2) and this is discussed in [3].
So, we take m ≥ 3.

Thus, we discuss about the graphs where m, n ≥ 3

4) Denote $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, then Km ^ Pn is the graph with V(Km ^ Pn) = $\{(u_i,v_j): i=1,2,\ldots,m; j=1,2,\ldots,n\}$ and the edge set being the set of elements of the form (u_i, v_j) $(u_{i'}, v_{j'})$ where i, $i' \in \{1, 2, \ldots, m\}$ with $i' \neq i$, j, $j' \in \{1, 2, \ldots, n\}$, $j' = 2$ when j =1, $j' = n-1$ when j = n and $j' = j-1$ or $j+1$ when $2 \leq j \leq n-1$.

Since $\deg_{K_m}(u_i) = m-1$ and $\deg_{P_n}(v_j) = 1$ or 2 according as j=1, n or j=2, …, (n – 1), it follows that

$$\deg_{K_m \wedge P_n}(u_i, v_j) = \begin{cases} 1(m-1) & \text{for } i=1, 2, \ldots, m;\ j=1 \text{ or } n, \end{cases}$$

$2(m - 1)$ for $i=1, 2, \ldots, m$; $j=2, 3, \ldots(n-1)$.

(Observe that the degree does not depend on 'n').

B. *Theorem.* $K_m \wedge P_n$ $(m, n \geq 3)$ (isomorphic to $P_n \wedge K_m$) is a simple, finite graph with mn vertices and $m(m-1)(n-1)$ edges.

1) *Proof.* Since K_m, P_n are simple, finite graphs and so is $K_m \wedge P_n$. It has 2m vertices of degree $(m - 1)$ and has $(n - 2)m$ vertices of degree $2(m - 1)$; it follows that the number of edges in $K_m \wedge P_n$ is $\frac{1}{2}[2m(m-1) + (n-2)m + 2(m-1)] = m(m-1)(n-1)$.

C. *Theorem.* $K_m \wedge P_n$ $(m, n \geq 3)$ is

connected b) bipartite and c) Eulerian iff m is odd.

1) *Proof.* Since K_m, P_n are connected graphs and K_m $(m \geq 3)$ contains the odd cycle K_3, by Result (2.4), it follows that $K_m \wedge P_n$ is connected. This proves (a).

In the usual notation, let

$V_1 = \{(u_i, v_j)$: $i = 1, 2, \ldots, m$, $j = 1, 3, \ldots, \overline{n-1}$ or n as according n is even or odd$\}$,

$V_2 = \{(u_i, v_j)$: $i = 1, 2, \ldots, m$, $j = 2, 4, \ldots, \overline{n-1}$ or n as according n is odd or even$\}$.

Clearly no two vertices of either V_1 or V_2 are adjacent in $K_m \wedge P_n$. This implies that $\{V_1, V_2\}$ is a bipartition of the vertex set of $K_m \wedge P_n$. Thus $K_m \wedge P_n$ is bipartite. This proves (b).

By the characterization Result (2.6), $K_m \wedge P_n$ is Eulerian iff each of its vertex is of even degree and $\Leftrightarrow$ m is odd. This proves (c).

Thus the proof of the theorem is complete.

D. *REMARK.* $|V_1| = mn/2 = |V_2|$ when n is even and $|V_1| = m(n + 1)/2$ & $|V_2| = m(n - 1)/2$ when n is odd.

E. *Theorem.* $K_m \wedge P_3$ $(m \geq 3)$ is a $((m - 1), 2(m - 1))$ - biregular graph and $W(K_m \wedge P_3) = m(7m + 1)$.

