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Quasi Jawarneh Manifolds

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Abstract: Quasi Jawarneh manifold have been defined, and some of its geometric properties are derived. Also some properties of Quasi Jawarneh Space-time have been obtained. Further a non-trivial example of Quasi Jawarneh manifold has been introduced to prove the typical existence.
Keywords: Riemannian manifold, Quasi Einstein manifold, killing vector field, cyclic type Ricci tensor, Codazzi type Ricci tensor, Jawarneh manifold, Space-time, General Relativity.
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I. INTRODUCTION

Chaki [1] introduced the notion of a quasi Einstein manifold, whose Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition:

A. $S(X,Y) = a\,g(X,Y) + b\,A(X)A(Y)$,

where a and b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

B. $g(X, \rho) = A(X)$,

for all vector fields X, ρ being a unit vector field. In a recent paper [2] the author introduced a new type of Riemannian manifold called Jawarneh manifold, i. e., a Riemannian manifold (M^n, g) (n>2), such that its curvature tensor R satisfies the relation:

C. $R(X,Y,Z) = k[S(Y,Z)X - S(X,Z)Y]$,

where k is constant and Q is the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S such that,

D. $S(X,Y) = g(QX,Y)$.

The object of this paper is to study a type of Riemannian manifold (M^n, g)(n>2) such that its curvature tensor R satisfies:

Section3 devoted to Quasi Jawarneh Space-time in which the necessary and sufficient condition for cyclic parallel Ricci tensor to be cyclic parallel energy momentum tensor have been established. Further it is shown that on Quasi Jawarneh Space-time the necessary and sufficient condition for Codazzi type Ricci tensor to be Codazzi type energy momentum tensor have been justified, and a Quasi Jawarneh Space-time with cyclic parallel Ricci tensor the generator ρ is a killing vector field. Last section will be devoted to a non trivial example of Quasi Jawarneh manifold to prove the existence of the manifold.

II. QUASI JAWARNEH MANIFOLDS

For more justification of the definition (1.5) we can easily verify that the quasi jawarneh manifold satisfy the anti-symmetric relation of Riemann curvature R(X,Y,Z) + R(Y,X,Z) = 0, and also satisfy the Bianchi first identity R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0.

Now contracting (1.5) with respect to X we get,

$$S(Y,Z) = \frac{b(n-1)}{1-a(n-1)}A(Y)A(Z).$$

This shows that the Ricci tensor is proportional to the 1-form. Thus we can state,

A. Theorem2.1) Quasi Jawarneh manifold is a quasi-Ricci flat manifold.

Taking covariant derivative of the above equation we get,

2.2) $(\nabla_X S)(Y,Z) = \frac{b(n-1)}{1-a(n-1)}[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)]$.

If the manifold is of Codazzi type Ricci tensor then we have from (2.2) and (1.7),

2.3) $[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)] = [(\nabla_Y A)(X)A(Z) + (\nabla_Y A)(Z)A(X)]$.

Contracting this equation with respect to Y and Z we get $(\nabla_\rho A)(X) = 0$, and therefore we have $\nabla_\rho \rho = 0$. Thus we can state,

B. Theorem2.2) On Quasi Jawarneh manifold of Codazzi type Ricci tensor the integral curves of the vector field ρ are gradient and geodesic.

Now let Z = ρ in (2.3) and taking in account $(\nabla_X A)(\rho) = 0$ we can have,
2.4)$[(\nabla_X A)(Y) - (\nabla_Y A)(X)]A(\rho) = 0$.
Since $A(\rho) \neq 0$, (2.4) will reduce to,
2.5) $(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0$.
This means that the 1-form A is closed. Thus we can state,

C. Theorem2.3) On Quasi Jawarneh manifold of Codazzi type Ricci tensor the 1-form A is closed.

If we consider the manifold as of cyclic Ricci tensor then from (1.8) and (2.2) we have,

2.6) $\left[\left((\nabla_X A)(Y) + (\nabla_Y A)(X)\right)A(Z) + \left((\nabla_X A)(Z) + (\nabla_Z A)(X)\right)A(Y) + \left((\nabla_Z A)(Y) + (\nabla_Y A)(Z)\right)A(X)\right] = 0$.

