



iJRASET

International Journal For Research in
Applied Science and Engineering Technology



INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Volume: 5 Issue: X Month of publication: October 2017

DOI: <http://doi.org/10.22214/ijraset.2017.10050>

www.ijraset.com

Call:  08813907089

E-mail ID: ijraset@gmail.com

Boundary Value Problem of Fractional Differential Equation

Rajesh Pandey¹

¹Maharishi University of Information Technology Lucknow 226013, India.

Abstract: In this paper, we prove the existence of the solution for the boundary value problem of fractional differential equations of order $q \in (2, 3]$. The Krasnoselskii's fixed point theorem is applied to establish the results.

Keywords: Fractional differential equation, Krasnoselskii's fixed point theorem, Boundary value problem, Positive Solution, Gamma functions

I. INTRODUCTION

Fractional differential equations are the generalization of ordinary equation to arbitrary non-integer order, and have received more and more interest due to their wide applications in various branch of science & engineering, such as physics, chemistry, biophysics, capacitor theory, blood flow phenomena, electrical circuits, control theory, etc, also recent investigations have demonstrated that the dynamics of many systems are described more accurately by using fractional differential equations. Nikolai was concerned with the nonlinear differential equation of fractional order

$$D_{0+}^q u(t) = f(t, u(t), u'(t)) \quad a.e. t \in (0, 1),$$

Where D_{0+}^q is Riemann-Liouville (R-L) fractional order derivatives, subject to the boundary conditions $u(0) = u(1) = 0$. Zhang has given the existence of positive solution to the equations

$$\begin{cases} {}^c D^q u(t) + f(t, u(t)) = 0, 0 < t < 1, \\ u(0) + u'(0) = u(1) + u'(1) = 0, \end{cases}$$

By the use of classical fixed point theorems, where ${}^c D^q$ denotes Caputo fractional derivative with $1 < q \leq 2$. Chen considered the existence of three positive solutions to three-point boundary value problem of the following fractional differential equation

$$\begin{cases} D_{0+}^q u(t) + f(t, u(t)) = 0, 0 < t < 1 \\ u(0) = 0, D_{0+}^p u(t) \Big|_{t=1} = \alpha D_{0+}^p u(t) \Big|_{t=\xi} \end{cases}$$

Where $1 < q \leq 2, 0 < p < 1, 1 + p \leq q$, and D_{0+}^q is the R-L fractional order derivative. The multiplicity results of positive solutions to the equations are obtained by using the well-known Leggett-Williams fixed-point theorem on convex cone, we study the existence of positive solution to two point BVP of nonlinear fractional equation.

$$\begin{cases} D_{0+}^q u(t) + \lambda f(t, u(t)) = 0, 0 < t < 1, \\ u(0) = D_{0+}^p u(t) \Big|_{t=0} = D_{0+}^p u(t) \Big|_{t=1} = 0 \end{cases} \quad (1.1)$$

Where $q, p \in R, 2 < q \leq 3, 1 < p \leq 2, 1 + p \leq q, D_{0+}^q$ is the R-L fractional order derivative, and $f \in C([0, 1] \times [0, \infty), [0, \infty))$, $\lambda > 0$.

II. NOTATIONS AND DEFINITIONS

Definition 1. The R-L fractional integrals $I_{0+}^p f$ of order $p \in R (p > 0)$ defined by

$$I_{0+}^p f(x) := \frac{1}{\Gamma(p)} \int_0^x \frac{f(t) dt}{(x-t)^{1-p}}, (x > 0).$$

Here $\Gamma(p)$ is the Gamma function.

Definition 2. The R-L fractional derivatives $D_{0+}^p f$ order $p \in R (p > 0)$ is defined by

$$D_{0+}^p f(x) = \left(\frac{d}{dx}\right)^n I_{0+}^{n-p} f(x)$$

$$= \frac{1}{\Gamma(n-p)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)dt}{(x-t)^{p-n+1}}, \quad (n = 1[p] + 1, x > 0),$$

Where p means the integral part of p .

III.MAIN RESULTS

Lemma 1.If $q_1 > q_2 > 0$, then, for $f(x) \in L_p(0,1), (1 \leq p \leq \infty)$ the relations $D_{0+}^{q_2} I_{0+}^{q_1} f(x) = I_{0+}^{q_1-q_2} f(x), I_{0+}^{q_1} I_{0+}^{q_2} f(x) = I_{0+}^{q_1+q_2} f(x)$ and $D_{0+}^{q_1} I_{0+}^{q_1} f(x) = f(x)$ hold a.e. on $[0,1]$.