1) *PROOF.* By Th.(4.3), it follows that the graph is bipartite with a bipartition $\{V_1, V_2\}$, where

$V_1 = \{(u_i, v_j)$: $i = 1,2, \ldots, m$; $j = 1,3\}$

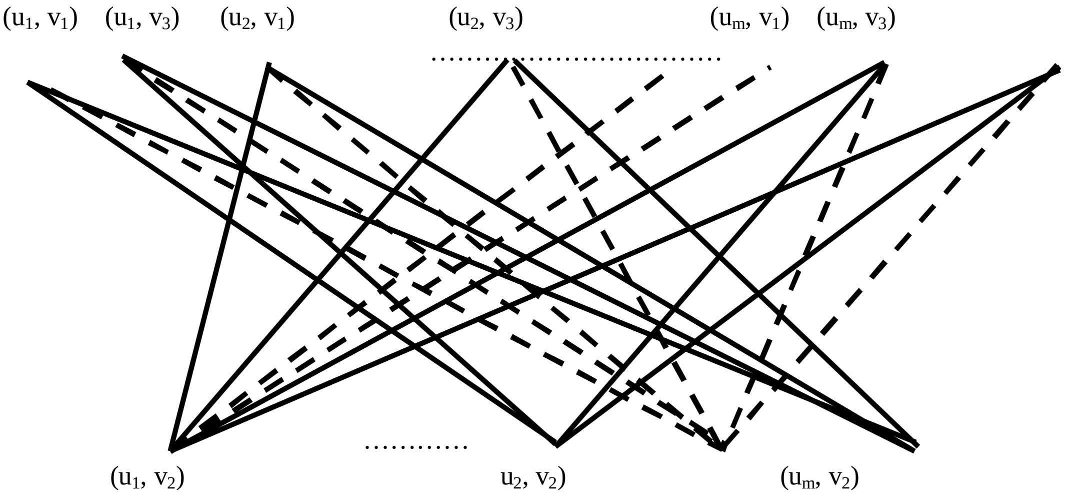
and

$V_2 = \{(u_i, v_2)$: $i = 1,2, \ldots, m\}$.

Clearly every vertex of V_1 is of degree $(m - 1)$ and that of V_2 is $2(m - 1)$. Thus the graph is a $((m - 1, 2(m - 1))$ – biregular graph.

Clearly $|V_1| = 2m$ and $|V_2| = m$.

Its diagrammatic representation is



Now,

$d\{(u_1, v_1), (u_1, v_1)\} = 0$

$d\{(u_1, v_1), (u_i, v_1)\} = 2$ $(i=2, \ldots, m)$,

$d\{(u_1, v_1), (u_i, v_3)\} = 2$ $(i=2, \ldots, m)$,

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$d\{(u_1, v_1), (u_1, v_2)\} = 3$
and
$d\{(u_1, v_1), (u_i, v_2)\} = 1 \quad (i=2,3, \ldots, m);$

$$\therefore \sum_{i=1}^{m}\sum_{j=1}^{3} d\{(u_1, v_1), (u_i, v_1)\} = 0 + 2(m-1) + 2(m) + 3 + (m-1) = 5m.$$

Since, interchanging any two vertices in V_1, does not affect the graph follows that we get the same sum for all the 2m points in V_1.
Also
$d\{(u_1, v_2), (u_1, v_1)\} = 3 = d\{(u_1, v_2), (u_1, v_3)\},$
$d\{(u_1, v_2), (u_i, v_j)\} = 1$ for i=2 , … m and j=1, 3
and $d\{(u_1, v_2), (u_1, v_2) = 0\}$, $d\{(u_1, v_2), (u_i, v_2)\} = 2$ for i= 2 ,…, m – 1.

$$\text{Thus } \sum_{i=1}^{m}\sum_{j=1}^{n} d\{ (u_1,v_2),(u_i, v_j)\} = (3+3) + (2m-2)\,1 + 0 + (m-1)(2)$$
$$= 6 + 2m - 2 + 2m - 2$$
$$= 4m + 2.$$

As in V_1, we get the same sum for all points of V_2.
Thus $W(K_m \wedge P_3) = ½ [(2m)(5m) + m(4m+2)]$
$= 5m^2 + m(2m+1)$
$= m(7m + 1).$