Contracting with respect to Y and Z we can get,

2.7) $(\nabla_\rho A)(X) = 0$,

which imply that $\nabla_\rho \rho = 0$. Thus we can state,

D. Theorem2.4) On Quasi Jawarneh manifold with cyclic parallel Ricci tensor the integral curves of the vector field ρ are gradient and geodesic.

From (1.8) and (2.2) we can also have,
2.8) $(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = \frac{b(n-1)}{1-a(n-1)}\left[\left((\nabla_X A)(Y) + (\nabla_Y A)(X)\right)A(Z) + \left((\nabla_X A)(Z) + (\nabla_Z A)(X)\right)A(Y) + \left((\nabla_Z A)(Y) + (\nabla_Y A)(Z)\right)A(X)\right]$.
Now let us consider the generator vector field ρ to be a killing vector field of the manifold then we have,
2.9) $(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0$.
Using this equation on (2.8) we get,
2.10) $(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0$.
Thus we can state,

E. Theorem2.4) On Quasi Jawarneh manifold if the vector field ρ is a killing vector field then the Ricci tensor is cyclic parallel.

Now contracting (2.1) we get,
2.11) $r = \frac{b(n-1)}{1-a(n-1)}$.
From which we have,
2.12) dr(X) = 0.
That is the curvature tensor of Quasi Jawarneh manifold is constant.

III. QUASI JAWARNEH SPACE-TIME

Einstein equation in general relativity flows given in the form,

) $S(Y,Z) - \frac{r}{2} g(Y,Z) + \lambda g(Y,Z) = kT(Y,Z)$,

for all vector fields Y and Z, k is the gravitational constant, λ is the cosmological constant and T is the energy momentum tensor of type (0,2). The energy momentum tensor T describes the matter of the space-time by distribution of the matter and the energy due to physical considerations.

From (3.1) and (2.1) we can have,

3.2) $\frac{2\lambda-r}{2}\, g(Y,Z) + \frac{b(n-1)}{1-a(n-1)} A(Y)A(Z) = kT(Y,Z).$

Let the generator ρ of the space-time be a killing vector field then we have,

3.3) $(\mathcal{L}_\rho g)(Y,Z) = 0,$

where $\mathcal{L}$ denotes the Lie derivative. Therefore from (3.2) we get,

3.4) $\frac{2\lambda-r}{2}\, (\mathcal{L}_\rho g)(Y,Z) = k(\mathcal{L}_\rho T)(Y,Z)$.

Thus in view of (3.3) and (3.4) taking in account that $k \neq 0$ we can state,

A. *Theorem 3.1)* On Quasi Jawarneh Space-time the generator ρ is a killing vector field if and only if the Lie derivative of the energy momentum tensor is zero.

Now taking covariant derivative of (3.1) we get,

3.5) $(\nabla_X S)(Y,Z) = \frac{1}{2}\, dr(X)g(Y,Z) + k(\nabla_X T)(Y,Z).$

Which by virtue of (2.12) yields,

3.6) $(\nabla_X S)(Y,Z) = k(\nabla_X T)(Y,Z).$

Hence from (1.8) and (3.6) we have,

3.7) $(\nabla_X T)(Y,Z) + (\nabla_Y T)(X,Z) + (\nabla_Z T)(X,Y) = 0,$

That is the energy momentum tensor T is cyclic parallel. Conversely if (3.7) holds then from (3.6) we can have (1.8). Thus we can state,

B. *Theorem 3.2)* On Quasi Jawarneh Space-time the Ricci tensor is cyclic parallel if and only if the energy momentum tensor is cyclic parallel.

Further using (3.6) on (1.7) we get,

3.8) $(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z),$

which means that the energy momentum tensor T is of Codazzi type.

Conversely if (3.8) holds then from (3.6) we get (1.7). Thus we can state,

C. *Theorem 3.3)* On Quasi Jawarneh Space-time the Ricci tensor is of Codazzi type if and only if the energy momentum tensor is of Codazzi type.

