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# Pseudo-Complete Color Critical Graphs

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**Abstract:** A pseudo-complete coloring of a graph  $G$  is an assignment of colors to the vertices of  $G$  such that for any two distinct colors, there exist adjacent vertices having those colors. The maximum number of colors used in a pseudo-complete coloring of  $G$  is called the pseudo-achromatic number of  $G$  and is denoted by  $\psi_s(G)$ . A graph  $G$  is called edge critical if  $\psi_s(G - e) < \psi_s(G)$  for any edge  $e$  of  $G$ . A graph  $G$  is called vertex critical if  $\psi_s(G - v) < \psi_s(G)$  for every vertex  $v$  of  $G$ . These graphs are generally called as pseudo-achromatic number critical graphs (called shortly as PAN Critical graphs). In this paper, we investigate the properties of these critical graphs. We also investigate the locally critical elements of graphs.

**Keywords:** Pseudo-complete coloring, Pseudo-achromatic number, critical graph

## I. INTRODUCTION

By a graph we mean a finite undirected graph without loops, multiple edges or isolated vertices.

An assignment of colors to the vertices of a graph is called a proper coloring, if any two adjacent vertices receive distinct colors. An assignment of colors to the vertices of a graph  $G$  is called a pseudo-complete coloring, if for any two distinct colors; there exist adjacent vertices having those colors. A proper and pseudo-complete coloring of  $G$  is called a complete coloring of  $G$ .

The minimum number of colors used in a proper coloring of  $G$  is called the chromatic number of  $G$  and is denoted by  $\chi(G)$ . The maximum number of colors used in a complete coloring of  $G$  is called the achromatic number of  $G$  and is denoted by  $\psi(G)$  [3]. The maximum number of colors used in a pseudo-complete coloring of  $G$  is called the pseudo-achromatic number of  $G$  and is denoted by  $\psi_s(G)$  [5]. A graph which admits a pseudo-complete coloring by  $k$  colors is called a  $k$ - pseudo-complete colorable graph. Several bounds for these coloring parameters were obtained in [3, 4, 5, 6] and a detailed study of this parameter and critical graphs with respect to it were studied by Suresh Kumar in his doctoral dissertation [7].

The concept of critical graphs with respect to chromatic number was introduced and studied by Dirac [1, 2]. In this paper, we introduce the concept of critical graphs with respect to pseudo-achromatic number, obtain characterizations of them and determine the pseudo-achromatic number of several classes of graphs.

Let  $n$  be any positive real number. Then,  $[n]$  denote the greatest integer less than or equal to  $n$  and  $\{n\}$  denotes the smallest integer greater than or equal to  $n$ .

For terms not defined explicitly here, reader can refer Harary [8].

## II. CRITICAL PATHS AND CRITICAL CYCLES

A. *Definition 2.1.* A graph  $G$  is called  $k$ -edge critical if  $\psi_s(G) = k$  and  $\psi_s(G - e) < k$ , for every edge  $e \in E(G)$ . A graph  $G$  is called  $k$ -vertex critical if  $\psi_s(G) = k$  and  $\psi_s(G - v) < k$  for every vertex  $v$  of  $G$ . A graph  $G$  is called  $k$ -contraction critical (shortly,  $k$ -con-critical) if  $\psi_s(G) = k$  and  $\psi_s(G||e) < k$  for every edge  $e$  of  $G$ , where  $G||e$  denotes the graph obtained from  $G$  by contracting the edge  $e$ . Following observations are quite useful later.

B. *Proposition 2.2.* A graph  $G$  is  $k$ -edge critical if  $G$  is  $k$ -pseudo-complete colorable and  $|E(G)| = \binom{k}{2}$

C. *Proposition 2.3.* A  $k$ -edge critical graph is  $k$ -con-critical and a  $k$ -con-critical graph is  $k$ -vertex-critical.

D. *Remark 2.4.* None of the statements in Proposition 2.3 can be reversed. For example,  $C_8$  is 4-con-critical but not edge critical. Also,  $C_4$  is 3-vertex critical, but not con-critical.

