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Fixed Point Theorem and Consequences in D^* -Metric Spaces

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Abstract: The purpose of this paper is to establish a fixed point theorem for quasi-contractions on D^* -metric spaces and obtain certain consequences

Keywords: Quasi-contraction, generalized quasi-contraction, orbit of x under f of length n

I. INTRODUCTION

The purpose of this paper is to establish a fixed point theorem for quasi-contractions on D^* -metric spaces and obtain certain consequences. In fact, we prove that the fixed point theorem for quasi-contractions on metric spaces, proved by Lj. B. Ćirić [1] as a particular case of the main result of this paper. The notion of Quasi-contraction defined for selfmaps of metric spaces given by Lj. B. Ćirić [2] has been extended to the selfmaps of D^* -metric spaces. The notion of quasi-contractions has been extended to include a wider class of selfmaps of metric spaces by Fisher [3] and we distinguish them as generalized quasi-contractions. Here we define below generalized quasi-contractions among selfmaps of D^* -metric spaces (X, D^*) .

II. PRELIMINARIES

A. *Definition:* A selfmap f of a D^* -metric space (X, D^*) is called a Quasi-contraction, if there is a number q with $0 \leq q < 1$ such that

$$D^*(fx, fy, fy) \leq q \cdot \max \{ D^*(x, y, y), D^*(x, fx, fx), D^*(y, fy, fy),$$

$$D^*(x, fy, fy), D^*(y, fx, fx) \}$$

for all $x, y \in X$.

B. *Definition:* A selfmap f of a D^* -metric space (X, D^*) is called a **generalized quasi-contraction**, if for some fixed positive integers k and l , there is a number q with $0 \leq q < 1$ such that

$$D^*(f^k x, f^l y, f^l y) \leq q \cdot \max \{ D^*(f^r x, f^s y, f^s y), D^*(f^r x, f^{r'} x, f^{r'} x),$$

$$D^*(f^s y, f^{s'} y, f^{s'} y) : 0 \leq r, r' \leq k; 0 \leq s, s' \leq l \}$$

for all $x, y \in X$.

C. *Definition:* Let f be a selfmap of a D^* -metric space (X, D^*) and $x \in X$, $n \geq 1$ be an integer. The orbit of x under f of length n , denoted by $O_f(x : n)$, is defined by

$$O_f(x : n) = \{x, fx, f^2x, \dots, f^n x\}$$

We define the diameter $\delta(A)$ of a set A in a D^* -metric space (X, D^*) by $\delta(A) = \sup_{x, y \in A} \{D^*(x, y, y)\}$

The following Lemmas are use full in proving fixed point theorems of quasi-contractions on D^* -metric spaces:

Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) and n be a positive integer. Then for each $x \in X$ and all integers $i, j \in \{1, 2, 3, \dots, n\}$,

$$D^*(f^i x, f^j x, f^j x) \leq q \cdot \delta [O_f(x:n)] < \delta [O_f(x:n)].$$

Let $x \in X$ be arbitrary, $n \geq 1$ be an integer and $i, j \in \{1, 2, 3, \dots, n\}$. Then $f^{i-1}x, f^{j-1}x, f^i x, f^j x \in O_f(x:n)$ and since f is a quasi-contraction,

$$\begin{aligned} D^*(f^i x, f^j x, f^j x) &= D^*(ff^{i-1}x, ff^{j-1}x, ff^{j-1}x) \\ &\leq q \cdot \max \{ D^*(f^{i-1}x, f^{j-1}x, f^{j-1}x), D^*(f^{i-1}x, f^i x, f^i x), \\ &\quad D^*(f^{j-1}x, f^j x, f^j x), D^*(f^{i-1}x, f^j x, f^j x), \\ &\quad D^*(f^{j-1}x, f^i x, f^i x) \} \\ &\leq q \cdot \text{Sup} \{ D^*(u, v, v) : u, v \in O_f(x:n) \} \\ &= q \cdot \delta [O_f(x:n)] \\ &< \delta [O_f(x:n)] \end{aligned}$$

D. *Lemma:* Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) and $x \in X$, then for every positive integer n , there exists positive integer $k \leq n$, such that

$$D^*(x, f^k x, f^k x) = \delta [O_f(x:n)]$$

1) *Proof:* If possible assume that the result is not true. This implies that there is positive integer m such that for all $k \leq m$, we have $D^*(x, f^k x, f^k x) \neq \delta [O_f(x:m)]$. Since $O_f(x:m)$ contains x and $f^k x$ for $k \leq m$, it follows that

$$D^*(x, f^k x, f^k x) < \delta [O_f(x:m)]$$

Since $O_f(x:m)$ is closed, there exists $i, j \in \{1, 2, 3, \dots, m\}$ such that $D^*(x, f^i x, f^j x) = \delta [O_f(x:m)]$, contradicting the Lemma 3.2.1. Therefore

$$D^*(x, f^k x, f^k x) = \delta [O_f(x:n)] \text{ for some } k \leq n.$$

E. *Lemma:* Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) , then

$$\delta [O_f(x:\infty)] \leq \frac{1}{1-q} D^*(x, fx, fx) \text{ for all } x \in X.$$