Also from (3.6) and (2.2) we can have,

3.9) $(\nabla_X T)(Y,Z) = \frac{b(n-1)}{k[1-a(n-1)]}[(\nabla_X A)(Y)A(Z) + ((\nabla_X A)(Z)A(Y)].$

If we take the energy momentum tensor to be of Codazzi type and using (3.9) then we have,

3.10) $(\nabla_X A)(Y)A(Z) + ((\nabla_X A)(Z)A(Y) = (\nabla_Y A)(X)A(Z) + ((\nabla_Y A)(Z)A(X).$

Contracting this equation with respect to Y and Z we can get $(\nabla_\rho A)(X) = 0$. Which implies that $\nabla_\rho \rho = 0$. Thus we can state,

D. *Theorem 3.4)* On Quasi Jawarneh Space-time if Ricci tensor is of Codazzi then the integral curves of the vector field ρ are geodesic.

Now if the Ricci tensor of Quasi Jawarneh Space-time is cyclic parallel then by virtue of (2.2) we have,

3.11) $\left[\left((\nabla_X A)(Y) + (\nabla_Y A)(X)\right)A(Z) + \left((\nabla_X A)(Z) + (\nabla_Z A)(X)\right)A(Y) + ((\nabla_Z A)(Y) + (\nabla_Y A)(Z))A(X)\right] = 0.$

Setting X= ρ in the above equation and using (2.7) we get,

3.12) $(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0,$

which implies,

3.13) $g(Z,\nabla_Y \rho) + g(Y,\nabla_Z \rho) = 0.$

Thus we can state,

E. *Theorem 3.5)* On Quasi Jawarneh Space-time with cyclic parallel Ricci tensor, the generator ρ is a killing vector field.

Lastly if we consider the energy momentum tensor to be cyclic parallel and using (3.9) we can have,

3.14) $(\nabla_X A)(Y)A(Z) + ((\nabla_X A)(Z)A(Y) + (\nabla_Y A)(X)A(Z) + ((\nabla_Y A)(Z)A(X) + (\nabla_Z A)(X)A(Y) + ((\nabla_Z A)(Y)A(X) = 0.$

Contracting this equation with respect to Y and Z we can get $(\nabla_\rho A)(X) = 0$. Which implies that $\nabla_\rho \rho = 0$. Thus we can state,

IV. EXAMPLE OF QUASI JAWARNEH MANIFOLD

Let us consider R^4 endowed with the Riemannian metric [4],

3.1) $d^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2$,

where i, j = 1,2,3,4. Then it is known [4] that the only non vanishing Ricci tensors and the curvature tensors are,

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}\ ;\ \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3(x^4)^{\frac{1}{3}}},$$

3.2) $R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x^4)^{\frac{2}{3}}}$,

3.3) $S_{11} = S_{22} = S_{33} = \frac{-2}{9(x^4)^{\frac{2}{3}}}$; $S_{44} = \frac{-2}{3(x^4)^2}$.

And the scalar curvature r = $\frac{-4}{3(x^4)^2}$.

Let us define A_i , a and b as follows:

3.4) $A_i = \frac{1}{3(x^4)}$, for i = 1, 2, 3, 4.

3.5) a = 1 ; b = 4 .

To verify the definition by (1.5) we have to verify only the following relations:

3.6) $R_{1441} = a[S_{44}g_{11}] + bA_4A_4g_{11}$,

3.7) $R_{2442} = a[S_{44}g_{22}] + bA_4A_4g_{22}$,

3.8) $R_{3443} = a[S_{44}g_{33}] + bA_4A_4g_{33}$.

Using (3.2), (3.3), (3.4) and (3.5) on (3.6) we get,

R.H.S. = $a[S_{44}g_{11}] + bA_4A_4g_{11}$

$= [\frac{-2}{3(x^4)^2}(x^4)^{\frac{4}{3}}] + 4[\frac{1}{9(x^4)^2}(x^4)^{\frac{4}{3}}]$

$= \frac{-2}{9(x^4)^{\frac{2}{3}}}$ = L.H.S.