E. *Theorem 2.5.* Let  $G$  be a  $k$ -pseudo-complete colorable graph. Then,  $|V(G)| \geq k \left\lceil \frac{k-1}{\Delta} \right\rceil$  where  $\Delta$  is the maximum degree of a vertex of  $G$  and  $\{n\}$  is the smallest integer less than or equal to  $n$ . Proof. Consider any  $k$ -pseudo-complete coloring of  $G$ . Then for any color  $c$ , there exist  $k-1$  edges in  $G$  such that one end vertex of each of these edges receive the color  $c$ . Hence there must be at least  $k \left\lceil \frac{k-1}{\Delta} \right\rceil$  vertices with color  $c$ , so that  $|V(G)| \geq k \left\lceil \frac{k-1}{\Delta} \right\rceil$ .

F. *Corollary 2.6.*  $\psi_s(G) = \max \left\{ k : k \left\lceil \frac{k-1}{\Delta} \right\rceil \leq |V(G)| \right\}$  for any graph  $G$  with the maximum degree,  $\Delta$ .

G. *Corollary 2.7.* If  $G$  is the Petersen Graph,  $\psi_s(G) = 5$ .

1) *Proof.* Clearly,  $E(G)$  can be partitioned into a 2-factor, say  $\{ (v_1v_2v_3v_4v_5v_1), (v_6v_7v_8v_9v_{10}v_6) \}$  and a 1-factor,  $\{ v_1v_6, v_2v_9, v_3v_7, v_4v_{10}, v_5v_8 \}$ . Now the function  $f: V(G) \rightarrow \{1,2,3,4,5\}$  defined by  $f(v_i) = i, 1 \leq i \leq 5$  and  $f(v_i) = 2i - 11 \pmod{5}, 6 \leq i \leq 10$  assigns a 5-pseudo-complete coloring for  $G$ . Hence,  $\psi_5(G) \geq 5$ . Also, by Corollary 2.6,  $\psi_5(G) \leq 5$ .

H. *Corollary 2.8.* If  $G$  is the 3-cube,  $Q_3, \psi_5(G) = 4$

1) *Proof.* By Corollary 2.6,  $\psi_5(G) \leq 4$ . The function  $f: V(G) \rightarrow \{1,2,3,4\}$  defined below assigns a 4-pseudo-complete coloring to  $Q_3$  so that the corollary follows.

$$f(0,0,0) = f(1,1,0) = 1, f(0,0,1) = f(1,1,1) = 2$$

$$f(0,1,1) = f(1,0,0) = 3, f(0,1,0) = f(1,0,1) = 4$$

I. *Corollary 2.9.* For the 4-cube  $Q_4, \psi_5(Q_4) = 8$

1) *Proof.* By Corollary 2.6,  $\psi_5(Q_4) \leq 8$ . The function  $f: V(Q_4) \rightarrow \{1,2,3,4, 5,6, 7, 8\}$  defined below assigns a 8-pseudo-complete coloring to  $Q_4$  so that the corollary follows.

$$f(0,1,1,0) = f(1,0,0,1) = 1, f(1,1,1,0) = f(0,0,0,1) = 2$$

$$f(0,1,0,0) = f(1,0,1,0) = 3, f(0,0,1,0) = f(1,1,1,1) = 4$$

$$f(1,1,0,0) = f(0,1,1,1) = 5, f(1,0,0,0) = f(0,1,1,1) = 6$$

$$f(0,1,0,1) = f(1,0,1,1) = 7, f(0,0,0,0) = f(1,1,0,1) = 8$$

J. *Corollary 2.10.* For the 5-cube  $Q_5, \psi_5(Q_5) = 11$ .

1) *Proof.* By Corollary 2.6,  $\psi_5(Q_5) \leq 11$ . An 11-pseudo complete coloring of  $Q_5$  is shown below, so that the corollary follows.

$$f(0,1,1,0,0) = f(1, 0, 0,1,0) = 1, f(1,1,1,0,0) = f(1,1,1,0,1) = f(0,0,0,1,1) = 2$$

$$f(0,1,0,0,0) = f(1,0,1,0,0) = 3, f(0,0,1,0,0) = f(1,1,1,1,0) = f(0,0,1,0,1) = 4$$

$$f(1,1,0,0,0) = f(0,1,1,1,0) = f(0,1,1,1,1) = 5, f(1,0,0,0,0) = f(0,0,1,1,0) = f(0,0,1,1,1) = 6, f(1,0,1,1,0) = f(0,1,0,1,1) = 7, f(0,0,0,0,0) = f(1,1,0,1,0) = f(0,0,0,0,1) = f(1,1,0,1,1) = 8$$

$$f(1,0,0,0,1) = f(0,1,0,1,0) = f(0,1,1,0,1) = 9$$

$$f(1,1,0,0,1) = f(1,0,0,1,1) = f(1,0,1,0,1) = f(1,0,1,1,1) = 10$$

$$f(0,0,0,1,0) = f(0,1,0,0,1) = f(1,1,1,1,1) = 11$$

These corollaries motivate the following conjecture:

K. *Conjecture 2.11.*  $\psi_5(Q_n) = \max \left\{ k: k \binom{k-1}{\Delta} \leq 2^n \right\}$

L. *Corollary 2.12.* For  $n \geq 2, \psi_5(K_{n,n}) = n + 1$  n + 1.

1) *Proof.* By Corollary 2.6,  $\psi_5(K_{n,n}) \leq n + 1$ . Also if  $X = \{x_1, x_2, \dots, x_n\}, Y = \{y_1, y_2, \dots, y_n\}$  is a bipartition of  $K_{n,n}$  then the function  $f: V(K_{n,n}) \rightarrow \{1, 2, \dots, n + 1\}$  defined by  $f(x_1) = 1, f(y_1) = 2,$  and  $f(x_i) = f(y_i) = i + 1, 2 \leq i \leq n$  gives a pseudo complete coloring of  $G$  so that  $\psi_5(K_{n,n}) = n + 1$  1

Now we proceed to characterize critical cycles and critical paths.

M. *Theorem 2.13.* Let  $n(k)$  denote the integer  $\binom{k}{2}$  or  $\binom{k}{2} + 1$ , according as  $k$  is odd or even. Then a cycle  $C_n$  is  $k$ -con-critical if and only if  $n = n(k)$ .

1) *Proof.* Let  $C_n = \langle v_1, v_2, \dots, v_{n(k)}, v_1 \rangle$  and  $Z_n = \{1, 2, \dots, n\}$

We first prove that  $C_n$  is  $k$ -pseudo-complete colorable.

Case 1.  $k$  is even.

Define a function  $f: V(C_{n(k)}) \rightarrow Z_n$  as follows:

$$f(v_i) = \begin{cases} (1 + (-1)^{n+1} \{i/2\}) \pmod{k} & \text{if } 1 \leq i \leq k \\ (\{i/k\} + (-1)^{1+g_i} \{g_i/2\}) \pmod{k} & \text{otherwise} \end{cases}$$

Where  $g_i = (i - \{i/k\}(k - 2) + k - 3) \pmod{k}$

Let  $j_1, j_2 \in Z_k, j_1 < j_2$  Suppose  $j_2 - j_1 = k/2$  let

$$\text{If } j_1 = 1, f(\{v_k, v_{k+1}\}) = \{j_1, j_2\}$$

$$\text{If } j_1 \geq 2, f(\{v_{j_1-2j_1+3}, v_{j_1-2j_1+4}\}) = \{j_1, j_2\}$$

Now, suppose  $j_2 - j_1 \neq k/2$ . Put  $n = \left\{ \frac{(j_1+j_2)(\text{mod } k)}{2} \right\}$

Choose  $j$  to be the least positive integer such that  $f(v_{(n-1)k+j}) = j_1$  or  $j_2$ . Then it can be easily verified that  $f(\{v_{(n-1)k+j}, v_{(n-1)k+j+1}\}) = \{j_1, j_2\}$ . Thus in all cases,  $f$  assigns a  $k$ -pseudo-complete coloring to  $C_n(k)$ .

Case 2.  $k$  is odd.

Define a function  $f: V(C_{n(k)}) \rightarrow Z_k$  as follows:

$$f(v_i) = \begin{cases} k & \text{if } i \equiv 1 \pmod{k} \\ \left( \binom{i}{k} + (-1)^{g_i} \left[ \frac{g_i}{2} \right] \right) \pmod{k-1} & \text{otherwise} \end{cases}$$

$$\text{where } g_i = \left( i - \left\{ \frac{i}{k} \right\} k + k - 1 \right) \pmod{k}$$

Let  $j_1, j_2 \in Z_k$  and  $j_1 < j_2$ . Suppose  $j_2 = k$ .

$$\text{If } j_1 \leq \left\{ \frac{k}{2} \right\}, \text{ then } f(v_{(j_1-1)k+1}, v_{(j_1-k)k+2}) = \{j_1, j_2\}$$

$$\text{If } j_1 > \left\{ \frac{k}{2} \right\}, \text{ then } f(v_{(j_1-(k/2))k}, v_{(j_1-(k/2))k+1}) = \{j_1, j_2\}$$

Now, suppose  $j_1 < k$ . Put  $n = \left\{ \frac{(j_1+j_2)(\text{mod } k-1)}{2} \right\}$ . Choose  $j$  to be the least positive integer such that  $f(v_{(n-1)k+j}) = j_1$  or  $j_2$

$$\text{Then, } f(v_{(n-1)k+j}, v_{(n-1)k+j+1}) = \{j_1, j_2\}$$

Thus in all cases,  $f$  assigns a  $k$ -pseudo-complete coloring to  $C_{n(k)}$ . Hence it follows from Theorem 2.5 that  $C_{n(k)}$  is  $k$ -con-critical.