F. *Result.* $W(K_m \wedge P_n) = \dfrac{m}{6}[mn (n^2 + 5) + 6(n-2)]$. $(m \geq 3 \text{ \& } n \geq 3$ and n is even).

In the usual notation, $K_m \wedge P_n$ is a bipartite graph with a bipartition, (X, Y) where
$X = \{(u_i, v_j):\ i =1, 2, \ldots, m;\ j = 1, 3, \ldots, (n-1)\},$
and
$Y = \{(u_i, v_j):\ i =1, 2, \ldots, m;\ j = 2, 4, \ldots, n\}.$
Clearly $|X| = |Y|= mn/2$. As the graph is symmetric w.r.t X and Y, we observe that

$$\sum_{i'=1}^{m}\ \sum_{j'odd}\ \sum_{i=1}^{m}\ \sum_{j=1}^{n} d\{(u_{i'}, v_{j'}), (u_i,v_j)\} = \sum_{i'=1}^{m}\ \sum_{j'even}\ \sum_{i=1}^{m}\ \sum_{j=1}^{n} d\{(u_{i'}, v_{j'}), (u_i,v_j)\}$$

(That means sum taken over the vertices in X is same as the sum taken over the vertices in Y).
On Calculation,

$$d\{(u_1, v_1), (u_i, v_1)\} = \begin{cases} 0 & \text{if } i = 1, \\ 2 & \text{if } i \neq 1. \end{cases}$$

$$d\{(u_1, v_1), (u_i, v_2)\} = \begin{cases} 3 & \text{if } i = 1, \\ 1 & \text{if } i \neq 1. \end{cases}$$

$d\{(u_1, v_1), (u_i, v_j)\} = (j-1)$ for all i and j = 3, …, n.

$$\sum_{i=1}^{m}\ \sum_{j=1}^{n} d\{(u_1, v_1), (u_i, v_j)\} = [\ 1(0) + (m-1)2 + 1(3) + (m-1)1 + m \sum_{j=3}^{n} (j-1)]$$
$$= [(2m-2) + 3 + (m-1) + m\sum_{j=2}^{n-1} j]$$

$$= \frac{m}{2}(n^2 - n + 4) \longrightarrow \text{(i)}$$

Since u_1 is adjacent with all $u_{i'}$ ($i' \neq 1$), it follows that we get the same sum when u_1 is replaced by $u_{i'}$.
Further
$d\{(u_1, v_3), (u_i, v_j)\} = (j - 3)$ for all i and j = 5, …, n (when $n \geq 5$).

$$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{n} d\{(u_1, v_3), (u_i, v_j)\} = \frac{m\left(n^2 - 5n + 16\right) + 4}{2} \longrightarrow \text{(ii)}$$

For $j' = 5, 7, \ldots, (m - 1)$ (when $m \geq 8$)
$d\{(u_1, v_{j'}), (u_i, v_1)\} = (j' - 1)$ for all i,
$d\{(u_1, v_{j'}), (u_i, v_2)\} = (j' - 2)$ for all i,

$d\{(u_1, v_{j'}), (u_i, v_{j'-2})\} = 2$ for all i.

$$d\{(u_1, v_{j'}), (u_i, v_{j'-1})\} = d\{(u_1, v_{j'}), (u_i, v_{j'+1})\} = \begin{cases} 3 & \text{if } i = 1, \\ 1 & \text{if } i \neq 1. \end{cases}$$

if i = 1,

$$d\{(u_1, v_{j'}), (u_i, v_{j'-1})\} = d\{(u_1, v_{j'}), (u_i, v_{j'+1})\} = \begin{cases} 2 & \text{if } i \neq 1. \end{cases}$$

$d\{(u_1, v_{j'}), (u_i, v_{j'+2})\} = 2$ for all i,

$d\{(u_1, v_{j'}), (u_i, v_n)\} = (n - j')$ for all i.

