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# Approximation Methods and Condition Sequences

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**Abstract:**The condition number of a bounded operator or a matrix  $A$  plays a vital role in the problem of solving the equation  $Ax = b$ . For an approximation method  $(A_n)$  for  $A$ , the behavior of the corresponding sequence  $(k(A_n))$  of condition numbers – simply known as the condition sequence – has much impact on the behavior of  $A$ . In this paper an attempt is made to expose some of the connections between the applicability and stability of approximation methods and the corresponding condition sequences.

## I. CONDITION NUMBERS

A. *Definition :* A bounded operator  $A$  defined on a Banach or Hilbert space  $X$  is said to be invertible if  $A$  is 1 – 1 and onto and  $A^{-1}$  is also a bounded operator.

For an invertible bounded operator  $A$ , the condition number  $k(A)$  of  $A$  is defined as

$$k(A) = \|A\| \|A^{-1}\|. \quad (1)$$

Suppose  $A$  is an infinite matrix. Then  $A$  can be regarded as an operator in  $\ell^2$ , where  $\ell^2$  is the space of all scalar sequences  $x = (x_1, x_2, x_3, \dots)$  such that  $\sum_{j=1}^{\infty} |x_j|^2$  is convergent. For  $x = (x_1, x_2, x_3, \dots) \in \ell^2$ , the norm of  $x$  is defined as

$$\|x\| = \left( \sum_{j=1}^{\infty} |x_j|^2 \right)^{1/2}. \quad (2)$$

The domain  $D(A)$  of  $A$  is the set of all  $x \in \ell^2$  such that  $Ax \in \ell^2$ .

$A$  is said to be bounded if  $D(A) = \ell^2$  and there exists  $\alpha < \infty$  such that

$$A(x) \leq \alpha \|x\| \text{ for all } x \in \ell^2. \quad (3)$$

We call a matrix  $A$  invertible if it is non singular; that is, if there exists a matrix  $A'$  such that

$$AA' = A'A = I, \quad (4) \text{ where } I \text{ is the identify matrix}$$

and  $A^{-1}$  is bounded.

B. *Definition :* For a bounded matrix  $A$ , its norm is defined by

$$\|A\| = \sup \{ \|Ax\| / x \in \ell^2 \|x\| \leq 1 \}. \quad (5)$$

Now (1) serves as the definition of condition numbers for matrices as well.

1) *Note :* The norm given in (5) is the standard norm for matrices. However, there are other norms as well for matrices and condition numbers may be defined accordingly

2) *Remark :* We have  $1 = \|I\| \leq \|AA^{-1}\| \leq \|A\| \|A^{-1}\| = k(A)$ . Thus  $k(A) \geq 1$  for any  $A$ .

C. *Condition Sequences*

Suppose  $(A_n)$  is a sequence of bounded invertible operators (matrices) such that  $A_n \rightarrow A$  in some sense. Discussion of the connection between the behavior of  $A$  and the corresponding condition sequence  $(k(A_n))$  is a vital area of research.

One of the basic questions before us is that whether  $k(A_n)$  converges to  $k(A)$ , when  $k(A) < \infty$ ? The answer is yes if both  $(A_n)$  converges to  $A$  and  $A_n^{-1}$  converges to  $A^{-1}$  in norm.

In that case,  $\|A_n\| \rightarrow \|A\|$  and  $\|A_n^{-1}\| \rightarrow \|A^{-1}\|$ , as 'norm' is continuous.

So,  $k(A_n) = \|A_n\| \|A_n^{-1}\| \rightarrow \|A\| \|A^{-1}\| = k(A)$ .

But, norm convergence cannot be expected always. Let us try to answer the question for approximation methods.

### D. Definition

Let  $H$  be an infinite dimensional Hilbert space and  $A$  be a bounded operator on  $H$ . Let  $(H_n)$  be an increasing sequence of finite dimensional subspaces of  $H$  and  $(A_n)$  be a sequence of bounded operators, with  $A_n \in B(H_n)$ , for each  $n$ , such that  $A_n P_n \rightarrow A$  pointwise. Then, the sequence  $(A_n)$  is called an approximation method for  $A$ .

