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Pseudo-Complete Color Critical Graphs

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Abstract: A pseudo-complete coloring of a graph G is an assignment of colors to the vertices of G such that for any two distinct colors, there exist adjacent vertices having those colors. The maximum number of colors used in a pseudo-complete coloring of G is called the pseudo-achromatic number of G and is denoted by $\psi_s(G)$. A graph G is called edge critical if $\psi_s(G-e)<\psi_s(G)$ for any edge e of G. A graph G is called vertex critical if $\psi_s(G-v) <. \psi_s(G)$ for every vertex v of G. These graphs are generally called as pseudo-achromatic number critical graphs (called shortly as PAN Critical graphs). In this paper, we investigate the properties of these critical graphs. We also investigate the locally critical elements of graphs.
Keywords: Pseudo-complete coloring, Pseudo-achromatic number, critical graph

I. INTRODUCTION

By a graph we mean a finite undirected graph without loops, multiple edges or isolated vertices.

An assignment of colors to the vertices of a graph is called a proper coloring, if any two adjacent vertices receive distinct colors. An assignment of colors to the vertices of a graph G is called a pseudo-complete coloring, if for any two distinct colors; there exist adjacent vertices having those colors. A proper and pseudo-complete coloring of G is called a complete coloring of G.

The minimum number of colors used in a proper coloring of G is called the chromatic number of G and is denoted by $\chi(G)$. The maximum number of colors used in a complete coloring of Gis called the achromatic number of G and is denoted by $\psi(G)$ [3]. The maximum number of colors used in a pseudo-complete coloring of Gis called the pseudo-achromatic number of G and is denoted by $\psi_s(G)$ [5]. A graph which admits a pseudo-complete coloring by k colors is called a k- pseudo-complete colorable graph. Several bounds for these coloring parameters were obtained in [3. 4, 5, 6] and a detailed study of this parameter and critical graphs with respect to it were studied by Suresh Kumar in his doctoral dissertation [7].

The concept of critical graphs with respect to chromatic number was introduced and studied by Dirac [1, 2]. In this paper, we introduce the concept of critical graphs with respect to pseudo-achromatic number, obtain characterizations of them and determine the pseudo-achromatic number of several classes of graphs.

Let n be any positive real number. Then, [n] denote the greatest integer less than or equal to n and {n} denotes the smallest integer greater than or equal to n.

For terms not defined explicitly here, reader can refer Harary [8].

II. CRITICAL PATHS AND CRITICAL CYCLES

A. *Definition2.1.* A graph G is called k-edge critical if$\psi_s(G)$= k and $\psi_s(G-e)$ <k, for every edge e E E(G). A graph G is called k-vertex critical if $\psi_s(G) = k$ and $\psi_s(G-v) < k$ for every vertex v of G. A graph G is called k-contraction critical (shortly, k-con-critical) if$\psi_s(G) = k$ and $\psi_s(G||e) < k$ for every edge e of G, where G||e denotes the graph obtained from G by contracting the edge e. Following observations are quite useful later.

B. *Proposition 2.2.* A graph G is k-edge critical if f G is k-pseudo-complete colorable and $|E(G)| = \binom{k}{2}$

C. *Proposition 2.3.* A k-edge critical graph is k--con-critical and a k-con-critical graph is k-vertex-critical.

D. *Remark 2.4.*None of the statements in Proposition 2.3can be reversed. For example, C8is 4-con-critical but not edge critical. Also, C4is 3-vertex critical, but not con-critical.

E. *Theorem2.5.* Let G be a k-pseudo-complete colorable graph. Then, $|V(G)| \geq k\left\{\frac{k-1}{\Delta}\right\}$ where Δ is the maximum degree of a vertex of G and {n} is the smallest integer less than or equal to n. Proof. Consider any k-pseudo-complete coloring of G. Then for any color c, there exist k—1 edges in G such that one end vertex of each of these edges receive the color c. Hence there must be at least $k\left\{\frac{k-1}{\Delta}\right\}$vertices with color c, so that$|V(G)| \geq k\left\{\frac{k-1}{\Delta}\right\}$.

F. *Corollary 2.6.*$\psi_s(G) = max\left\{k: k\left\{\frac{k-1}{\Delta}\right\} \leq |V(G)|\right\}$for any graph G with the maximum degree, Δ.