$$\therefore \sum_{i=1}^{m} \sum_{j=1}^{n} d\{(u_1, v_{j'}), (u_i, v_j)\} = m[(j' - 1) + (j' - 2) + \ldots + 2] +$$

$$2\{1(3) + (m-1)(1)\} + \{1(0) + (m-1)(2) + m[2 + 3 + \ldots + (n - j')]$$

$$= m[2 + \ldots + (j' - 1)] + (4m + 2) + m[2 + \ldots + (n - j')]$$

$$= m\left[\frac{(j'-1)j'}{2} - 1\right] + (4m + 2) + m\left[\frac{(n-j')(n-j'+1)}{2} - 1\right]$$

$$= m\left(\frac{n^2 + n + 4}{2}\right) + 2 + m[j'^2 - (n+1)j'].$$

$$\therefore \sum_{j'=5,7,\ldots(m-1)} d\{(u_1, v_{j'}), (u_i, v_j)\}$$

$$= m\left\{\frac{(n^2 + n + 4)}{4} + 4\right\}(n - 4) - \frac{m(n+1)(n^2 - 16)}{4} - 10m + \frac{mn(n^2 - 1)}{6} \longrightarrow \text{(iii)}$$

Now follows from (i), (ii) &(iii),

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$$W(K_m \wedge P_n) = \frac{1}{2}(2)\,m \; \frac{m}{2}\Big[(n^2 - n + 4) + \frac{m\left(n^2 - 5n + 16\right) + 4}{2}$$

$$+ \Big\{\frac{m\left(n^2 + n + 4\right) + 4}{4}\Big\}(n-4) - \frac{m(n+1)\left(n^2 - 16\right)}{4}$$

$$- 10m + \frac{mn\left(n^2 - 16\right)}{6}\Big]$$

$$= \frac{1}{6}[m^2 (n^3 + 5n) + 6m (n-2)] \text{ (On simplification)}$$

$$= \frac{m}{6}[mn(n^2 + 5) + 6(n-2)].$$

This completes the proof of the result.

G. Open problem. To find a general formula for the Wiener Number of $K_m \wedge P_n$ when $m \geq 3$ and n is odd.

H. Result. $W(K_m \wedge P_5) = m(25m + 3)$ ($m \geq 3$).

1) ROOF. Clearly $K_m \wedge P_5$ is a bipartite graph with bipartition X, Y where

$X = \{(u_i, v_j): i = 1, 2, \ldots, m; j = 1, 3, 5\}$,

and

$Y = \{(u_i, v_j): i = 1, 2, \ldots, m; j = 2, 4\}$.

On calculation

$d\{(u_1, v_1), (u_1, v_1)\} = 0$, $d\{(u_1, v_1), (u_i, v_1)\} = 2$ for $i \neq 1$.

$$\left.\begin{array}{l} d\{(u_1, v_1), (u_i, v_3)\} = 2 \\ d\{(u_1, v_1), (u_i, v_5)\} = 4 \end{array}\right\} \text{for all i.}$$

Also $d\{(u_1, v_1), (u_i, v_j)\} = 3$ for $j = 2, 4$;

$$\left.\begin{array}{l} d\{(u_1, v_1), (u_i, v_4)\} = 3 \\ d\{(u_1, v_1), (u_i, v_2)\} = 1 \end{array}\right\} \text{for } i \neq 1.$$

So

$$\sum_{i=1}^{m}\sum_{j=1}^{5} d\{(u_1, v_1), (u_i, v_j)\} = 1(0) + \{(m-1) + m\}(2) + m(4) + \{2 + (m-1)3\} + (m-1)1$$

$$= (4m - 2) + 4m + (3m + 3) + (m - 1)$$

$$= 12m.$$

We observe that we get the same sum with all the 2m vertices (u_i, v_j) ($i = 1, 2, \ldots, m; j = 1, 4$).