1) *Proof:* Let $x \in X$ be arbitrary. Since $O_f(x:1) \subseteq O_f(x:2) \subseteq \dots \subseteq O_f(x:n) \subseteq O_f(x:n+1) \subseteq \dots$, we get that

$$\delta [O_f(x:1)] \leq \delta [O_f(x:2)] \leq \dots \leq \delta [O_f(x:n)] \leq \delta [O_f(x:n+1)] \leq \dots, \text{ showing}$$

$$\lim_{n \rightarrow \infty} \delta [O_f(x:n)] = \text{Sup} \{ \delta [O_f(x:n)] : n = 1, 2, 3, \dots \}.$$

III. MAIN RESULT

A. *Theorem:* Suppose f is a quasi-contraction with constant q on a D^* -metric space (X, D^*) and X is f -orbitally complete. Then f has a unique fixed point $u \in X$. In fact,

B. $u = \lim_{n \rightarrow \infty} f^n x$ for any $x \in X$

and

C. $D^*(f^n x, u, u) \leq \frac{q^n}{1-q} D^*(x, fx, fx)$ for all $x \in X, n \geq 1$.

1) *Proof:* Let x be an arbitrary point of X . We claim that $\{f^n x\}$ is a Cauchy sequence in X . Let m, n be any positive integers with $n < m$. Since f is quasi-contraction,

$$\begin{aligned} D^*(f^n x, f^m x, f^m x) &= D^*(ff^{n-1} x, ff^{m-1} x, ff^{m-1} x) \\ &\leq q \cdot \delta [O_f(f^{n-1} x : m - n + 1)] \end{aligned}$$

That is,

D. $D^*(f^n x, f^m x, f^m x) \leq q \cdot \delta [O_f(f^{n-1} x : m - n + 1)]$

According to the Lemma 2.3, there exists an integer k_1 , with $0 \leq k_1 \leq m - n + 1$, such that

E. $\delta [O_f(f^{n-1} x : m - n + 1)] = D^*(f^{n-1} x, f^{k_1} f^{n-1} x, f^{k_1} f^{n-1} x)$

Using Lemma 2.4, we get

$$\begin{aligned} D^*(f^{n-1} x, f^{k_1} f^{n-1} x, f^{k_1} f^{n-1} x) &= D^*(ff^{n-2} x, f^{k_1+1} f^{n-2} x, f^{k_1+1} f^{n-2} x) \\ &\leq q \cdot \delta [O_f(f^{n-2} x : k_1 + 1)] \\ &\leq q \cdot \delta [O_f(f^{n-2} x : m - n + 2)] \end{aligned}$$

(Since $k_1 + 1 \leq m - n + 2$)

Thus

$$F. \quad D^*(f^{n-1}x, f^{k_1} f^{n-1}x, f^{k_1} f^{n-1}x) \leq q \cdot \delta \left[O_f \left(f^{n-2}x : m-n+2 \right) \right]$$

From (3.4), (3.5) and (3.6) we get

$$\begin{aligned} D^*(f^n x, f^m x, f^m x) &\leq q \cdot \delta \left[O_f \left(f^{n-1}x : m-n+1 \right) \right] \\ &\leq q^2 \cdot \delta \left[O_f \left(f^{n-2}x : m-n+2 \right) \right] \end{aligned}$$

Proceeding in this manner, we obtain

$$D^*(f^n x, f^m x, f^m x) \leq q^n \cdot \delta \left[O_f \left(x : m \right) \right]$$

Using Lemma 2.3, we get

$$G. \quad D^*(f^n x, f^m x, f^m x) \leq \frac{q^n}{1-q} \cdot \delta \left[O_f \left(x : m \right) \right]$$

Letting $n \rightarrow \infty$ and since $\lim_{n \rightarrow \infty} q^n = 0$, we get that $\{f^n x\}$ is Cauchy sequence. Again X being f -orbitally complete and

$\{f^n x\}$ is a Cauchy sequence in $O_f(x : \infty)$, there is a point $u \in X$ such that $u = \lim_{n \rightarrow \infty} f^n x$.

We shall now show that u is a fixed point of f .

Consider,

$$\begin{aligned} D^*(u, fu, fu) &\leq D^*(u, f^{n+1}u, f^{n+1}u) + D^*(f^{n+1}u, fu, fu) \\ &= D^*(u, f^{n+1}u, f^{n+1}u) + D^*(ff^n u, fu, fu) \\ &\leq D^*(u, f^{n+1}u, f^{n+1}u) + q \cdot \max \{ D^*(f^n u, u, u), \\ &\quad D^*(f^n u, f^{n+1}u, f^{n+1}u), D^*(u, fu, fu), \\ &\quad D^*(f^n u, fu, fu), D^*(u, f^{n+1}u, f^{n+1}u) \} \end{aligned}$$

$$\leq D^*(u, f^{n+1}u, f^{n+1}u) + q \cdot \{ D^*(f^n u, f^{n+1}u, f^{n+1}u) + D^*(f^n u, u, u) \\ + D^*(u, fu, fu) + D^*(f^{n+1}u, u, u) \}$$