G. *Corollary 2.7.* If G is the Petersen Graph, $\psi_s(G) = 5$.

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1) *Proof*. Clearly, E(G) can be partitioned into a 2-factor, say { (v1v2v3v4v5v1), (v6v7v8v9viov6)} and a 1-factor, {v1v6, v2v9, v3v7, v4v10, v5v8}. Now the function $f: V(G) \to \{1,2,3,4.5\}$definedby$f(v_I) = i, 1 \le i \le 5$ and$f(v_I) = 2i - 11(mod\ 5), 6 \le i \le 10$assigns a 5-pseudo-complete coloring for G Hence,$\psi_s(G) \ge 5$. Also, by Corollary 2.6, $\psi_s(G) \le 5$.

H. *Corollary 2.8*.IfG is the 3-cube, Q$_3$,$\psi_s(G) = 4$

1) *Proof*. By Corollary 2.6,$\psi_s(G) \le 4$. The function $f: V(G) \to \{1,2,3,4\}$defined below assigns a 4-pseudo-complete coloring to Q$_3$ so that the corollary follows.

$$f(0,0,0) = f(1,1,0) = 1\ ,\ f(0,0,1) = f(1,1,1) = 2$$
$$f(0,1,1) = f(1,0,0) = 3\ ,\ f(0,1,0) = f(1,0,1) = 4$$

I. *Corollary 2.9.For the 4-cube Q_4, $\psi_s(Q_4) = 8$*

1) *Proof.* By Corollary 2.6, $\psi_s(Q_4) \le 8$. The function *f : V (Q$_4$)*→{1,2,3,4, 5,6, 7, 8} defined below assigns a 8-pseudo-complete coloring to Q$_4$ so that the corollary follows.

$$f(0,1,1,0) = f(1,0,0,1) = 1\ ,\ f(1,1,1,0) = f(0,0,0,1) = 2$$
$$f(0,1,0,0) = f(1,0,1,0) = 3\ ,\ f(0,0,1,0) = f(1,1,1,1) = 4$$
$$f(1,1,0,0) = f(0,1,1,1) = 5\ ,\ f(1,0,0,0) = f(0,1,1,1) = 6$$
$$f(0,1,0,1) = f(1,0,1,1) = 7\ ,\ f(0,0,0,0) = f(1,1,0,1) = 8$$

J. *Corollary 2.10.For the 5-cube* Q$_5$,$\psi_s(Q_5) = 11$.

1) *Proof.* By Corollary 2.6, $\psi_s(Q_5) \le 11$. An 11-pseudo complete coloring of Q$_5$ is shown below, so that the corollary follows.

f(0,1,1,0,0) = f(1, 0, 0,1,0) =1, f(1,1,1,0,0) =f(1,1,1,0,1) = f(0,0,0,1,1) =2

f(0,1,0,0,0) = f(1,0,1,0,0) = 3, $f(0,0,1,0,0) = f(1,1,1,1,0) = f(0,0,1,0,1) = 4$

$f(1,1,0,0,0) = f(0,1,1,1,0) = f(0,1,1,1,1) = 5$, $f(1,0,0,0,0) = f(0,0,1,1,0) = f(0,0,1,1,1) = 6$, $f(1,0,1,1,0) = f(0,1,0,1,1) =$ 7, $f(0,0,0,0,0) = f(1,1,0,1,0) = f(0,0,0,0,1) = f(1,1,0,1,1,) = 8$

$f(1,0,0,0,1) = f(0,1,0,1,0) = f(0,1,1,0,1) = 9$

$f(1,1,0,0,1) = f(1,0,0,1,1) = f(1,0,1,0,1) = f(1,0,1,1,1) = 10$

$f(0,0,0,1,0) = f(0,1,0,0,1) = f(1,1,1,1,1) = 11$

Thesecorollaries motivate the following conjecture:

K. *Conjecture 2.11.*$\psi_s(Q_n) = max\left\{k: k\left\{\frac{k-1}{\Delta}\right\} \le 2^n\right\}$

L. *Corollary 2.12*. For $n \ge 2, \psi_s(K_{n,n}) = n + 1$ n + 1.

1) Proof. By Corollary 2.6, $\psi_s(K_{n,n}) \le n + 1$. Also if $X = \{x_1, x_2, \ldots. x_n\}, Y = \{y_1, y_2, \ldots. y_n\}$is a bipartition of K$_{n,n}$ then the function$f: V(K_{n,n}) \to \{1,2, \ldots. n + 1\}$defined by $f(x_1) = 1, f(y_1) = 2,)$ *and* $f(x_i) = f(y_i) = i + 1, 2 \le i \le$ $n$gives a pseudo complete coloring of G so that $\psi_s(K_{n,n}) = n + 1$ 1

Now we proceed to characterize critical cycles and critical paths.