Now

$d\{(u_1, v_3), (u_i, v_j)\} = 2$ for all i and $j = 1, 5$

$d\{(u_1, v_3), (u_1, v_3)\} = 0$ and $d\{(u_1, v_3), (u_i, v_3)\} = 2$ for $i \neq 1$.

$d\{(u_1, v_3), (u_1, v_j)\} = 1$ for $j = 2, 4$,

$d\{(u_1, v_3), (u_i, v_j)\} = 1$ for $i \neq 1$ and $j = 2, 4$.

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So

$$\sum_{i=1}^{m}\sum_{j=1}^{5} d\{(u_1, v_3), (u_i, v_j)\} = 1(0) + \{2m + (m-1)\}2 + 2(3) + 2(m-1)(1)$$

$$= (6m-2) + 6 + (2m-2)$$

$$= 8m + 2.$$

We observe that we get the same sum with all the m vertices (u_i, v_3) (i=1, 2, …, m).

Further

$d\{(u_1, v_2), (u_i, v_j)\} = 3$ for $j = 1, 3$.

$d\{(u_1, v_2), (u_i, v_j)\} = 1$ for $i \neq 1$ and $j = 1, 3$.

$d\{(u_1, v_2), (u_i, v_5)\} = 3$ for all i.

$d\{(u_1, v_2), (u_1, v_2)\} = 0$; $d\{(u_1, v_2), (u_i, v_2)\} = 2$ for $i \neq 1$,

$d\{(u_1, v_2), (u_i, v_4)\} = 2$ for all i.

So

$$\sum_{i=1}^{m}\sum_{j=1}^{5} d\{(u_1, v_2), (u_i, v_j)\} = (2+m)(3) + 2(m-1)(1) + 1(0) + \{(m-1) + m\}(2)$$

$$= (6 + 3m) + (2m-2) + (4m-2)$$

$$= 9m + 2.$$

We observe that we get the same sum with all the 2m vertices (u_i, v_j) (i=1,2,…,m; j=2, 4).

Hence

$$W(K_m \wedge P_5) = \frac{1}{2}[2m(12m) + m(8m+2) + 2m(9m+2)]$$

$$= \frac{1}{2}[50m^2 + 6m]$$

$$= m(25m + 3).$$

V. RESULTS ON $C_M \wedge P_N$ (M, N BEING POSITIVE INTEGERS WITH $M \geq 3$)

Initially we have

A. *Observations.*

1) $C_m \wedge P_1$ is an empty graph (with m vertices).
   So we take $n \geq 2$.
2) $C_m \wedge P_2 = C_m \wedge K_2 = K_2 \wedge C_m$ and this is considered in § 2
3) $C_3 \wedge P_n = K_3 \wedge P_n$ and this is considered in § 4 when n=3 or 4.
   So, we are left with the graphs for which $m \geq 4$ and $n \geq 3$

Denote $V(C_m) = \{u_1, u_2, …, u_m\}$ and $V(P_n) = \{v_1, v_2, …, v_n\}$. Then $C_m \wedge P_n$ is the graph with $V(C_m \wedge P_n) = \{(u_i, v_j)$: i = 1, 2, …, m; j = 1, 2, …, n$\}$ and the edge set being the set of edges of the form $(u_i, u_j)\ (u_{i'}, v_{j'})$ where $i, i^{|} \in \{1, 2, …, m\}$ with $i^{|} = i-1$ or $i+1$ under the convention $u_0 = u_m$ and $u_{m+1} = u_1$, $j, j^{|} \in \{1, 2, …, n\}$ with $j^{|} = 2$ when $j = 1$, $j^{|} = n-1$ when $j=n$ and $j = j+1$ or $j-1$ when $2 \leq j \leq n-1$.

e) Since $\deg_{C_m}(u_i) = 2$ and $\deg_{P_n}(v_j) = 1$ or 2 according as $j \in \{1, n\}$ or $2 \leq i \leq (n-1)$ it follows that

$$\deg_{C_m \wedge P_n}(u_i, v_j) = \begin{cases} 2 & \text{for } i=1, 2, …, m;\ j=1 \text{ or } n, \\ 4 & \text{for } i=1, 2, …, m;\ 2 \leq j \leq n-1. \end{cases}$$

(Thus the degree of each vertex is even & is independent of m and n).