Conversely, suppose  $C_n$  is  $k$ -con-critical. Since  $C_{n(k)}$  is  $k$ -pseudo-complete colorable,  $n < n(k)$  and by Theorem 2.5,  $n > n(k)$ .

*N. Corollary 2.14.* Let  $n(k)$  denote the integer  $\binom{k}{2}$  or  $\binom{k}{2} + \frac{k}{2}$ , according as  $k$  is odd or even. Then  $\psi_s(C_n) = \max\{k: n(k) \leq n\}$

*O. Corollary 2.15.* Let  $m(k)$  denote the integer  $\binom{k}{2} + 1$  or  $\binom{k}{2} + \frac{k}{2}$ , according as  $k$  is odd or even. Then a path  $P_n$  is  $k$ -con-critical if and only if  $n = m(k)$ .

If  $P_n$  is  $k$ -pseudo-complete colorable, then  $n > m(k)$ , by from Theorem 2.5. But since  $C_{n(k)}$  is  $k$ -pseudo-complete colorable, so is  $P_{m(k)}$ . Hence  $P_n$  is  $k$ -con-critical if and only if  $n = m(k)$ .

*Corollary 2.16.* Let  $m(k)$  denote the integer  $\binom{k}{2} + 1$  or  $\binom{k}{2} + \frac{k}{2}$ , according as  $k$  is odd or even. Then  $\psi_s(P_n) = \max\{k: m(k) \leq n\}$

The following propositions can easily be deduced from Theorem 2.5, Theorem 2.13 and Corollary 2.15.

*P. Proposition 2.17.* A cycle,  $C_n$  is  $k$ -vertex critical iff

$$n = \begin{cases} \binom{k}{2} + \frac{k}{2} & \text{if } k \text{ is even} \\ \binom{k}{2} \text{ or } \binom{k}{2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

*Q. Proposition 2.18.* A Path,  $P_n$  is  $k$ -vertex critical iff

$$n = \begin{cases} \binom{k}{2} + \frac{k}{2} & \text{if } k \text{ is even} \\ \binom{k}{2} + 1 & \text{if } k \text{ is odd} \end{cases}$$

*R. Theorem 2.19.* There is no  $k$ -edge critical cycle, if  $k$  is even

Suppose  $C_n$  is a  $k$ -edge critical cycle and  $k$  is even. By Theorem 2.13,  $n = \binom{k}{2} + \frac{k}{2}$ . Now, for any edge  $e$  of

$C_n, C_n \sim e$  (a path on  $\binom{k}{2} + \frac{k}{2}$  vertices) is not  $k$ -pseudo-complete colorable, a contradiction.

S. *Corollary 2.20.* A cycle,  $C_n$ , is  $k$ -edge critical if and only if  $k$  is odd and  $n = \binom{k}{2}$ .

T. *Theorem 2.21.* There is no  $k$ -edge critical path, if  $k \geq 4$  is even.

Suppose  $P_n$  is  $k$ -edge critical and  $k \geq 4$  is even. By Corollary 2.15,  $n = \binom{k}{2} + \frac{k}{2}$  so that,  $|E(P_n)| > \binom{k}{2}$ , a contradiction to Proposition.2.2.

U. *Corollary. 2.22.* A path  $P_n$  is  $k$ -edge critical if and only if either  $k$  is odd and  $n = \binom{k}{2} + 1$  or  $n=k=2$

V. *Remark 2.23.* Corollaries 2.14, 2.7, 2.8, 2.9, 2.10 and 2.12 show that the regular graphs such as cycles, Petersen graph,  $n$ -cubes for  $n = 3, 4, 5$  and  $K_{n,n}$  are solutions of the equation:

$$\psi_s(G) = \max\{k : k \{(k-1)/\Delta\} \leq |V(G)|\}$$

$$O_s(G) = \max\{k : k [(k-1)A_{i,c_i}V(G)_1]\}$$

This motivates us to propose the following conjecture for which Conjecture 2.11 is a special case.