M. *Theorem 2.13*. Let n(k) denote the integer$\binom{k}{2}$ or $\binom{k}{2} + 1$, according as*k*is odd or even. Then a cycle C$_n$ is k-con-critical if and only if n = n(k).

1) *Proof*. Let $C_n = \langle v_1, v_2, \ldots. v_{n(k)}. v_1\rangle$ and $Z_n = \{1,2 \ldots. n\}$

We first prove that C$_n$ is k-pseudo-complete colorable.

Case 1.kis even.

Define a function $f: V(C_{n(k)}) \to Z_n$ as follows:

$$f(v_i) = \begin{cases} (1 + (-1)^{n+1}\{i/2\})(mod\ k) & if\ 1 \le i \le k \\ (\{i/k\} + (-1)^{1+g_i}\{g_i/2\})(mod\ k) & otherwise \end{cases}$$

Where $g_i = (i - \{i/k\}(k-2) + k - 3)(mod\ k)$

Let $j_1, j_2 \in Z_{k,} j_1 < j_2$ Suppose $j_2 - j_1 = k/2$ let

$$If\ j_1 = 1, f(\{v_k, v_{k+1}\}) = \{j_1, j_2\}$$
$$If\ j_1 \geq 2, f(\{v_{j_1-2j_1+3}, v_{j_1-2j_1+4}\}) = \{j_1, j_2\}$$

Now, suppose $j_2 - j_1 \neq k/2$.Put $n = \left\{\frac{(j_1+j_2)(mod\ k)}{2}\right\}$

Choose j to be the least positive integer such that $f(v_{(n-1)k+j})$=j1 or j2. Then it can be easily verified that $f(\{v_{(n-1)k+j}, v_{(n-1)k+j+1}\}) = \{j_1, j_2\}$. Thus in all cases, f assigns a k-pseudo-complete coloring to Cn(k).

Case 2.kis odd.

Define a function $f: V(C_{n(k)}) \to Z_k$as follows:

$$f(v_i) = \begin{cases} k & if\ i \equiv 1(mod\ k) \\ \left(\left\{\frac{i}{k}\right\} + (-1)^{g_i}\left[\frac{g_i}{2}\right]\right)(mod\ k-1) & otherwise \end{cases}$$

where$g_i = \left(i - \left\{\frac{i}{k}\right\}k + k - 1\right)(mod\ k)$

Let $j_1, j_2 \in Z_k\ and\ j_1 < j_2$. Suppose $j_2 = k$.

If $j_1 \leq \left\{\frac{k}{2}\right\}, then\ f(v_{(j_1-1)k+1}, v_{(j_1-k)k+2}) = \{j_1, j_2\}$

If $j_1 > \left\{\frac{k}{2}\right\}, then\ f(v_{(j_1-\{k/2\})k}, v_{(j_1-\{k/2\})k+1}) = \{j_1, j_2\}$

Now, suppose $j_1 < k$. Put $n = \left\{\frac{(j_1+j_2)(mod\ k-1)}{2}\right\}$. Choose j to be the least positive integer such that $f(v_{(n-1)k+j}) = j_1\ or\ j_2$

Then,$f(v_{(n-1)k+j}, v_{(n-1)k+j+1}) = \{j_1, j_2\}$

Thus in all cases, f assigns a k-pseudo-complete coloring to $C_{n(k)}$. Hence it follows from Theorem 2.5 that $C_{n(k)}$ is k-con-critical. Conversely, suppose Cn is k-con-critical. Since $C_{n(k)}$is k-pseudo-complete colorable, n<n(k) and by Theorem 2.5, n>n(k).

N. *Corollary 2.14.* Letn(k) denote the integer$\binom{k}{2}\ or\ \binom{k}{2} + \frac{k}{2}$, according as k is odd or even. Then $\psi_s(C_n) = max\{k: n(k) \leq n\}$

O. *Corollary 2.15.* Letm(k) denote the integer $\binom{k}{2} + 1\ or\ \binom{k}{2} + \frac{k}{2}$, according as k is odd or even. Then a path P_n is k-con-critical if and only if n = m(k).

If P_n is k-pseudo-complete colorable, then n >m(k), by from Theorem 2.5. But since $C_{n(k)}$is k-pseudo-complete colorable, so is $P_{m(k)}$. Hence P_n is k-con-critical if and only if n = m(k).