B. Theorem.

For m, $n \geq 3$, $C_m \wedge P_n$ (isomorphic to $P_n \wedge C_m$) is a simple, finite graph such that the degree of each vertex is either 2 or 4 with mn vertices and $2m(m - 1)$ edges and is bipartite.

Since C_m , P_n are simple, finite and so is $C_m \wedge P_n$. Clearly it has mn vertices. From observation (5.1)(e), it follows that the degree of each verte is either 2 or 4. Further, there are 2m vertices of degree 2 and $(n - 2)m$ vertices of degree 4. Hence, the number of edges is $\frac{1}{2}[2m(2) + (n - 2)m(4)] = \frac{1}{2}[4m + 4mn - 8m]$

$$= 2mn - 2m = 2m(n - 1).$$

Let $V(C_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n\} = \{v_1, v_2, \ldots, v_n\}$. Denote

$V_1 = \{(u_i, v_j): i = 1, 2, \ldots, m, j = 1, 3, \ldots, \overline{n-1}$ or n according as n is even or odd$\}$

and $V_2 = \{(u_i, v_j): i = 1, 2, \ldots, m, j = 2, 4, \ldots, \overline{n-1}$ or n according as n is odd or even$\}$.

Clearly, no two vertices of either V_1 or V_2 are adjacent in $C_m \wedge P_n$. Now, follows that $\{V_1, V_2\}$ is a bipartition of this graph. Hence, the graph is bipartite.

This completes the proof of the theorem.

C. Observations

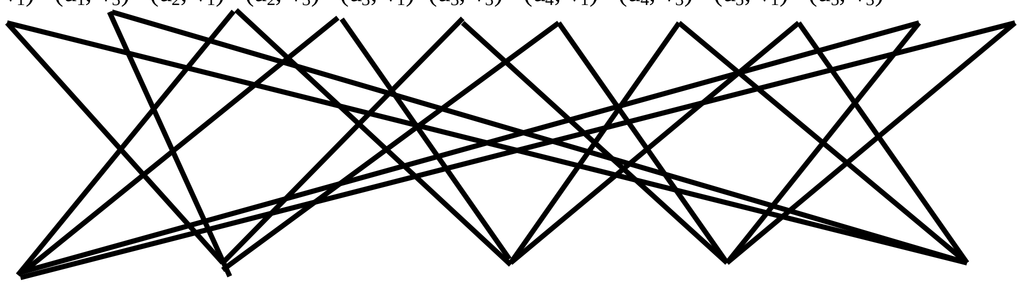
1) follows that is $C_m \wedge P_n$ is connected when and only when m is odd.
2) Since C_m , P_n are connected, P_n does not contain any cycle and C_m does not contain an odd cycle when m is even, by Result (2.5), it follows that $C_m \wedge P_n$ contain exactly two components, when m is even.
3) Since, each vertex is $C_m \wedge P_n$ is of even degree, it follows that $C_m \wedge P_n$ is Eulerian when m is odd and is a union of two disjoint Eulerian graphs when m is even. (Since each component is Eulerian).
4) $C_m \wedge P_n$ $(m \geq 4, n \geq 3)$ is not connected when m is even and is connected when m is odd $(\Rightarrow m \geq 5)$.