W. *Conjecture 2.24.* For any  $n$ -regular graph  $G$ ,

$$\psi_s(G) = \max\{k : k \{(k-1)/n\} \leq |V(G)|\}$$

### III. CHARACTERIZATION OF PAN CRITICAL GRAPHS

Let  $v$  be a vertex of  $G$  having degree  $d$  and let  $n$  be an integer such that  $1 \leq n \leq d$ . Then an  $n$ -splitting of  $v$  is the replacement of  $v$  by a set  $S$  of  $n$  independent vertices such that  $N(S) = N(\{v\})$ , degree of  $u \in S$  is  $\geq 1$  for all  $u \in S$  and  $\sum_{u \in S} \deg u = d$ , where  $N(S)$  denotes the set of all neighbors of the vertices in  $S$ .

Clearly, the complete graph on  $k$  vertices and the graph consisting of  $\binom{k}{2}$  disjoint copies of  $K_2$  are  $k$ -edge critical. The following theorem shows that any  $k$ -edge critical graph can be obtained from these graphs by simple operations, which preserve edge criticality.

A. *Theorem 3.1.* The following statements are equivalent:

G is a  $k$ -edge critical graph.

G can be obtained from  $K_k$  by a sequence of  $n$ -splitting operations.

Let  $H$  be the graph consisting of disjoint copies of  $K_2$  and let  $C$  be a  $k$ -pseudo-complete coloring of  $H$ . Then  $G$  can be obtained from  $H$  by a sequence of identifications of vertices of same color.

1) *Proof.* Let  $G$  be a  $k$ -edge critical graph and let  $c_1, c_2, \dots, c_k$  be the colors used in a  $k$ -pseudo-complete coloring of  $G$ . Let  $d_{i1}, d_{i2}, \dots, d_{im_i}$  denote the degrees of the vertices of  $G$  with colors  $c_i, 1 \leq i \leq k$ . Clearly  $\sum_{j=1}^{m_i} d_{ij} = d_i$ . Now, let  $V(K_k) = \{v_1, v_2, \dots, v_k\}$ . Then  $G$  can be obtained from  $K_k$  by performing an  $m_i$ -splitting operation on  $v_i$ , for each  $i, 1 \leq i \leq k$ , such that the  $m_i$  vertices which replace  $v_i$  have degrees  $d_{i1}, d_{i2}, \dots, d_{im_i}$ . Hence (1) implies (2). Also, since an  $n$ -splitting operation preserves the edge criticality and  $K_k$  is  $k$ -edge critical, (2) implies (1).

If  $G$  is  $k$ -edge critical,  $G$  is  $k$ -pseudo-complete colorable and  $G$  has exactly  $\binom{k}{2}$  edges. Now  $H$  can be obtained from  $G$  by recursively performing 2-splitting operations and each of these operations can be reversed by an identification of a pair of vertices of same color. Hence (1) implies (3). Also, since an identification of vertices of same color preserves the edge criticality and  $H$  is  $k$ -edge critical, (3) implies (1).

B. *Corollary 3.2.* A graph  $G$  is 2-edge critical if and only if  $G = K_2$ .

C. *Corollary 3.3.* A graph  $G$  is 3-edge critical if and only if  $G = K_3, P_4, 3K_2$  or  $K_2 \cup K_1, 2$

D. *Corollary 3.4.* A necessary condition for a cycle,  $C_n$  with a chord, to be  $k$ -edge critical is that  $k$  is odd and  $n = \binom{k}{2} - 1$

If a cycle  $C_n$  with a chord is  $k$ -edge critical, then the path  $P_{n+2}$  is also  $k$ -edge critical, because  $P_{n+2}$  can be obtained from  $C_n$  with a

chord by a 2-splitting operation at each of the end vertices of the chord. So, by Corollary 2.22,  $k$  is odd and  $n = \binom{k}{2} - 1$

E. Corollary 3.5. A necessary condition for a  $\theta$ -graph  $\theta(a,b,c)$  consisting of 3 internally disjoint paths of lengths  $a$ ,  $b$  and  $c$  where  $2 \leq a \leq b \leq c$ , to be

$k$ -edge critical is that  $k$  is odd and  $a + b + c = \binom{k}{2}$

The following theorem shows that there is no forbidden-subgraph characterization of any of the three types of critical graphs.