Similarly we can show (3.6) and (3.7) are true, whereas the other cases are trivially true. Hence R^4 along with the metric g defined by (3.1) is Quasi Jawarneh manifold. Thus we can state,

A. Theorem3.1) A Riemannian manifold (M^4, g) endowed with the metric (3.1) is a Quasi Jawarneh manifold with non-constant scalar curvature.

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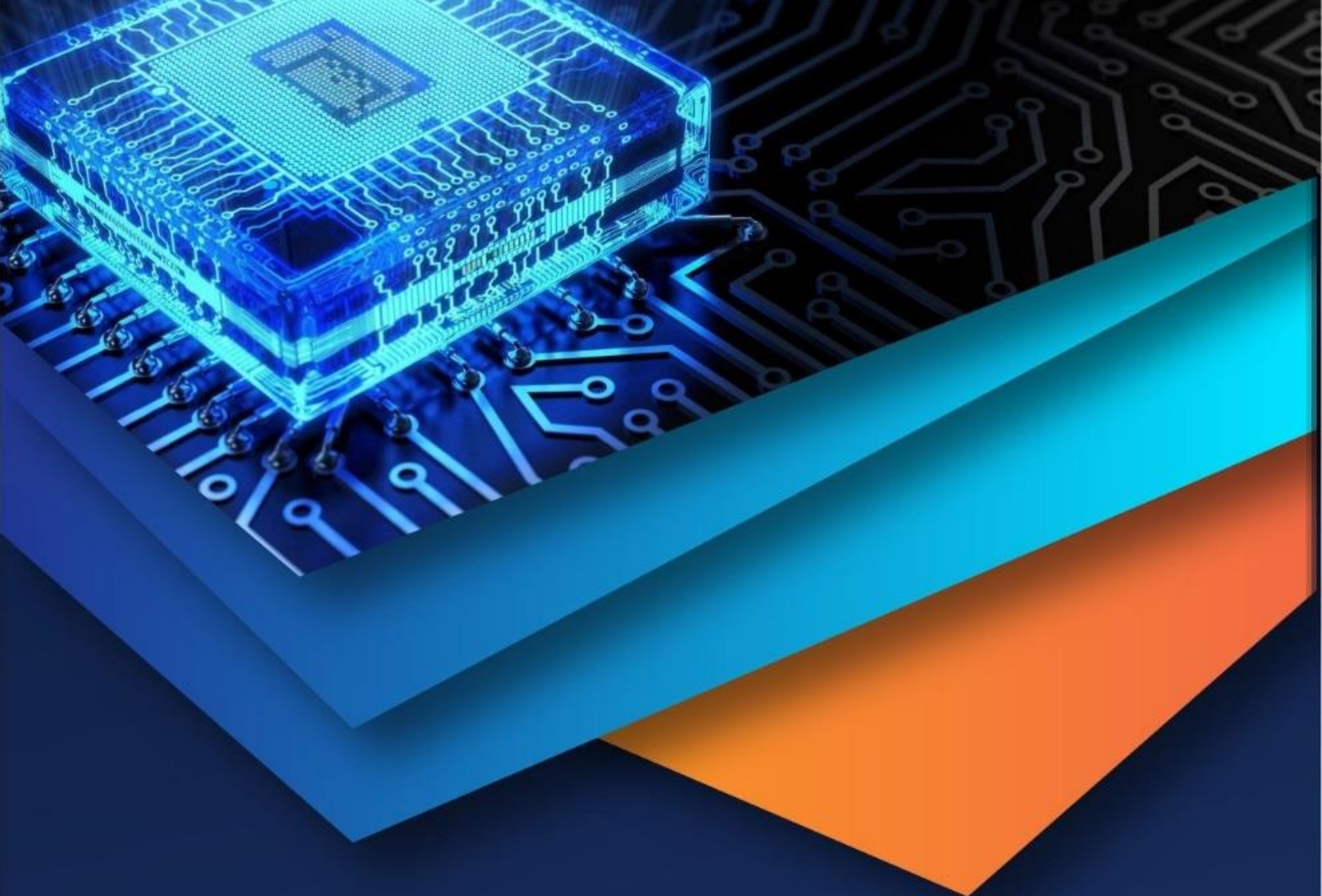

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