Corollary 2.16.Let m(k) denote the integer $\binom{k}{2} + 1\ or\ \binom{k}{2} + \frac{k}{2}$, according as k is odd or even. Then $\psi_s(P_n) = max\{k: m(k) \leq n\}$

The following propositions can easily be deduced from Theorem 2.5, Theorem 2.13 and Corollary 2.15.

P. *Proposition 2.17.* A cycle, C_n is k-vertex critical iff

$$n = \begin{cases} \binom{k}{2} + \frac{k}{2} & if\ k\ is\ even \\ \binom{k}{2}\ or\ \binom{k}{2} + 1 & if\ k\ is\ odd \end{cases}$$

Q. *Proposition 2.18.* A Path, P_n is k-vertex critical iff

$$n = \begin{cases} \binom{k}{2} + \frac{k}{2} & if\ k\ is\ even \\ \binom{k}{2} + 1 & if\ k\ is\ odd \end{cases}$$

R. *Theorem 2.19.* There is no k-edge critical cycle, if k is even

Suppose$C_n$is a k-edge critical cycle and k is even. By Theorem 2.13, $n = \binom{k}{2} + \frac{k}{2}$. Now, for any edge e of

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$C_n, C_n \sim e$(a path on $\binom{k}{2} + \frac{k}{2}$vertices) is not k-pseudo-complete colorable, a contradiction.

S. Corollary 2.20. A cycle,C_n, is k-edge critical if and only if k is odd and$n = \binom{k}{2}$.

T. Theorem 2.21. There is no k-edge critical path, if $k \geq 4$is even.

Suppose P_n is k-edge critical and $k \geq 4$is even. By Corollary 2.15, $n = \binom{k}{2} + \frac{k}{2}$ so that, $|E(P_n)| > \binom{k}{2}$, a contradiction toProposition.2.2.

*U. Corollary. 2.22.*A path P_n is k-edge critical if and only if either k is odd and $n = \binom{k}{2} + 1$or n=k=2

V. Remark 2.23. Corollaries 2.14, 2.7, 2.8, 2.9, 2.10 and 2.12 show that the regular graphs such as cycles, Petersen graph, n-cubes for n = 3, 4, 5 and $K_{n,n}$are solutions of the equation:

$$\psi_s(G) = max\{k: k\{(k-1)/\Delta\}\} \leq |V(G)|$$

$O_s(G)$ = max{k : k [(k _ i)Ai C iv(G)I}.

This motivates us to propose the following conjecture for which Conjecture 2.11 is a special case.

W. Conjecture 2.24. For any n-regular graph G,

$$\psi_s(G) = max\{k: k\{(k-1)/\mathrm{n}\}\} \leq |V(G)|$$

III. CHARACTERIZATION OF PAN CRITICAL GRAPHS

Let v be a vertex of G having degree d and let n be an integer such that1 ≤ ≤ . Then an n-splitting of v is the replacement of v by a set S of n independent vertices such that N(S) = N({v}), degree of ≥ 1 for all ∈ and $\sum_{u \in S} \deg u = d$, where N(S) denotes the set of all neighbors of the vertices in S.

Clearly, the complete graph on k vertices and the graph consisting of $\binom{}{2}$disjoint copies of $_2$are k-edge critical. The following theorem shows that any k-edge critical graph can be obtained from these graphs by simple operations, which preserve edge criticality.

A. Theorem 3.1. The following statements are equivalent:

G is a k-edge critical graph.

G can be obtained from Kkby a sequence of n-splitting operations.

Let H be the graph consisting of disjoint copies of K2 and let C bea k-pseudo-complete coloring of H. Then G can be obtained from H by a sequence of identifications of vertices of same color.

1) Proof. Let G be a k-edge critical graph and let $c_1, c_2 \ldots, c_k$be the colors used in a k-pseudo-complete coloring of G. Let $d_{i1}, d_{i2}, \ldots d_{im_i}$denote the degreesof the vertices of G with colors $c_i, 1 \leq i \leq k$. Clearly $\sum_{j=1}^{m_i} d_{ij=1}$. Now, let$V(K_k) = \{v_1, v_2, \ldots v_k\}$. Then G can be obtained from Kk by performing an mi-splitting operation on vi, for each i, $1 \leq i \leq k$, such that the mj vertices which replace vi have degrees $d_{i1}, d_{i2}, \ldots d_{im_i}$. Hence (1) implies (2). Also, since an n-splitting operation preserves the edge criticality and Kkis k-edge critical, (2) implies (1).