D. Open problem. To find a general formula for the Wiener number of $C_m \wedge P_n$ for m odd & ≥ 5 and $n \geq 3$. We end up this by finding the following:

C. Result. $W(C_5 \wedge P_3) = 280$.

1) *Justification.* A diagrammatic representation of $C_5 \wedge P_3$ is

(u_1, v_1) (u_1, v_3) (u_2, v_1) (u_2, v_3) (u_3, v_1) (u_3, v_3) (u_4, v_1) (u_4, v_3) (u_5, v_1) (u_5, v_3)



(u_1, v_2) (u_2, v_2) (u_3, v_2) (u_4, v_2) (u_5, v_2)

We observe that the graph is symmetric w.r.t. the vertices of degree two, namely (u_i, v_j) $(i = 1, 2, \ldots, 5; j = 1, 3)$ as well as w.r.t. the vertices of degree 4, namely (u_i, v_j) $(i= 1, 2, \ldots, 5, j = 2)$.

Now,

$d\{(u_1, v_1), (u_1, v_1)\} = 0$, $d\{(u_1, v_1), (u_1, v_3)\} = 2$,

$d\{(u_1, v_1), (u_i, v_1)\} = 2 = d\{(u_1, v_1), (u_i, v_3)\}$ (i =3, 4),

$d\{(u_1, v_1), (u_i, v_1)\} = 4 = d\{(u_1, v_1), (u_i, v_3)\}$ (i = 2, 5);

Also

$d\{(u_1, v_1), (u_1, v_2)\} = 5$,

$d\{(u_1, v_1), (u_i, v_2)\} = 2$ (i=3, 4),

$d\{(u_1, v_1), (u_i, v_2)\} = 1$ (i=2, 5) .

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So $\sum_{i=1}^{5} \sum_{j=1}^{4}$ d{(u_1, v_1), (u_i, v_j)} = 1(0) + 2(1) + 7(2) + 4(4) + 1(5)

= 37.

There are 10 points having the same sum.

Further

d{(u_1, v_2), (u_1, v_j)} = 5 (j = 1, 3),

d{(u_1, v_2), (u_i, v_j)} = 1 (i = 2, 5 and j = 1, 3),

d{(u_1, v_2), (u_i, v_j)} = 3 (i = 3, 4 and j = 1, 3).

d{(u_1, v_2), (u_1, v_2)} = 0,

d{(u_1, v_2), (u_i, v_2)} = 2 (i = 3, 4),

d{(u_1, v_2), (u_i, v_2)} = 4 (i = 2, 5).

So

$\sum_{i=1}^{5} \sum_{j=1}^{4}$ d{(u_1, v_2), (u_i, v_2)} = 2(5) + 4(1) + 4(5) + 1(10) +2(2) + 2(4)

= 38.

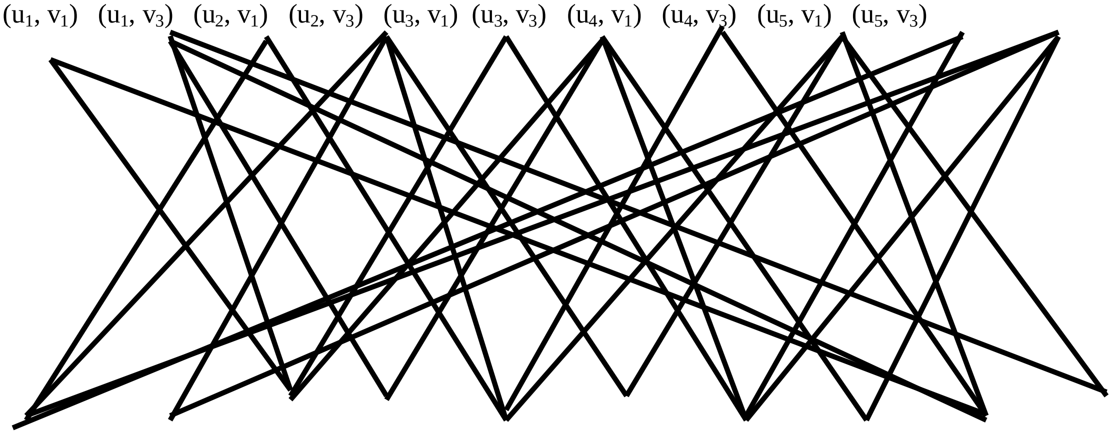
There are '5' points having the same sum.