F. Theorem 3.6. Any graph  $G$  is an induced subgraph of some connected edge critical graph.

1) Proof. Let  $V(G) = \{v_1, v_2, \dots, v_n\}$ . For each  $i$ ,  $1 \leq i \leq n$ , such that degree of  $v_i$  is not equal to  $n-1$ , add a new vertex  $u_i$  and join  $u_i$  to all the vertices  $v_j$  with  $j > i$  and is not adjacent to  $v_i$ . Let  $G'$  denote the resulting graph. Clearly  $G'$  is  $n$ -edge critical. Also  $G'$  is connected, unless  $G$  is the disjoint union of two complete graphs. If  $G'$  is disconnected, we add a new vertex  $u_0$  to  $G'$  and join  $u_0$  to all the vertices of  $G$ . The resulting graph is connected and  $(n+1)$ -edge critical.

Now, we investigate the local criticality concepts such as critical vertices, critical edges and non-contractible edges.

2) Local Criticality

Now, we investigate the local criticality concepts such as critical vertices, critical edges and non-contractible edges of a graph.

G. Definition 3.7. Let  $G$  be a graph. A vertex  $v \in V(G)$  is called a critical vertex of  $G$ , if  $\psi_s(G - v) < \psi_s(G)$ . An edge  $e \in E(G)$  is called a critical edge of  $G$ , if  $\psi_s(G - e) < \psi_s(G)$ . An edge  $e \in E(G)$  is called a non-contractible edge of  $G$ , if  $\psi_s(G - e) = \psi_s(G)$ .

H. Remark 3.8. We observe that  $e = uv$  is a critical edge of a graph  $G$  with  $\psi_s(G) = k$  if and only if in any  $k$ -pseudo-complete coloring of  $G$ , there exist two colors  $c_1$  and  $c_2$  such that  $u$  and  $v$  are the only adjacent vertices, having the colors  $c_1$  and  $c_2$  respectively. Hence the end vertices of a critical edge are critical. However, critical vertices need not be the end vertices of critical edges. For example, in the graph  $P_{11} \pm K_1$ , the unique vertex of degree 11 is critical, but none of the edges incident at it is critical.

I. Remark 3.9. Since  $G - e$  can be obtained from  $G \setminus e$ , by a 2-splitting operation, any critical edge is non-contractible. Again, a non-contractible edge need not be critical. For example in the graph  $P_H$  any edge incident at the unique vertex of degree 11 is non-contractible, but is not critical.

J. Remark 3.10. We observe that an edge  $e = \{u, v\}$  of  $G$  is non-contractible if and only if in any  $k$ -pseudo-complete coloring of  $G$ ,  $u$  and  $v$  have distinct colors where  $k = \psi_s(G)$ . Hence, the end vertices of a non-contractible edge are critical. Again, a critical vertex of  $G$  need not be an end vertex of some non-contractible edge of  $G$ . For example, every vertex of the cycle  $C_4$  is critical, but none of its edges is non-contractible. The next proposition gives a condition for a critical vertex to be an end vertex of some non-contractible edge.

K. Proposition 3.11. Let  $G$  be a graph on  $p$  vertices and let  $v$  be a vertex of  $G$  with degree  $p-1$ . Then, any vertex  $u \neq v$  is critical if and only if the edge  $e = uv$  is non-contractible.

1) Proof. If a graph  $G$  has  $p$  vertices and a vertex  $v$  of  $G$  has degree  $p-1$ , then clearly  $v$  is a critical vertex of  $G$ . Suppose that a vertex  $u$  of  $G$  is critical,  $u \neq v$  and  $e = uv$  is not non-contractible. Then,  $\psi_s(G) - 1 \leq \psi_s(G - \{u, v\}) \leq \psi_s(G - v) \leq \psi_s(G) - 1$  so that  $\psi_s(G - \{u, v\}) = \psi_s(G - v)$  Thus,  $u$  is not a critical vertex of  $G - v$  and hence  $u$  is not a critical vertex of  $G$ , a contradiction. Converse follows from Remark 3.10.

L. Corollary 3.12. If  $G$  is  $k$ -con-critical, then  $G + K_1$  is  $(k + n)$ -con-critical.



*M. Remark 3.13.* If  $G$  is a  $k$ -vertex critical graph, then  $G+K_{r_i}$  is  $(k+n)$ -vertex critical. However, if  $G$  is  $k$ -edge critical, then  $G+K_{r_i}$  need not be edge critical.

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