Letting $n \rightarrow \infty$ and since $\lim_{n \rightarrow \infty} f^n x = u$, we get

$$D^*(u, fu, fu) = 0 \text{ and hence } fu = u, \text{ showing that } u \text{ is fixed point of } f.$$

To prove the uniqueness, let u, u' be two fixed points of f . That is, $fu = u, fu' = u'$

$$D^*(u, u', u') = D^*(fu, fu', fu') \\ \leq q \cdot \max \{ D^*(u, u', u'), D^*(u, fu, fu), D^*(u', fu', fu'), \\ D^*(u, fu', fu'), D^*(u', fu, fu) \} \\ D^*(u, u', u') \leq q \cdot \max \{ D^*(u, u', u'), D^*(u, fu, fu), D^*(u', fu', fu'), \\ D^*(u, fu', fu'), D^*(u', fu, fu) \}$$

$$\text{That is, } D^*(u, u', u') \leq q \cdot D^*(u, u', u')$$

Since $q < 1$, $D^*(u, u', u') = 0$, which implies that $u = u'$.

Letting $n \rightarrow \infty$ in (3.7) we get (3.3). This completes the proof of the theorem.

H. Theorem: Suppose f is a selfmap of a D^* -metric space (X, D^*) and X is f -orbitally complete. If there is a positive integer k such that f^k is a quasi-contraction with constant q . Then f has a unique fixed point $u \in X$. In fact,

$$I. \quad u = \lim_{n \rightarrow \infty} f^n x \text{ for any } x \in X$$

and

$$J. \quad D^*(f^n x, u, u) \leq \frac{q^n}{1-q} a(x) \text{ for all } x \in X, n \geq 1,$$

where $a(x) = \max \{ D^*(f^i x, f^{i+k} x, f^{i+k} x) : i = 1, 2, 3, \dots \}$ and $m = \left[\frac{n}{k} \right]$, the greatest integer not exceeding $\frac{n}{k}$.

1) *Proof:* Suppose f^k is a quasi-contraction of a D^* -metric space (X, D^*) . It has unique fixed point by Theorem 3.1. Let u be a fixed point of f^k . Then we claim that fu is also a fixed point of f^k . In fact,

$$f^k(fu) = f^{k+1}u = f(f^k u) = fu$$

By the uniqueness of fixed point of f^k , it follows that $fu = u$, showing that u is a fixed point of f . Uniqueness of the fixed point of f can be proved as in the Theorem 3.3.1.

To prove (3.10), let n be any integer. Then by the division algorithm, we have, $n = mk + j, 0 \leq j < k, m \geq 0$

Therefore $x \in X, f^n x = (f^k)^m f^j x$, since f^k is a quasi-contraction,

$$\begin{aligned} D^*(f^n x, u, u) &\leq \frac{q^m}{1-q} D^*(f^j x, f^k f^j x, f^k f^j x) \\ &\leq \frac{q^m}{1-q} \max \{ D^*(f^i x, f^k f^i x, f^k f^i x) : i = 0, 1, 2, \dots, k-1 \} \\ &\leq \frac{q^m}{1-q} \max \{ D^*(f^i x, f^{k+i} x, f^{k+i} x) : i = 0, 1, 2, \dots, k-1 \} \end{aligned}$$

proving (3.10). Letting $m \rightarrow \infty$, we get that $\lim_{n \rightarrow \infty} f^n x = u$, since $q^m \rightarrow 0$ as $m \rightarrow \infty$, proving (3.9). This completes the proof of the theorem.

K. Theorem: Let f be a quasi-contraction with constant q on a metric space (X, d) and X be f -orbitally complete, then f has a unique fixed point $u \in X$. In fact,

$$L. \quad u = \lim_{n \rightarrow \infty} f^n x \quad \text{for all } x \in X$$

and

$$M. \quad d(f^n x, u) \leq \frac{q^n}{1-q} \cdot d(x, fx) \quad \text{for all } x \in X, n \geq 1.$$

1) *Proof:* If (X, d) is a f -orbitally complete metric space, then it can be proved that (X, D_1^*) is a f -orbitally complete D^* -metric space and hence f -orbitally complete for any selfmap f of X . Also if f is a quasi-contraction with constant q of (X, d) , then the condition of quasi-contraction can be written as

$$\begin{aligned} D_1^*(fx, fy, fy) &\leq q \cdot \max \{ D_1^*(x, y, y), D_1^*(x, fx, fx), D_1^*(y, fy, fy), \\ &\quad D_1^*(x, fy, fy), D_1^*(y, fx, fx) \} \end{aligned}$$



for all $x, y \in X$, since $D_1^*(x, y, y) = d(x, y)$; so that f is a quasi-contraction on (X, D_1^*) . Thus f is a quasi-contraction on the f -orbitally complete D^* -metric space (X, D_1^*) and hence the conclusions of Theorem 3.1 hold for f ; which are the conclusions of the theorem.

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