Hence, W(C_5 ^ P_3) = (1/2) [10(37) + 5(38)] = 280.

D. *Result.* W(C_5 ^ P_4) = 540.

A diagrammatic representation of C_5 ^ P_4 is

(u_1, v_1) (u_1, v_3) (u_2, v_1) (u_2, v_3) (u_3, v_1) (u_3, v_3) (u_4, v_1) (u_4, v_3) (u_5, v_1) (u_5, v_3)



(u_1, v_2) (u_1, v_4) (u_2, v_2) (u_2, v_4) (u_3, v_2) (u_3, v_4) (u_4, v_2) (u_4, v_4) (u_5, v_2) (u_5, v_4)

We observe that the graph is symmetric w.r.t. the vertices of degree two, namely (u_i, v_j) (i = 1, 2, …, 5; j = 1, 3) as well as w.r.t. the vertices of degree 4, namely (u_i, v_j) (i= 1, 2, …, 5, j = 2,3).

Now

d{(u_1, v_1), (u_1, v_1)} = 0, d{(u_1, v_1), (u_1, v_3)} = 2,

d{(u_1, v_1), (u_i, v_j)} = 4 (i =2, 5; j = 1, 3),

d{(u_1, v_1), (u_i, v_j)} = 2 (i =3, 4; j = 1, 3);

d{(u_1, v_1), (u_i, v_2)} = 1 (i =2, 5),

d{(u_1, v_1), (u_i, v_j)} = 3 (i =3, 4; j = 2, 4),

d{(u_1, v_1), (u_i, v_4)} = 3 (i =1, 2),

d{(u_1, v_1), (u_1, v_j)} = 5 (j = 2, 4).

So

$$\sum_{i=1}^{5} \sum_{j=1}^{4} d\{(u_1, v_1), (u_i, v_j)\} = 1(0) + 2(1) + 5(2) + 6(3) + 4(4) + 2(5)$$

$$= 2 + 10 + 18 + 16 + 10$$

$$= 56.$$

There are 10 such points. We get the same sum for all these points.

Also

$d\{(u_1, v_3), (u_1, v_1)\} = 2$, $d\{(u_1, v_3), (u_1, v_3)\} = 0$,

$d\{(u_1, v_3), (u_i, v_j)\} = 2$ $(i = 3, 4; j = 1, 3)$,

$d\{(u_1, v_3), (u_i, v_j)\} = 4$ $(i = 2,5; j = 1, 3)$,

$d\{(u_1, v_3), (u_i, v_j)\} = 5$ $(i = 2,5; j = 2, 4)$,

$d\{(u_1, v_3), (u_i, v_j)\} = 1$ $(i = 2,5; j = 2, 4)$,

$d\{(u_1, v_3), (u_i, v_j)\} = 3$ $(i = 3,4; j = 2, 4)$,

So $$\sum_{i=1}^{5} \sum_{j=1}^{4} d\{(u_1, v_3), (u_i, v_j)\} = 1(0) + 4(1) + 5(2) + 4(3) + 4(4) + 2(5)$$

$$= 4 + 10 + 12 + 16 + 10$$

$$= 52.$$

There are 10 such points. We get the same sum for all these points.

Hence, $W(C_5 \wedge P_4) = \frac{1}{2}(10)[56 + 52]$

$$= 5(108) = 540.$$

VI. CONCLUSIONS.

As there is significant use of Tensor product graphs in computational Chemistry, an attempt is made to obtain Wiener index of $K_m \wedge K_n$, $P_m \wedge P_n$ and $C_m \wedge C_n$ in the preceeding paper [see 3]. Now we attempted to determine the Wiener index of $K_m \wedge P_n$, $K_m \wedge P_n$ and $C_m \wedge P_n$ wherever possible.

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