Lemma 2. Let $q > 0, n = [q] + 1, f(x) \in L_1(0,1)$, then the equality

$$I_{0+}^q D_{0+}^q f(x) = f(x) + \sum_{i=1}^n C_i t^{q-n}.$$

Lemma 3.Let $y \in C[0,1], 2 < q \leq 3, 1 < p \leq 2, 1 + p \leq q$, then the problem

$$D_{0+}^q u(t) + y(t) = 0, 0 < t < 1, \tag{3.1}$$

subject to the boundary conditions

$$u(0) = D_{0+}^p u(t) \Big|_{t=0} = D_{0+}^p u(t) \Big|_{t=1} = 0 \tag{3.2}$$

has the unique solution $u(t) = \int_0^1 G(t,s)ds$, where

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}, & 0 \leq s \leq t \leq 1 \\ t^{q-1}(1-s)^{q-p-1} & 0 \leq t \leq s \leq 1 \end{cases}$$

And that $G(t,s)$ has the following properties

(i) $G(t,s) \in C([0,1] \times [0,1])$ and $G(t,s) > 0$ for $t,s \in (0,1)$, and $\max_{0 \leq t \leq 1} G(t,s) = G(s,s), s \in (0,1)$.

(ii) There exists a positive function $\varphi \in C((0,1) \times (\tau, \infty))$ such that

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t,s) = \varphi(s) \tilde{G}(s,s) \geq \inf_{0 < s < 1} \varphi(s) \max_{0 \leq t \leq 1} G(t,s) = \tau G(s,s),$$

Where

$$\tilde{G}(s,s) = \frac{s^{q-p}(1-s)^{p-q-1}}{\Gamma(q)}, s, \tau \in (0,1), \tau = \inf_{0 < s < 1} \varphi(s).$$

Proof. Applying the operator I_{0+}^q to both sides of the equation (1.1), and using Lemma 2, we have

$$u(t) = -I_{0+}^q y(t) + c_1 t^{q-1} + c_2 t^{q-2} + c_3 t^{q-3} \tag{3.3}$$

In view of the boundary condition $u(0) = 0$, we find that $C_3 = 0$ hence

$$u(t) = -I_{0+}^q y(t) + C_1 t^{q-1} + C_2 t^{q-2},$$

then, noting the relation $D_{0+}^{q_2} I_{0+}^{q_1} f(x) = I_{0+}^{q_1-q_2} f(x)$ in Lemma 1, we obtain

$$D_{0+}^p u(t) = -I_{0+}^{q-p} y(t) + C_1 \frac{\Gamma(q)}{\Gamma(q-p)} t^{q-1-p} + C_2 \frac{\Gamma(q-1)}{\Gamma(q-p-1)} t^{q-p-2},$$

In accordance with the equation (3.1), we can calculate out that

$$C_1 = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-p-1} y(s) ds, c_2 = 0.$$

Substituting the values of C_1, C_2 and C_3 in (3.2) we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds + \frac{t^{q-1}}{\Gamma(q)} \int_0^1 (1-s)^{q-p-1} y(s) ds \\ &= \frac{1}{\Gamma(q)} \left\{ \int_0^t [t^{q-1}(1-s)^{q-p-1} y - (t-s)^{q-1}] y(s) ds + \int_t^1 [t^{q-1}(1-s)^{q-p-1}] y(s) ds \right\} \\ &= \int_0^1 G(t,s) y(s) ds \end{aligned}$$

Next we prove the properties of $G(t, s)$.