Here  $P_n$  is the projection operator on  $H$  with  $R(P_n) = H_n$  for each  $n$ , where  $R(P_n)$  denotes the range of  $P_n$ . An approximation method  $(A_n)$  is said to be meaningful if  $A_n$  is invertible in  $B(H_n)$  for all  $n \geq n_0$  for some positive integer  $n_0$ . A meaningful approximation method  $(A_n)$  is said to be a stable method if

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty .$$

By passing to a subsequence of  $(A_n)$ , if necessary, we may assume that  $(A_n)$  is invertible for all  $n$  for a stable or applicable approximation method  $(A_n)$ . (6)

An approximation method  $(A_n)$  is said to be an applicable method for  $A$  if  $A_n^{-1} P_n y \rightarrow A^{-1} y$  for all  $y \in H$ .

The same definitions can be generalized to study approximation methods for densely defined closed (possibly unbounded) operators  $A$  as well. But, choosing an appropriate convergence is a challenging task here, since one cannot expect the usual convergence for unbounded operators. Now let us define the condition sequence corresponding to an approximation method.

### E. Definition

Let  $(A_n)$  be a meaningful approximation method for  $A$ . The sequence  $(k(A_n)) = (k(A_1), k(A_2), \dots)$  is called the condition sequence corresponding to  $(A_n)$ .

If  $(k(A_n))$  converges to  $k(A)$ , we may say that  $(A_n)$  is a well – conditioned approximation method for  $A$ .

1) *Theorem:* Let  $A \in B(H)$ . Put  $A_n = A P_n|_{H_n}$ , where  $H_1 \subset H_2 \subset \dots \subset H$  and  $\cup H_n$  is dense in  $H$ . Then,  $\lim_{x \rightarrow \infty} \|A_n\| = \|A\|$ .

2) *Proof:* As  $\cup H_n$  is dense in  $H$ ,  $A_n P_n x \rightarrow Ax$  for all  $x \in H$ .

We claim that  $\|A_n\| \rightarrow \|A\|$ .

Let  $\epsilon > 0$  be given. Then,  $\|A\| - \epsilon < \|A\|$ . Since  $\|A_n\| = \sup \{ \|Ax\| / \|x\| : x \in H, \|x\| \leq 1 \}$ , there exists  $x_0 \in H$  with  $\|x_0\| \leq 1$  such that

$$\|A_n\| - \epsilon < \|Ax_0\|. \tag{8}$$

But  $Ax_0 = \lim_{x \rightarrow \infty} A_n P_n x_0$ .

$$\text{So, } \|Ax_0\| = \left\| \lim_{n \rightarrow \infty} A_n P_n x_0 \right\| = \lim_{x \rightarrow \infty} \|A_n P_n x_0\|. \tag{9}$$

From (8) and (9),  $\lim_{n \rightarrow \infty} \|A_n P_n x_0\| > \|A\| - \epsilon$ .

Hence, there exists a positive integer  $n_0$  such that

$$\|A_n P_n x_0\| > \|A\| - \epsilon \text{ for all } n \geq n_0. \tag{10}$$

But  $\|A_n P_n x_0\| \leq \|A_n P_n\| \|x_0\| \leq \|A_n P_n\| \leq \|A_n\| \|P_n\| \leq \|A_n\|$ , since  $\|P_n\| \leq 1$  and  $\|x_0\| \leq 1$ .

Therefore, (10) becomes

$$\|A\| - \epsilon < \|A_n P_n\| \leq \|A_n\| \text{ for all } n \geq n_0. \tag{11}$$

Now, Fix  $n$ .

For  $x \in H_n$ ,  $P_n x = x$ .

So,  $A_n x = P_n A P_n x = P_n A x$ .

Therefore,  $\|A_n x\| = \|P_n A x\| \leq \|P_n\| \|A x\| \leq \|A x\|$ , as  $\|P_n\| \leq 1$ .