If G is k-edge critical, G is k-pseudo-complete colorable and G has exactly $\binom{k}{2}$edges. Now H can be obtained from G by recursively performing 2-splitting operations and each of these operations can be reversed by an identification of a pair of vertices of same color. Hence (1) implies (3). Also, since an identification of vertices of same color preserves the edge criticality and H is k-edge critical, (3) implies (1).

B. Corollary 3.2. A graph G is 2-edge critical if and only if G = K2.

C. Corollary 3.3. A graph G is 3-edge critical if and only if G = K3, P4, 3K2 or K2UK1,2

D. Corollary 3.4. A necessary condition for a cycle,$C_n$with a chord, to be kedge critical is that k is odd and $n = \binom{k}{2} - 1$

If a cycle C_n with a chord is k-edge critical, then the path P_{n+2} is also k-edge critical, because P_{n+2}can be obtained from $C_n$with a chord by a 2-splitting operation at each of the end vertices of the chord. So, by Corollary 2.22, k is odd and $n = \binom{k}{2} - 1$

E. Corollary 3.5. A necessary condition for a θ-graph θ(a,b,c) consisting of 3 internally disjoint paths of lengths a, b and c where $2 \leq a \leq b \leq c$, to be

k-edge critical is that k is odd and a + b + c $=\binom{k}{2}$

The following theorem shows that there is no forbidden-subgraph characterization of any of the three types of critical graphs.

F. *Theorem 3.6***.** Any graph G is an induced subgraph of some connected edge critical graph.

1) *Proof. Let* V(G) = {v1, v2, , vn}. For each i, $1 \leq i \leq n$, such that degreeof vi is not equal to n—1, add a new vertex ui and join ui to all the vertices $v_j$with j >i and is not adjacent to vi. Let G' denote the resulting graph. Clearly G' is n-edge critical. Also G' is connected, unless G is the disjoint union of two complete graphs. If G' is disconnected, we add a new vertex u0 to G' and join u0 to all the vertices of G. The resulting graph is connected and (n+1)-edge critical.

Now, we investigate the local criticality concepts such as critical vertices, critical edges and non-contractible edges.

2) *Local Criticality*

Now, we investigate the local criticality concepts such as critical vertices, critical edges and non-contractible edges of a graph.

G. *Definition 3.7.* Let G be a graph. A vertex v E V(G) is called a critical vertex of G. if 7/),(C — v) <V),(G). An edge e EE(G) is called a critical edge of G, if O_s(G e) < i./),(G). An edge e eE(G) is called a non-contractible edge of G, if 0_5(G1ie) < 0,(G).

H. *Remark 3.8.* We observe that e = uv is a critical edge of a graph *G* with 1/),(G) = *k* if and only if in any k-pseudo-complete coloring of *G*, there exist two colors c1 and c2 such that u and v are the only adjacent vertices, having the colors c_1 and c2 respectively. Hence the end vertices of a critical edge are critical. However, critical vertices need not be the end vertices of critical edges. For example, in the graph P11 ± K1, the unique vertex of degree 11 is critical, but none of the edges incident at it is critical.

I. *Remark 3.9.* Since *G—e* can be obtained from G||e, by a 2-splitting operation, any critical edge is non-contractible. Again, a non-contractible edge need not be critical. For example in the graph P_H any edge incident at the unique vertex of degree 11 is non-contractible, but is not critical.

J. *Remark 3.10.* We observe that an edge e ={u, v} of Gis non-contractible if and only if in any k-pseudo-complete coloring of *G*, u and v have distinct colors where $k = \psi_s(G)$. Hence, the end vertices of a non-contractible edge arecritical. Again, a critical vertex of *G* need not be an end vertex of some non-contractible edge of *G*. For example, every vertex of the cycle C_4 is critical, but none of its edges is non-contractible. The next proposition gives a condition for a critical vertex to be an end vertex of some non-contractible edge.

K. *Proposition 3.11.* Let G be a graph on p vertices and let v be a vertex of G with degree p —1. Then, any vertex $u \neq v$ is critical if and only if the edge e = uvis non-contractible.

1) *Proof.*If a graph G has p vertices and a vertex v of G has degree p-1, then clearly v is a critical vertex of G. Suppose that a vertex u of G is critical, $u \neq v$ and e = uv is not non-contractible. Then, $\psi_s(G) - 1 \leq \psi_s(G - \{u, v\}) \leq \psi_s(G - v) \leq \psi_s(G) - 1$so that $\psi_s(G - \{u, v\}) = \psi_s(G - v)$ Thus, u is not a critical vertex of G—v and hence u is not a critical vertex of G, a contradiction. Converse follows from Remark 3.10.