For a given $s \in (0, 1)$, $G(s, t)$ is the decreasing with respect to t for $s \leq t$ while increasing for $t \leq s$. Thus, we have

$$\max_{0 \leq t \leq 1} G(t, s) = G(s, s) = \frac{s^{q-1}(1-s)^{q-p-1}}{\Gamma(q)} \leq \frac{s^{q-p}(1-s)^{q-p-1}}{\Gamma(q)} = \tilde{G}(s, s),$$

for $s \in (0, 1)$. Then we set

$$g_1(t, s) = \frac{t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}}{\Gamma(q)}, g_2(t, s) = \frac{t^{q-1}(1-s)^{q-p-1}}{\Gamma(q)},$$

from the two equation above we have

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s) = \frac{1}{\Gamma(q)} \begin{cases} 0.75^{q-1}(1-s)^{q-p-1} - (0.75-s)^{q-1}, & 0 < s \leq r \\ 0.25^{q-1}(1-s)^{q-p-1}, & r \leq s < 1 \end{cases}$$

where $\frac{1}{4} \leq r < \frac{3}{4}$ is the unique solution of the equation.

$$0.75^{q-1}(1-s)^{q-p-1} - (0.75-s)^{q-1} = 0.25^{q-1}(1-s)^{q-p-1}$$

Finally, we consider a function $\varphi(s)$ defined by

$$\varphi(s) = \frac{\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s)}{\tilde{G}(s, s)} = \begin{cases} \frac{0.75^{q-1}(1-s)^{q-p-1} - (0.75-s)^{q-1}}{s^{q-p}(1-s)^{q-p-1}}, & 0 < s \leq r, \\ \frac{0.25^{q-1}}{s^{q-p}}, & 0 \leq s < 1. \end{cases}$$

when $p > q - 1$ we find from the continuity of $\varphi(s)$ and $\lim_{s \rightarrow 0^+} \varphi(s) = +\infty$ that there exists \tilde{r} small enough such that $\varphi'(s) < 0$ for

$s \in (0, \tilde{r}]$ hence, we set

$$0 < r = \inf_{0 < s < 1} \varphi(s) = \min \left\{ \varphi(\tilde{r}), m, \frac{1}{4^{q-1}} \right\} < 1,$$

here, $m = \min_{\tilde{r} \leq s \leq r} \varphi(s)$.

when $q = p - 1$, we have $\lim_{s \rightarrow 0^+} \varphi(s) = \frac{4}{3}(q-1)$, then we set

$$0 < \tau = \inf_{0 < s < 1} \varphi(s) = \min \left\{ \inf_{0 < s < r} \varphi(s), \frac{3}{4}(q-1), \frac{1}{4^{q-1}} \right\} < 1.$$

Thus

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} G(t, s) \geq \varphi(s) \tilde{G}(s, s) \geq \inf_{0 < s < 1} \varphi(s) \max_{0 \leq t \leq 1} G(t, s) = \tau G(s, s).$$

This completes the proof. Therefore the solution $u \in C_{[0,1]}$ of the problem (1.1) can be written by

$$u(t) = \lambda \int_0^1 G(t, s) f(s, u(s)) ds.$$

IV. CONCLUSIONS

The paper proves the existence of the solution for boundary value problem of fractional differential equations of the order $q \in (2, 3]$. and three Lemmas are established.

V. ACKNOWLEDGMENT

My thanks are due to Dr. G.C Chaubey Ex Associate Professor & Head department of Mathematics TDPG College Jaunpur and Professor B. Kunwar Department of Mathematics IET, Lucknow for their encouragement and for providing necessary support. I am extremely grateful for their constructive support.



REFERENCES

- [1] S.Q. Zhang, Existence results of positive solutions to the boundary value problem for fractional differential equation, *Positivity* 13 (2009), 583-599.
- [2] A.A. Kilbasa, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equation*, Elsevier, Amsterdam, 2006.
- [3] S. Zhang, Positive solutions solution for some class of the nonlinear, *J. Math. Anal. Appl.* 278(2003), 136-148.
- [4] Y. Zhang, Z. Bai : Existence of solutions for nonlinear fractional three point boundary value problems at resonance. *J. Appl. Math. Comput.* 36, 417-440 (2011).
- [5] A.P Chen, Y.S. Tian, Existence of Three Positive Solutions to Three Point Boundary Value Problem of Nonlinear Fractional Differential Equation, *Differ. Equ. Dyn. Syst.* 18 (2010), 327-339.
- [6] Yi Chen, Xianhua Tang: Positive solutions of fractional differential equations at resonance on the half-line. *Boundary Value Problems* (2012). doi: 10.1186/1687-2770-2012-64.



10.22214/IJRASET



45.98



IMPACT FACTOR:
7.129



IMPACT FACTOR:
7.429



INTERNATIONAL JOURNAL FOR RESEARCH

IN APPLIED SCIENCE & ENGINEERING TECHNOLOGY

Call : 08813907089  (24*7 Support on Whatsapp)