That is,  $\|A_n\| \leq \|A\|$ . Thus,  $\|A_n\| \leq \|A\|$  for all  $n$ . (12)

From (11) and (12),  $\|A_n\| - \epsilon \leq \|A_n\| \leq \|A\| < \|A\| + \epsilon$  for all  $n \geq n_0$ .

Hence  $\lim_{n \rightarrow \infty} \|A_n\| = \|A\|$ . □

The sequence  $(A_n)$ , where  $A_n = A P_n|_{H_n}$  is known as the projection method.

Let  $A \in B(H)$  be invertible. Put  $A_n = A P_n|_{H_n}$  where  $H_1 \subset H_2 \subset \dots \subset H$  and  $\cup H_n$  is dense in  $H$ . Suppose  $(A_n)$  is an applicable method for  $A$ . Then,  $(A_n)$  is well conditioned.

By Theorem 2.3,  $\lim_{n \rightarrow \infty} \|A_n\| = \|A\|$ .

As  $(A_n)$  is an applicable method for  $A$ ,  $A_n^{-1} P_n x \rightarrow A^{-1} x$  for all  $x \in H$ .

Then, replacing  $A_n$  by  $A_n^{-1}$  and  $A$  by  $A^{-1}$  in the proof of Theorem 2.3 we get

$$\lim_{n \rightarrow \infty} \|A_n^{-1}\| = \|A^{-1}\|.$$

Now,  $\lim_{n \rightarrow \infty} \|A_n\| \|A_n^{-1}\| = \left(\lim_{n \rightarrow \infty} \|A_n\|\right) \left(\lim_{n \rightarrow \infty} \|A_n^{-1}\|\right) = \|A\| \|A^{-1}\|$ .

That is,  $\lim_{n \rightarrow \infty} k(A_n) = k(A)$  and hence  $(A_n)$  is a well conditioned approximation method. □

Suppose  $(A_n)$  is an approximation method for  $A$ . that is,  $A_n P_n \rightarrow A$  strongly. Then,  $(A_n)$  is an applicable method for  $A$  if and only if  $(A_n)$  is well conditioned. This is a basic result for approximation methods. One can refer [2] for a proof.

Using this result we are able to prove the following result for condition sequences of monotonic approximation methods

**E. Definition**

An approximation method  $(A_n)$  is said to be monotonic if  $\|A_n\| \leq \|A_m\|$  for  $n < m$ .

**F. Theorem**

Let  $(A_n)$  be a monotonic approximation method for  $A$ . Then,  $(A_n)$  is a stable method if and only if  $(k(A_n))$  is a bounded sequence.

*1) Proof:* Suppose  $(A_n)$  is stable.

Without loss of generality, we may assume that  $A_n$  is invertible for all  $n$  and  $\|A_n^{-1}\| \leq \alpha$ .

Since  $(A_n)$  is an approximation method for  $A$ ,  $A_n P_n x \rightarrow Ax$  for all  $x \in X$ .

By the Uniform Boundedness Theorem,  $\|A_n\| \leq \beta$  for all  $n$ , for some  $\beta < \infty$ .

Hence  $\|A_n\| \|A_n^{-1}\| \leq \beta \alpha$  for all  $n$  and  $k(A_n)$  is bounded.

Conversely suppose that  $k(A_n)$  is bounded.

Then there exists  $\gamma < \infty$  such that  $k(A_n) \leq \gamma$  for all  $n$ .

That is  $\|A_n\| \|A_n^{-1}\| \leq \gamma$  (13)

As  $(A_n)$  is monotonic,  $\|A_1\| \leq \|A_2\| \leq \dots$

Thus,  $\|A_n\| \geq \|A_1\| > 0$  for all  $n$ , since  $A_1 \neq 0$ , as  $A_1$  is invertible.

Now (13) implies,  $\|A_n^{-1}\| \leq \frac{\gamma}{\|A_n\|} \leq \frac{\gamma}{\|A_1\|}$  for all  $n$ .

Thus  $(A_n)$  is stable.  $\square$

Hence  $(A_n)$  is an applicable method for  $A$ .

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