L. *Corollary 3.12.* If G is k-con-critical, then G + is (k + n)-con-critical.

*M. Remark 3.13.*If G is a k-vertex critical graph, then $G+K_{ri}$is (k+n)-vertex critical. However, if G is k-edge critical, then $G+K_{ri}$ need not be edge critical.

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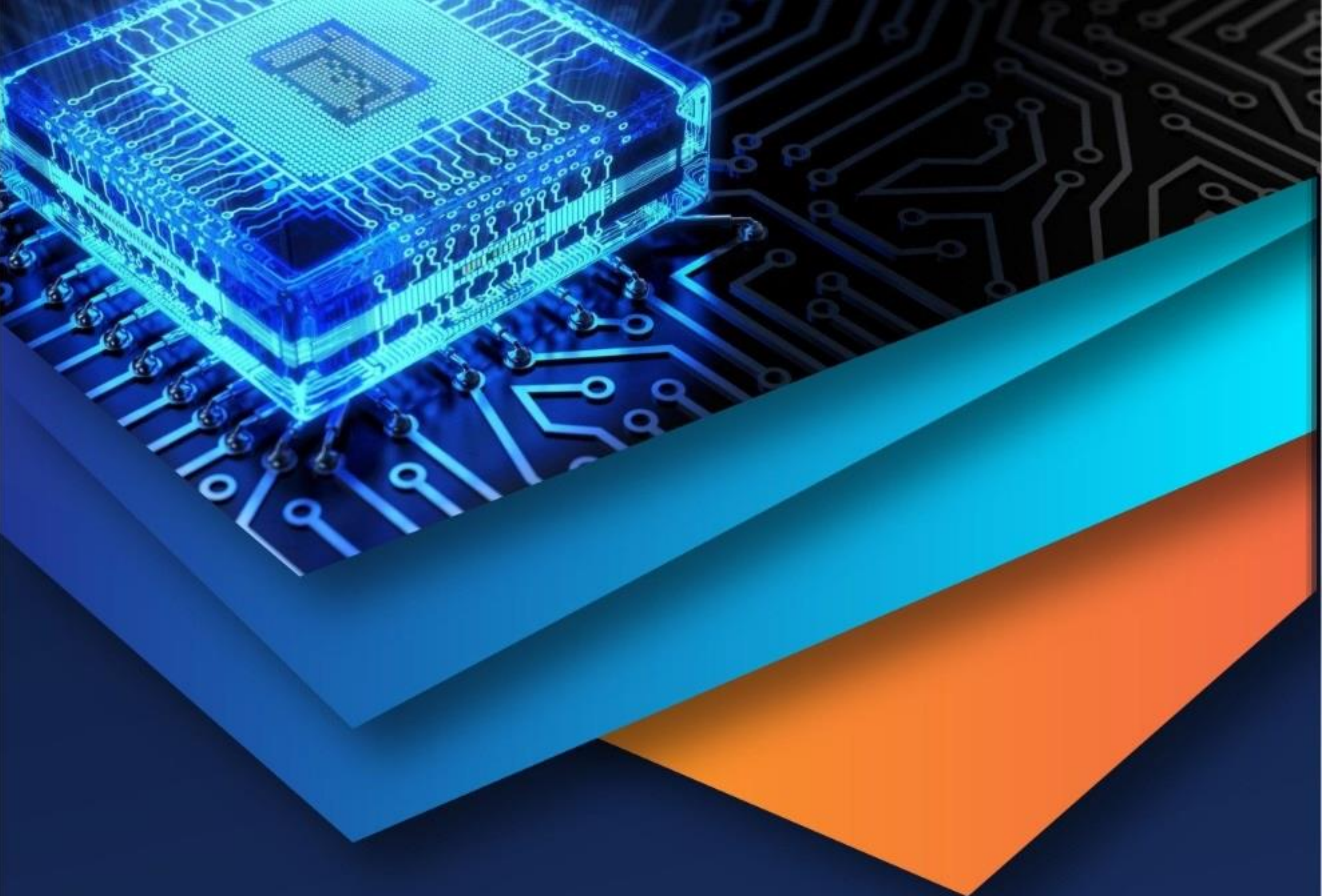

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